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ON THE CONE CONJECTURE FOR CALABI–YAU MANIFOLDS WITH PICARD NUMBER TWO

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Abstract. Following a recent work of Oguiso, we calculate explicitly the groups of automorphisms and birational automorphisms on a Calabi–Yau manifold with Picard number two. When the group of birational automorphisms is infinite, we prove that the Cone conjecture of Morrison and Kawamata holds.

Contents

1. Introduction

The Cone conjecture of Morrison and Kawamata is concerned with the structure of the nef and the movable cones on a Calabi–Yau manifold in presence of automorphisms or birational automorphisms. To be more precise, consider a Calabi–Yau manifold X with nef cone $\mathrm{Nef}(X)$, the movable cone $\overline{\mathrm{Mov}}(X)$, and effective cone $\mathrm{Eff}(X)$. A Calabi–Yau manifold in our context is a projective manifold X with trivial canonical bundle, such that $H^1(X,\mathcal{O}_X)=0$. As usual, $\mathrm{Aut}(X)$, respectively $\mathrm{Bir}(X)$, denotes the group of automorphisms, respectively birational automorphisms of X. Then, the Cone conjecture can be stated as follows.

Conjecture 1.1. *Let X be a Calabi–Yau manifold.*

(1) *There exists a rational polyhedral cone Π which is a fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)\cap\mathrm{Eff}(X)$, in the sense that*

$$\mathrm{Nef}(X)\cap\mathrm{Eff}(X)=\bigcup_{g\in\mathrm{Aut}(X)} g^*\Pi,$$

and $\mathrm{int}\,\Pi\cap\mathrm{int}\,g^*\Pi=\emptyset$ *unless* $g^*=\mathrm{id}$.

(2) *There exists a rational polyhedral cone Π' which is a fundamental domain for the action of $\mathrm{Bir}(X)$ on $\overline{\mathrm{Mov}}(X)\cap\mathrm{Eff}(X)$.*

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There is also the following weaker form.

Conjecture 1.2. *Let X be a Calabi–Yau manifold.*

(1) *There exists a (not necessarily closed) cone Π which is a weak fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)\cap\mathrm{Eff}(X)$, in the sense that*

$$\mathrm{Nef}(X)\cap\mathrm{Eff}(X)=\bigcup_{g\in\mathrm{Aut}(X)} g^*\Pi,$$

$\mathrm{int}\,\Pi\cap\mathrm{int}\,g^*\Pi=\emptyset$ *unless* $g^*=\mathrm{id}$, *and for every* $g\in\mathrm{Aut}(X)$, *the intersection* $\Pi\cap g^*\Pi$ *is contained in a rational hyperplane.*

(2) *There exists a polyhedral cone Π' which is a weak fundamental domain for the action of $\mathrm{Bir}(X)$ on $\overline{\mathrm{Mov}}(X)\cap\mathrm{Eff}(X)$.*

For the study of the Cone conjectures, the action

$$r\colon\ \mathrm{Bir}(X)\to\mathrm{GL}(N^1(X))$$

on the Neron–Severi group $N^1(X)$ is important. We denote by $\mathcal{B}(X)$ its image, and by $\mathcal{A}(X)$ the image of the automorphism group.

Based on and inspired by recent work of Oguiso [6], we prove the following results.

Theorem 1.3. *Let X be a Calabi–Yau manifold of Picard number 2. Then, either $|\mathcal{A}(X)|\le 2$, or $\mathcal{A}(X)$ is infinite; and either $|\mathcal{B}(X)|\le 2$, or $\mathcal{B}(X)$ is infinite.*

In fact, we explicitly calculate the groups $\mathcal{A}(X)$ and $\mathcal{B}(X)$, and for more detailed information we refer to Section 3. The consequences for the Cone conjectures can be summarized as follows.

Theorem 1.4. *Let X be a Calabi–Yau manifold with Picard number 2. Then*

(1) *If the group $\mathrm{Bir}(X)$ is finite, then the weak Cone conjecture holds on X.*
(2) *If the group $\mathrm{Bir}(X)$ is infinite, then the Cone conjecture holds on X.*

Oguiso in [6] showed that there are indeed Calabi–Yau 3-folds X with $\rho(X)=2$ and with infinite $\mathrm{Bir}(X)$, as well as hyperkähler 4-folds X with $\rho(X)=2$ and with infinite $\mathrm{Aut}(X)$.

2. Preliminaries

In this section, we give some basic definitions and gather results which we need in this paper.

A *Calabi–Yau manifold* of dimension n is a projective manifold X with trivial canonical bundle $K_X\simeq\mathcal{O}_X$ such that $H^1(X,\mathcal{O}_X)=0$. In particular, we do not require X to be simply connected.

