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ZARISKI *F*-DECOMPOSITION AND LAGRANGIAN FIBRATION ON HYPERKÄHLER MANIFOLDS

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Abstract. For a compact hyperkähler manifold X, we show certain Zariski decomposition for every pseudo-effective $\mathbb{R}$-divisor. We also prove that any sequence of D-flops between projective hyperkähler manifolds terminates after finitely many steps.

1. Introduction

It is a classical result that a pseudo-effective divisor on a compact complex surface has a Zariski decomposition. When X is a compact complex manifold of dimension >2 and D a pseudo-effective $\mathbb{R}$-divisor on X, we may consider two types of Zariski-decompositions: either in the sense of Fujita or in the sense of Cutkosky–Kawamata–Moriwaki as defined before Theorem 3.1. In general, such decompositions may not exist. On the other hand, there are also positive results for big divisors or for toric varieties; see [10, Remark 7-3-6, Th. 7-3-7] and [14, Chapter II, Remark 1.17].

We show the existence of such decompositions on a projective hyperkähler manifold (cf. the definition in 2.1).

Below is our first main theorem, a special case of the more elaborated Theorem 3.1.

Theorem 1.1. *Let X be a projective hyperkähler manifold and D a pseudo-effective $\mathbb{R}$-divisor on X. Then there are a birational map $\sigma_1 : X_1 \dashrightarrow X$ from a projective hyperkähler manifold X_1 and a birational morphism $\sigma_2 : X_2 \to X$ from a projective manifold X_2 such that, for each $k \in \{1, 2\}$, $D_k := \sigma_k^* D$ has a Zariski–Fujita decomposition*

$$D_k = P_k + N_k$$

in the sense of Fujita [7]. *Namely, we have:*

(i) *the divisor P_k is nef, i.e., $P_k \in \bar{K}(X)$, the closure of the Kähler cone $K(X)$; and*
(ii) *the divisor N_k is effective; $F \geq \tau^* N_k$ holds whenever there are a birational morphism $\tau : X' \to X_k$ and divisor $F \geq 0$ with $\tau^* D_k - F$ nef.*

We remark that the decomposition $D_k = P_k + N_k$ $(k = 1, 2)$ in Theorem 1.1 is also in the sense of Cutkosky–Kawamata–Moriwaki; see [10, Def. 7-3-2, 7-3-5] or the paragraph before Theorem 3.1 for the definition.

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Our next main theorem is used in the implication "Theorem 3.1 ⇒ Theorem 1.1", and is a consequence of Theorem 4.1: the termination of flops between projective hyperkähler manifolds.

Theorem 1.2. *Let X be a projective hyperkähler manifold, and P an effective $\mathbb{R}$-divisor in the closed movable cone* $\overline{\mathrm{Mov}}(X)$ (cf. the definition in 2.1)*. Then there is a birational map $\tau : X' \dashrightarrow X$ from a projective hyperkähler manifold X' such that τ^*P is nef.*

2. Preliminary results

2.1. Conventions and terminology

We use the conventions in [8, 11] and Hartshorne's book.

Let X be a compact complex Kähler manifold. For an $\mathbb{R}$-divisor Δ on X, we set $H^0(X, \Delta) := H^0(X, \lfloor\Delta\rfloor)$, with $\lfloor\Delta\rfloor$ the integral part (or round down) of Δ.

Set $H^{1,1}(X, \mathbb{R}) := H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$. The (closed) *nef cone* $\bar{K}(X)$ in $H^{1,1}(X, \mathbb{R})$ is the closure of the *Kähler cone* $K(X)$ of X. An element in $\bar{K}(X)$ is called a *nef class*. Let $\mathrm{NS}(X)$ be the *Néron–Severi group*. Set $\mathrm{NS}_{\mathbb{Q}}(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{N}^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We have $\mathrm{N}^1(X) \subseteq H^{1,1}(X, \mathbb{R})$.

The (closed) *pseudo-effective divisor cone* $\mathrm{PE}(X)$ in $\mathrm{N}^1(X)$ is the closure of effective $\mathbb{R}$-divisor classes on X. The *closed movable cone* $\overline{\mathrm{Mov}}(X)$ in $\mathrm{N}^1(X)$ is generated by the classes of fixed-component free divisors. We have $\overline{\mathrm{Mov}}(X) \subseteq \mathrm{PE}(X)$.

We now briefly recall some definitions related to hyperkähler manifolds, and refer to [8, pages 171, 176, 182–184, 223–224] for details. The compact complex Kähler manifold X is called *hyperkähler* (or irreducible holomorphic symplectic) if it is simply connected such that $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate two-form σ. It follows then $\dim X = 2n$ for some integer $n \geq 1$. We normalize σ so that $\int_X (\sigma\bar{\sigma})^n = 1$.