Let $N^1(X)$ be the Neron–Severi group, generated by the classes of the line bundles on X and let $N^1(X)_\mathbb{R}$ be the corresponding real vector space in $H^2(X,\mathbb{R})$. As usual, $\mathrm{Nef}(X)\subseteq N^1(X)_\mathbb{R}$ denotes the closed cone of nef divisors, $\mathrm{Big}(X)$ stands for the open cone of big divisors, $\overline{\mathrm{Mov}}(X)$ is the closure of the cone generated by mobile divisors (that is, effective divisors whose base locus does not contain divisors), and $\mathrm{Mov}(X)$ is its interior. Finally, $\mathrm{Eff}(X)$ is the effective cone, and $\overline{\mathrm{Eff}}(X)$ is the pseudo-effective cone (the closure of the effective cone, or equivalently, the closure of the big cone).

On a normal $\mathbb{Q}$-factorial projective variety X with terminal singularities and nef canonical class, $\mathrm{Aut}(X)$ denotes the automorphism group and $\mathrm{Bir}(X)$ the group of birational automorphisms. We obtain a natural homomorphism

$$r\colon \mathrm{Bir}(X) \to \mathrm{GL}(N^1(X))$$

given by $g \mapsto g^*$.

Notation 2.1. Assume that a Calabi–Yau manifold X has Picard number $\rho(X) = 2$. We let ℓ_1, ℓ_2 be the two boundary rays of $\mathrm{Nef}(X)$, and let m_1, m_2 be the boundary rays of $\overline{\mathrm{Mov}}(X)$. We fix non-trivial elements $x_i \in \ell_i$ and $y_i \in m_i$. We set

$$\mathcal{A}(X) = r\big(\mathrm{Aut}(X)\big) \quad \text{and} \quad \mathcal{B}(X) = r\big(\mathrm{Bir}(X)\big).$$

It is well-known, see for instance [6, Proposition 2.4], that the group $\mathrm{Bir}(X)$ is finite if and only if $\mathcal{B}(X)$ is, and similarly for $\mathrm{Aut}(X)$ and $\mathcal{A}(X)$.

Recall also the following result [6, Proposition 3.1].

Proposition 2.2. *Let X be a Calabi–Yau manifold of dimension n such that $\rho(X) = 2$.*

(1) *If n is odd, or if one of the ℓ_i is rational, then every non-trivial element of $\mathcal{A}(X)$ has order* 2.
(2) *If one of the m_i is rational, then every non-trivial element of $\mathcal{B}(X)$ has order* 2.

As a consequence, by using Burnside's theorem, Oguiso obtains:

Theorem 2.3. *Let X be a Calabi–Yau manifold of dimension n such that $\rho(X) = 2$.*

(1) *If n is odd, then* $\mathrm{Aut}(X)$ *is finite.*
(2) *If n is even and one of the rays ℓ_i is rational, then* $\mathrm{Aut}(X)$ *is finite.*
(3) *If one of the rays m_i is rational, then* $\mathrm{Bir}(X)$ *is finite.*

Proposition 3.3 below makes this result more precise. In contrast to Theorem 2.3, Oguiso constructed an example of Calabi–Yau manifold with $\rho(X) = 2$ such that $\mathrm{Bir}(X)$ is infinite. In this example both rays m_i are irrational, and we recall it in Example 4.6.

If g is any element of $\mathcal{B}(X)$, then $\det g = \pm 1$ since g acts on the integral lattice $N^1(X)$. We introduce the notations

$$\mathcal{A}^+(X) = \{g \in \mathcal{A}(X) \mid \det g = 1\}$$

and

$$\mathcal{A}^-(X) = \{g \in \mathcal{A}(X) \mid \det g = -1\};$$

and similarly $\mathcal{B}^+(X)$ and $\mathcal{B}^-(X)$. Note that each $g \in \mathcal{A}(X)$ restricts to an action on the set $\ell_1 \cup \ell_2$, and each $g \in \mathcal{B}(X)$ restricts to an action on the set $m_1 \cup m_2$. Moreover, since the cone $\overline{\mathrm{Eff}}(X)$ does not contain lines, this "restricted" action completely determines g. Additionally, each $g \in \mathcal{A}(X)$ is completely determined by gx_1 since $\det g = \pm 1$. Similarly, each $g \in \mathcal{B}(X)$ is completely determined by gy_1.

We frequently and without explicit mention use the following well-known lemma, see for instance [4, Lemma 1.5].

Lemma 2.4. *Let X be a Calabi–Yau manifold. Then, $g \in \mathrm{Bir}(X)$ is an automorphism if and only if there exists an ample divisor H on X such that g^*H is ample.*

We also use the following result [3, Theorem 5.7], [4, Corollary 2.7], [2, Theorem 3.8].

Theorem 2.5. *Let X be a Calabi–Yau manifold. Then, the cones* $\operatorname{Nef}(X)$ *and* $\overline{\operatorname{Mov}}(X)$ *are locally rational polyhedral in* $\operatorname{Big}(X)$.

3. Calculating Aut(X) and Bir(X)

In this section we calculate explicitly the groups $\mathcal{A}(X)$ and $\mathcal{B}(X)$ on a Calabi–Yau manifold with Picard number 2. We start with some elementary observations.

Lemma 3.1. *Let X be a Calabi–Yau manifold such that $\rho(X) = 2$. If $g \in \mathcal{B}^-(X)$, then $g^2 = \mathrm{id}$.*

Proof. By assumption there exist $\alpha > 0$ and $\beta > 0$ such that $gy_1 = \alpha y_2$ and $gy_2 = \beta y_1$. However, then $g^2 y_1 = \alpha\beta y_1$ and $g^2 y_2 = \alpha\beta y_2$, and we have $g^2 \in \mathcal{A}^+(X)$. Therefore $\det(g^2) = (\alpha\beta)^2 = 1$, so $\alpha\beta = 1$. Thus, g^2 is the identity. □

Lemma 3.2. *Let X be a Calabi–Yau manifold such that $\rho(X) = 2$. Then, $\mathcal{B}^-(X) = \mathcal{B}^+(X)g$ for any $g \in \mathcal{B}^-(X)$. Similarly, $\mathcal{A}^-(X) = \mathcal{A}^+(X)h$ for any $h \in \mathcal{A}^-(X)$.*

In particular, if $\mathcal{B}(X)$ is infinite, so is $\mathcal{B}^+(X)$; and if $\mathcal{A}(X)$ is infinite, so is $\mathcal{A}^+(X)$.

Proof. Let $g, g' \in \mathcal{B}^-(X)$. Then, $g'g = f \in \mathcal{B}^+(X)$, and since $g^2 = \mathrm{id}$ by Proposition 2.2, we have $g' = fg \in \mathcal{B}^+(X)g$. The proof in the case of automorphisms is identical. □

Proposition 3.3. *Let X be a Calabi–Yau manifold such that $\rho(X) = 2$. If $\mathcal{A}(X)$ is finite, then $|\mathcal{A}^+(X)| = 1$ and $|\mathcal{A}(X)| \leq 2$. If $\mathcal{B}(X)$ is finite, then $|\mathcal{B}^+(X)| = 1$ and $|\mathcal{B}(X)| \leq 2$.*

In particular, if n is odd, or if one of the ℓ_i is rational, then $|\mathcal{A}(X)| \leq 2$.

Proof. Assume that $\mathcal{A}(X)$ is finite, and fix $g \in \mathcal{A}(X)$. If $g \in \mathcal{A}^+(X)$, then there exists $\alpha > 0$ such that $gx_1 = \alpha x_1$. Then, $g^m = \mathrm{id}$ for some positive integer m, hence $\alpha^m = 1$, and therefore $\alpha = 1$ and $\mathcal{A}^+(X) = \{\mathrm{id}\}$. Now $|\mathcal{A}(X)| \leq 2$ by Lemma 3.2. The proof for $\mathcal{B}(X)$ is the same, and the last claim follows from Theorem 2.3. □

Proposition 3.3 can also be directly deduced from the following elementary lemma, simplifying calculations in [6].

Lemma 3.4. *Let X be an n-dimensional Calabi–Yau manifold with $\rho(X) = 2$. Assume that $|\mathcal{A}^+(X)| \neq 1$. Then,*

$$x_1^m \cdot x_2^{n-m} = 0$$

for all m unless $n = 2m$.

If $n = 2m$, then $x_1^m \neq 0$ and $x_2^m \neq 0$.

Proof. Let f be a non-trivial element in $\mathcal{A}^+$. Then, $fx_1 = \alpha x_1$ and $fx_2 = \alpha^{-1} x_2$ with $\alpha > 0$, $\alpha \neq 1$. Then

$$(fx_1)^m \cdot (fx_2)^{n-m} = \alpha^{2m-n} x_1^m \cdot x_2^{n-m}.$$

On the other hand,

$$(fx_1)^m \cdot (fx_2)^{n-m} = x_1^m \cdot x_2^{n-m},$$

hence $x_1^m \cdot x_2^{n-m} = 0$ unless $n = 2m$.