There exists a primitive integral quadratic form $q_X(*)$ on $H^2(X, \mathbb{Z})$, the *Beauville–Bogomolov form*. Indeed, there is a positive constant a such that

$$q(L) = a \int_X L^2(\sigma\bar{\sigma})^{n-1}, \quad L \in H^{1,1}(X, \mathbb{C}).$$

This $q_X(*)$ is non-degenerate of signature $(3, b_2(X)-3)$. There is a so called Beauville–Fujiki number $c > 0$ such that

$$q(L)^n = cL^{2n}, \quad L \in H^2(X, \mathbb{C}).$$

The *positive cone* $C(X)$ in $H^{1,1}(X, \mathbb{R})$ is the connected component of the open cone $\{\alpha \in H^{1,1}(X, R) \mid q(\alpha) > 0\}$ that contains a Kähler class of X. The closure of $C(X)$ in $H^{1,1}(X, \mathbb{R})$ is denoted by $\bar{C}(X)$. The *birational Kähler cone* BK(X) in $H^{1,1}(X, \mathbb{R})$ is

$$\mathrm{BK}(X) = \cup_{f : X \dashrightarrow X'} f^*K(X')$$

where $f : X \dashrightarrow X'$ runs through all bimeromorphic maps $X \dashrightarrow X'$ from X to another compact hyperkähler manifold X'. The closure of $\mathrm{BK}(X)$ in $H^{1,1}(X, \mathbb{R})$ is denoted by $\overline{\mathrm{BK}}(X)$. It is known that $\mathrm{BK}(X) \subseteq C(X)$ and hence $\overline{\mathrm{BK}}(X) \subseteq \bar{C}(X)$.

For our compact hyperkähler X, it is known that $\overline{\mathrm{Mov}}(X) = \overline{\mathrm{BK}}(X) \cap \mathrm{N}^1(X)$; see also [14, Chapter III, Def. 1.13 and 3.9] and Remark 3.3 (3) below. Further, $\overline{\mathrm{BK}}(X)$ coincides with Boucksom's *modified nef cone* $\mathrm{MN}(X)$ (cf. [4, Prop. 4.4 and Lemma 4.9] and [8, Prop. 28.7]).

Lemma 2.2. *Let X be a compact hyperkähler manifold with $q(*)$ the primitive Beauville–Bogomolov quadratic form (and $q(*,*)$ its bilinear form) on $H^2(X,\mathbb{Z})$. Then we have:*

(1) *The birational Kähler cone $\overline{\mathrm{BK}}(X)$ is the intersection of $\bar{C}(X)$ and the dual of the pseudo-effective divisor (closed) cone $\mathrm{PE}(X) \subset H^{1,1}(X,\mathbb{R})$ with respect to $q(*,*)$. If $D \in \mathrm{PE}(X)$ and $q(D,E) \geq 0$ for every prime divisor E, then $D \in \overline{\mathrm{BK}}(X)$.*

(2) *If D_1, D_2 are distinct prime divisors, then $q(D_1,D_2) \geq 0$. Similarly, $q(D_1,D_3) \geq 0$ when $D_3|D_1$ is a pseudo-effective divisor on D_1.*

(3) *Suppose that E_i are prime divisors with negative definite matrix $(q(E_i,E_j))_{i,j}$. Then E_i are linearly independent in the Neron–Severi group $\mathrm{NS}_{\mathbb{Q}}(X)$, and the Iitaka D-dimension $\kappa(X, \sum E_i) = 0$. If $D \in \mathrm{PE}(X)$ and $E = \sum e_i E_i$ such that $q(D-E,E_j) \leq 0$ for all j, then $D - E \in \mathrm{PE}(X)$.*

(4) *$L \in H^{1,1}(X,\mathbb{R})$ belongs to $C(X)$ if and only if $q(L) > 0$ and $q(L,\omega) > 0$ for some Kähler class ω.*

(5) *If an effective $\mathbb{Q}$-divisor $L \in \mathrm{NS}_{\mathbb{Q}}(X)$ satisfies $q(L) > 0$, then X is projective, and L is big, i.e., $L = A + E$ for an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E; further, $|sL| = |M| + F$ for some integer $s \geq 1$, with M big, $|M|$ the movable part, and F the fixed part.*

(6) *If $\sigma : X' \dashrightarrow X$ is a bimeromorphic map from a compact hyperkähler manifold X', then it is isomorphic in codimenion one. Hence σ^* is well defined on $H^2(X,\mathbb{C})$ and compatible with its Hodge structure, the Beauville Bogomolov quadratic form $q(*)$ and the birational Kähler cone.*