For the second statement, observe that $x_1 + x_2$ is an ample class, hence

$$0 < (x_1 + x_2)^n = \binom{n}{m} x_1^m \cdot x_2^m,$$

and therefore the classes x_i^m are non-zero. □

Corollary 3.5. *Let X be a Calabi–Yau manifold of dimension n such that $\rho(X) = 2$. If the group $\mathrm{Aut}(X)$ is infinite, then the following holds.*

(1) *n is even and the rays ℓ_i are irrational.*
(2) $\mathrm{Nef}(X) = \overline{\mathrm{Eff}}(X)$, *and* $\mathrm{Nef}(X) \cap \mathrm{Eff}(X) = \mathrm{Amp}(X)$.
(3) $c_{n-1}(X) = 0$ *in* $H^{2n-2}(X, \mathbb{Q})$.

Proof. Claim (1) is Oguiso's Theorem 2.3.

For the first part of (2), if $\mathrm{Nef}(X) \neq \overline{\mathrm{Eff}}(X)$, then at least one boundary ray of $\mathrm{Nef}(X)$ is rational by Theorem 2.5. This contradicts (1). For the second part of (2), without loss of generality it suffices to show that x_1 is not effective. Otherwise, we can write $x_1 = \sum \delta_j D_j \geq 0$ as a sum of at least two prime divisors, since x_1 is irrational. However, then ℓ_1 is not an extremal ray of the cone $\mathrm{Nef}(X) = \overline{\mathrm{Eff}}(X)$, a contradiction.

For (3), note that $|\mathcal{A}^+(X)| \geq 2$ by Lemma 3.2. Pick a non-trivial element $f \in \mathcal{A}^+(X)$, and let $\alpha \neq 1$ be a positive number such that $fx_1 = \alpha x_1$. Then,

$$\alpha x_1 \cdot c_{n-1}(X) = fx_1 \cdot c_{n-1}(X) = x_1 \cdot c_{n-1}(X)$$

since the Chern class $c_{n-1}(X)$ is invariant under f. Thus, $x_1 \cdot c_{n-1}(X) = 0$; similarly we get $x_2 \cdot c_{n-1}(X) = 0$. Therefore $c_{n-1}(X) = 0$ as $\{x_1, x_2\}$ is a basis of $N^1_{\mathbb{R}}(X)$. □

Remark 3.6. (1) The same arguments as in Corollary 3.5 yield

$$c_{i_1}(X) \cdot \ldots \cdot c_{i_r}(X) = 0$$

if $i_1 + \cdots + i_r = n - 1$.

(2) We do not know of any example of a simply connected Calabi–Yau manifold X in the strong sense, (i.e., such that $H^q(X, \mathcal{O}_X) = 0$ for $1 \leq q \leq n-1$) of even dimension n such that $c_{n-1}(X) = 0$. One might wonder whether any simply connected irreducible projective manifold X of dimension n with $\omega_X \simeq \mathcal{O}_X$ and $c_{n-1}(X) = 0$ is a hyperkähler manifold.

In some further cases, the even dimensional case can be treated.

Theorem 3.7. *Let X be a Calabi–Yau manifold of even dimension n. If $\rho(X) = 2$ and if $c_2(X)$ can be represented by a positive closed $(2,2)$-form, then $\mathrm{Aut}(X)$ is finite.*

Proof. Arguing by contradiction, we suppose that there is an automorphism $f \in \mathcal{A}^+(X)$ of infinite order; cf. Lemma 3.2. Write $n = 2m$. Then, $x_1^m \neq 0$ and $x_2^m \neq 0$ by Lemma 3.4.

Suppose that m is even, and write $m = 2k$. Then

$$x_1^{2k} \cdot c_2(X)^k > 0$$

by our positivity assumption on $c_2(X)$. On the other hand,

$$x_1^{2k} \cdot c_2(X)^k = (fx_1)^{2k} \cdot c_2(X)^k = \alpha^{2k} x_1^{2k} \cdot c_2(X)^k$$

since $c_2(X)$ is invariant under f. Since $\alpha \neq 1$, this is a contradiction.

If m is odd, we write $n = 4s + 2$ and argue with $x_1^{2s} \cdot c_2(X)^{s+1}$. □

Notice that for every projective manifold X of dimension n with nef canonical bundle, the second Chern class $c_2(X)$ has the following positivity property (Miyaoka [5]):

$$c_2(X) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0$$

for all ample line bundles H_j.

Concerning bounds for $\mathcal{B}(X)$, we have:

Proposition 3.8. *Let X be a Calabi–Yau manifold such that $\rho(X) = 2$. Assume that* $\mathrm{Nef}(X) \not\subseteq \mathrm{Mov}(X)$. *Then, $\mathcal{A}^+(X) = \mathcal{B}^+(X)$. In particular, if the dimension of X is odd, then $|\mathcal{B}(X)| \leq 2$.*

Proof. The condition $\mathrm{Nef}(X) \not\subseteq \mathrm{Mov}(X)$ implies that one of the rays ℓ_i is an extremal ray of $\overline{\mathrm{Mov}}(X)$. Hence, without loss of generality, we may assume that $m_1 = \ell_1$. Let g be a non-trivial element of $\mathcal{B}^+(X)$. Then, $g\ell_1 = gm_1 = m_1$, and m_1 is an extremal ray of the cone

$$\mathbb{R}_+ m_1 + \mathbb{R}_+ g\ell_2 = \mathbb{R}_+ g\ell_1 + \mathbb{R}_+ g\ell_2 = g\,\mathrm{Nef}(X).$$

This implies that $g\,\mathrm{Nef}(X)$ intersects the interior of $\mathrm{Nef}(X)$, and hence $g \in \mathcal{A}(X)$ by Lemma 2.4. This proves the first claim.

The second claim then follows from Proposition 3.3. □

Theorem 3.9. *Let X be a Calabi–Yau manifold such that $\rho(X) = 2$. Then, either $|\mathcal{A}^+(X)| = 1$ or $\mathcal{A}^+(X) \simeq \mathbb{Z}$; and either $|\mathcal{B}^+(X)| = 1$ or $\mathcal{B}^+(X) \simeq \mathbb{Z}$.*

Proof. Assume that $|\mathcal{A}^+(X)| \geq 2$. For every $g \in \mathcal{A}^+(X)$, let α_g be the positive number such that $gy_1 = \alpha_g y_1$, and set

$$\mathcal{S} = \{\alpha_g \mid g \in \mathcal{A}^+(X)\}.$$

Note that $\mathcal{S}$ is a multiplicative subgroup of $\mathbb{R}^*$ and that the map

$$\mathcal{A}^+(X) \to \mathcal{S}, \quad g \mapsto \alpha_g$$

is an isomorphism of groups. We need to show that $\mathcal{S}$ is an infinite cyclic group.

We first show that $\mathcal{S}$ is, as a set, bounded away from 1. Otherwise, we can pick a sequence (g_i) in $\mathcal{A}^+(X)$ such that α_{g_i} converges to 1. Fix two integral linearly independent classes h_1 and h_2 in $N^1(X)_\mathbb{R}$. Then, $g_i h_1$ converge to h_1 and $g_i h_2$ converge to h_2. Since $g_i h_1$ and $g_i h_2$ are also integral classes and $N^1(X)$ is a lattice in $N^1(X)_\mathbb{R}$; this implies that $g_i h_1 = h_1$ and $g_i h_2 = h_2$ for $i \gg 0$, and hence $g_i = \mathrm{id}$ for $i \gg 0$.

Hence, the set $\mathcal{S}' = \{\ln \alpha \mid \alpha \in \mathcal{S}\}$ is an additive subgroup of $\mathbb{R}$ which is discrete as a set. Then, it is a standard fact that $\mathcal{S}'$, and hence $\mathcal{S}$ is isomorphic to $\mathbb{Z}$, cf. [1, 21.1].

The proof for the birational automorphism group is the same. □

4. Structures of $\mathrm{Nef}(X)$ and $\overline{\mathrm{Mov}}(X)$

Proposition 4.1. *Let X be a Calabi–Yau manifold such that $\rho(X) = 2$. If $\mathcal{A}(X)$ is finite, then the weak Cone conjecture holds for* $\mathrm{Nef}(X)$. *If $\mathcal{B}(X)$ is finite, then the weak Cone conjecture holds for* $\overline{\mathrm{Mov}}(X)$.

Proof. We only prove the statement about the nef cone, since the other statement is analogous. By Proposition 3.3, we have $|\mathcal{A}(X)| \leq 2$, hence we may assume that $|\mathcal{A}(X)| = 2$. Fix an integral class $x \in \operatorname{Nef}(X)$, let $g \in \mathcal{A}^-(X)$, and consider the class $y = x + gx \in \operatorname{Nef}(X)$. Then, y is fixed under the action of $\mathcal{A}(X)$. Since g acts on $N^1(X)$, both gx and y must be integral. It is then obvious that $\Pi = \ell_1 + \mathbb{R}_+ y$ is a fundamental domain for the action of $\mathcal{A}(X)$ on $\operatorname{Nef}(X)$. □

Remark 4.2. If X is a Calabi–Yau manifold of odd dimension such that $\rho(X) = 2$ and $\operatorname{Nef}(X) \not\subseteq \operatorname{Mov}(X)$, then the weak Cone conjecture holds for $\overline{\operatorname{Mov}}(X)$. The proof is analogous to that of Proposition 4.1, using Proposition 3.8.

Proposition 4.3. *Let X be a Calabi–Yau manifold such that $\rho(X) = 2$. Assume that* $\operatorname{Nef}(X) \subseteq \operatorname{Mov}(X)$*. Then, the Cone conjecture holds for* $\operatorname{Nef}(X)$*.*

Proof. By assumption, we have $\operatorname{Nef}(X) \subseteq \operatorname{Big}(X)$, and hence, the nef cone is rational polyhedral by Theorem 2.5. Then, argue as in the proof of Proposition 4.1. □

Lemma 4.4. *Let X be a Calabi–Yau manifold with $\rho(X) = 2$. Assume that* $\operatorname{Bir}(X)$ *is infinite. Then,* $\overline{\operatorname{Mov}}(X) \cap \operatorname{Eff}(X) = \operatorname{Mov}(X)$*.*

Proof. The rays of $\overline{\operatorname{Mov}}(X)$ are irrational by Proposition 2.2, and therefore $\overline{\operatorname{Mov}}(X) = \overline{\operatorname{Eff}}(X)$ by Theorem 2.5. We cannot have $y_1 \in \operatorname{Eff}(X)$: otherwise, we can write $y_1 = \sum \delta_i D_i \geq 0$ as a sum of at least two different prime divisors, since m_1 is irrational. However, then m_1 is not an extremal ray of the cone $\overline{\operatorname{Mov}}(X) = \overline{\operatorname{Eff}}(X)$, a contradiction. This concludes the proof. □

Theorem 4.5. *Let X be a Calabi–Yau manifold with $\rho(X) = 2$. If the group* $\operatorname{Bir}(X)$ *is infinite, then the Cone conjecture holds on X.*

Proof. (i) First we show that the Cone conjecture holds for $\operatorname{Nef}(X)$ in case $\operatorname{Aut}(X)$ is infinite.

Note that $\operatorname{Nef}(X) = \overline{\operatorname{Eff}}(X)$ and $\operatorname{Nef}(X) \cap \operatorname{Eff}(X) = \operatorname{Amp}(X)$ by Corollary 3.5(2), and in particular we have $\mathcal{A}(X) = \mathcal{B}(X)$. By Lemma 3.2 and Theorem 3.9, we know that $\mathcal{A}(X) = \mathcal{A}^+(X) \cup \mathcal{A}^-(X)$, where $\mathcal{A}^+(X) \simeq \mathbb{Z}$ and $\mathcal{A}^-(X) = \mathcal{A}^+(X)g$ for any $g \in \mathcal{A}^-(X)$.

Assume first that $\mathcal{A}(X) = \mathcal{A}^+(X) \simeq \mathbb{Z}$. Let h be a generator of $\mathcal{A}(X)$, let x be any point in $\operatorname{Amp}(X)$, and denote

$$\Pi = \mathbb{R}_+ x + \mathbb{R}_+ hx.$$

It is then straightforward to check that Π is a fundamental domain for the action of $\mathcal{A}(X)$ on $\operatorname{Amp}(X)$. Indeed, it is clear that the cones $h^k\Pi$ have disjoint interiors, and to see that they cover $\operatorname{Amp}(X)$, it suffices to notice that the rays $\mathbb{R}_+ h^k x$ converge to ℓ_1, ℓ_2, when $k \to \pm\infty$.

Now assume that $\mathcal{A}^-(X) \neq \emptyset$. Let f be a generator of $\mathcal{A}^+(X)$, let τ be an element of $\mathcal{A}^-(X)$, and let x be an integral class in $\operatorname{Amp}(X)$. Set

$$z_1 = x + \tau x \quad \text{and} \quad z_2 = z_1 + f z_1,$$

and note that z_1 and z_2 are integral classes since τ and f act on $N^1(X)$. Denote $\theta = f\tau \in \mathcal{A}^-(X)$. Then, $\tau^2 = \theta^2 = \mathrm{id}$ by Lemma 3.1, and hence

$$\theta\tau = (f\tau)\tau = f \quad \text{and} \quad \theta f = \theta(\theta\tau) = \tau.$$

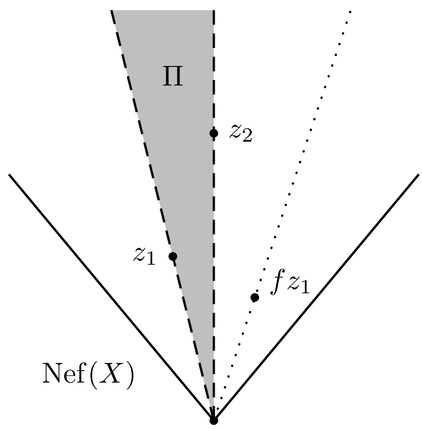
This implies

$$\tau z_1 = z_1, \quad \theta z_1 = f z_1, \quad \theta z_2 = z_2. \tag{4.1}$$

Now, let

$$\Pi = \mathbb{R}_+ z_1 + \mathbb{R}_+ z_2.$$

Then, Π is a rational polyhedral cone, and we claim that Π is a fundamental domain for the action of $\mathcal{A}(X)$ on $\operatorname{Amp}(X)$.



First, by (4.1) we have

$$\theta\Pi = \mathbb{R}_+ \theta z_1 + \mathbb{R}_+ \theta z_2 = \mathbb{R}_+ f z_1 + \mathbb{R}_+ z_2,$$

and thus

$$\Pi \cup \theta\Pi = \mathbb{R}_+ z_1 + \mathbb{R}_+ f z_1.$$

This implies

$$\bigcup_{k \in \mathbb{Z}} f^k(\Pi \cup \theta\Pi) = \operatorname{Amp}(X)$$

as in the first part of the proof, and therefore,

$$\bigcup_{g \in \mathcal{A}(X)} g\Pi = \operatorname{Amp}(X).$$

Second, assume that there exists $\lambda \in \mathcal{A}(X)$ such that $\operatorname{int}\Pi \cap \operatorname{int}\lambda\Pi \neq \emptyset$. Then, possibly after replacing λ by λ^{-1}, this implies that $\lambda z_1 \subseteq \operatorname{int}\Pi$ or $\lambda z_2 \subseteq \operatorname{int}\Pi$. If $\lambda z_1 \subseteq \operatorname{int}\Pi$, then by Lemma 3.2 there exists $k \in \mathbb{Z}$ such that $\lambda = f^k\tau$, hence $\lambda z_1 = f^k z_1 \in \operatorname{int}\Pi$ by (4.1), which is clearly impossible. Similarly, if $\lambda z_2 \subseteq \operatorname{int}\Pi$, again by Lemma 3.2 there exists $\ell \in \mathbb{Z}$ such that $\lambda = f^\ell\theta$, hence $\lambda z_2 = f^\ell z_2 \in \operatorname{int}\Pi$ by (4.1), a contradiction. This finishes the proof of (i).

(ii) Next, we show that the Cone conjecture holds for $\operatorname{Nef}(X)$ if $\operatorname{Aut}(X)$ is finite, but $\operatorname{Bir}(X)$ is infinite. Here, $\operatorname{Nef}(X) \subseteq \operatorname{Mov}(X)$ by Lemma 3.2 and Proposition 3.8. Then, the Cone conjecture for $\operatorname{Nef}(X)$ holds by Proposition 4.3.

(iii) Finally, note that $\overline{\operatorname{Mov}}(X) \cap \operatorname{Eff}(X) = \operatorname{Mov}(X)$ by Lemma 4.4; hence the proof of the Cone conjecture for $\operatorname{Mov}(X)$ is the same as that of (i) by a simple adaption. □

Example 4.6. We recall [6, Proposition 6.1]. Oguiso constructs a Calabi–Yau 3-fold X with Picard number 2, obtained as the intersection of general hypersurfaces in $\mathbb{P}^3 \times \mathbb{P}^3$ of bidegrees $(1,1)$, $(1,1)$, and $(2,2)$, which has the following properties: x_1 and x_2 are rational, $y_1 = (3+2\sqrt{2})x_2 - x_1$, $y_2 = (3+2\sqrt{2})x_1 - x_2$, there are two birational involutions τ_1 and τ_2 such that $\tau_1\tau_2$ is of infinite order, and the group $\operatorname{Bir}(X)$ is generated by $\operatorname{Aut}(X)$ and by τ_1 and τ_2.

We now show that Example 4.6 is a typical example of a Calabi–Yau manifold with Picard number 2 and with infinite group of birational automorphisms.

Theorem 4.7. *Let X be a Calabi–Yau manifold of dimension n and with $\rho(X) = 2$. Assume that $\operatorname{Bir}(X)$ is infinite.*

1. *Let f be a generator of $\mathcal{B}^+(X)$, and let $\alpha > 0$ be the real number such that $fy_1 = \alpha y_1$. Then, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.*
2. *Let $\{v, w\}$ be any integral basis of $N^1(X)_\mathbb{R}$. Then, $m_1 = \mathbb{R}_+(av + bw)$ and $m_2 = \mathbb{R}_+(cv + dw)$, where $a, b, c, d \in \mathbb{Q}(\alpha)$.*
3. *There exists a birational automorphism τ (possibly the identity) such that $\tau^2 \in \operatorname{Aut}(X)$, and a birational automorphism of infinite order σ such that the group $\operatorname{Bir}(X)$ is generated by $\operatorname{Aut}(X)$ and by τ and σ.*

Proof. By rescaling y_1 and y_2, we can assume that

$$h = y_1 + y_2$$

is a primitive integral class in $N^1(X)_\mathbb{R}$. Denote

$$h' = fh = \alpha y_1 + \frac{1}{\alpha} y_2 \quad \text{and} \quad h'' = f^2 h = \alpha^2 y_1 + \frac{1}{\alpha^2} y_2;$$

these are again primitive integral classes since $\mathcal{B}(X)$ preserves $N^1(X)$. Then, an easy calculation shows that

$$h + h'' = \frac{\alpha^2 + 1}{\alpha} h',$$

and hence the number $\frac{\alpha^2+1}{\alpha} = \alpha + \frac{1}{\alpha}$ is an integer. Since

$$y_1 = \frac{1}{\alpha^2 - 1}(\alpha h' - h),$$

and y_1 is not rational by Theorem 2.3, the number α cannot be rational, and (1) follows.

For (2) fix an integral basis $\{v, w\}$ of $N^1(X)_\mathbb{R}$, and write

$$y_1 = av + bw \quad \text{and} \quad y_2 = cv + dw.$$

Then,

$$h = (a + c)v + (b + d)w \quad \text{and} \quad h' = (\alpha a + c/\alpha)v + (\alpha b + d/\alpha)w.$$

Write $p = a + c$ and $q = \alpha a + c/\alpha$, and note that $p, q \in \mathbb{Z}$. Then, an easy calculation shows that $a, c \in \mathbb{Q}(\alpha)$, and similarly for b and d.

Finally, for (3), note that by Theorem 3.9 and Lemma 3.2, we have $\mathcal{B}(X) = \mathcal{B}^+(X) \cup \mathcal{B}^-(X)$, where $\mathcal{B}^+(X)$ is infinite cyclic with generator σ', and $\mathcal{B}^-(X) = \mathcal{B}^+(X)\tau'$ for any $\tau' \in \mathcal{B}^-(X)$. Pick $\tau, \sigma \in \operatorname{Bir}(X)$ such that

$$r(\tau) = \tau' \quad \text{and} \quad r(\sigma) = \sigma',$$

see Notation 2.1. Since $r(\tau^2) = \tau'^2 = \mathrm{id}$ by Lemma 3.1, it follows that τ^2 is an isomorphism by [6, Proposition 2.4]. Now if θ is any element of $\mathrm{Bir}(X)$, then there exist integers k and ℓ such that $r(\theta) = \sigma'^k \tau'^\ell = r(\sigma^k \tau^\ell)$, and we conclude again by [6, Proposition 2.4]. □

Remark 4.8. We are indebted to the referee for pointing out the following example, which provides a variety satisfying the assumptions of Theorem 4.7 in any dimension $n \geq 3$.

Let X be the complete intersection

$$H_1 \cap H_2 \cap \cdots \cap H_{n-1} \cap Q \subseteq \mathbb{P}^n \times \mathbb{P}^n,$$

where $n \geq 3$, where H_i are general hypersurfaces of bidegree $(1, 1)$, and where Q is a general hypersurface of bidegree $(2, 2)$. Then, X is a simply connected Calabi–Yau n-fold with Picard number two. More precisely, $\mathrm{Pic}(X) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$, where L_1 and L_2 are pullbacks of the hyperplane classes of factors $\mathbb{P}^n$. Consider the two birational involutions ι_1, ι_2 induced by the two natural projections of X to $\mathbb{P}^n$. Then, $\iota_1\iota_2$ is a birational automorphism of X of infinite order. The last statement can be checked by computing $(\iota_1\iota_2)^* L_i$ as in [6, Proposition 6.1].

Remark 4.9. One can obtain a similar description of the cone $\mathrm{Nef}(X)$ when the automorphism group of X is infinite.

Basically there are two types of simply connected irreducible Calabi–Yau manifolds: those which do not carry any holomorphic forms of intermediate degree — these manifolds are often simply called Calabi–Yau manifolds — and hyperkähler manifolds carrying a non-degenerate holomorphic 2-form. While in the hyperkähler case the nef cone can be irrational by [6, Proposition 1.3], it is believed that the nef cone of a "strict" Calabi–Yau manifold with, say, $\rho(X) = 2$, must be rational. The evidence is provided by the fact that in odd dimensions $\mathrm{Aut}(X)$ is finite, and then the Cone conjecture would imply the rationality. In even dimensions, we saw that an infinite automorphism group on a strict Calabi–Yau manifold with Picard number two is possible only in very special circumstances.

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