(7) *If $0 \neq L \in \bar{C}(X)$ and $D \in \mathrm{PE}(X)$ satisfy $q(L,D) = 0$, then either D and L are parallel in $H^{1,1}(X,\mathbb{R})$ and $q(D) = 0$, or $q(D) < 0$.*

(8) *If $L_1 \equiv L_2$ (numerical equivalence) for two $\mathbb{Q}$-divisors L_i, then $L_1 \sim_{\mathbb{Q}} L_2$.*

Proof. For (1), see [8, Prop. 28.7]. Note that $q(D) \geq 0$ in the second part.

For (2), since $\sigma\bar{\sigma}|D_1 \cap D_2$ is weakly positive, one has

$$q(D_1,D_2) = a \int_{D_1 \cap D_2} (\sigma\bar{\sigma}|D_1 \cap D_2)^{n-1} \geq 0.$$

For the first part of (3), suppose $E' := \sum a_i E_i \equiv \sum b_j E_j =: E''$ (numerical equivalence) for some $a_i \geq 0$, $b_j \geq 0$ and $E_i \neq E_j$. Then $0 \geq q(E') = q(E',E'') \geq 0$ by (2), and hence $E' = 0 = E''$ by the negative-definite assumption. For the second part, write $D = \lim_{t\to\infty} D(t)$ and $D(t) = \sum d(t)_i E_i + D(t)'$ with $d(t)_i \geq 0$, where $D(t)'$ is an effective divisor and contains no any E_i. Since $D(t)$ has a limit and intersecting with a power of a Kähler class, we see that $d(t)_i$ are bounded and we let $\lim_{t\to\infty} d(t)_i = d_i \geq 0$. Then

$$\begin{aligned} 0 \geq q(D-E,E_j) &= \lim_{t\to\infty} q(D(t)-E,E_j) \\ &\geq \lim_{t\to\infty} q\left(\sum (d(t)_i - e_i)E_i, E_j\right) = q\left(\sum (d_i - e_i)E_i, E_j\right). \end{aligned}$$

Hence $\sum (d_i - e_i)E_i \geq 0$ by Zariski's lemma as in [12, Lemma 3.2]. Thus

$$D - E = \lim_{t\to\infty} (D(t) - E) = \sum (d_i - e_i)E_i + \lim_{t\to\infty} D(t)' \in \mathrm{PE}(X).$$

(4) is due to the very definition of the positive cone $C(X)$ (containing the Kähler cone $K(X)$).

The first part of (5) follows from [8, Prop. 26.13] and the claim in its proof, while the second part follows from the first part.

(6) is proved in [8, Prop. 21.6 and 25.14] (cf. (1)).

For (7), see [1, Chapter IV, (7.2)]) and note: $q(*)$ has signature $(1, b_2(X) - 3)$ on $H^{1,1}(X, \mathbb{R})$.

(8) is true because the irregularity $h^1(X, \mathcal{O}_X) = 0$. □

3. Proof of Theorems and their consequences

In this section, we prove the results in the introduction as well as Theorem 3.1. Theorem 3.1 implies Theorem 1.1, thanks to Theorem 1.2.

The part (I) below has been given an analytic proof by Boucksom [4, Section 4]. The result in [4] is broader since Boucksom decomposes every pseudo-effective classes in $H^{1,1}(X, \mathbb{R})$ (which may not be a divisor class) as the sum of a modified nef class and the class of an effective exceptional divisor. The purpose of including the Part (I) here is to stress that it has also an algebraic constructive proof; to be precise, we will see that it follows also from the original Zariski's lemmas [17] as in Fujita [6] for surfaces (cf. also [12, Chapter I, Section 3.1–3.7]).

The part (II) below shows, under certain condition $(*)$ (always true for projective X by Theorem 1.2), the existence of the *Zariski decomposition in the sense of Fujita* [7], i.e., requiring (II) (i-ii) in Theorem 3.1 (II); and hence is also the *Zariski decomposition in the sense of Cutkosky–Kawamata–Moriwaki*, i.e., requiring (i), (iii) in Theorem 3.1 (II) and (ii)': N_k is effective.

Curious readers may try to extend Theorem 3.1 (II) to deal with the modified nef classes, by using the characterization of Demailly–Paun for nef classes.

Theorem 3.1. *Let X be a compact hyperkähler manifold and $D \in \mathrm{PE}(X)$ a pseudo-effective divisor. Then we have:*

(I) *There is the following* Zariski q-decomposition $D = P_D + N_D$ *such that:*
  (i) *the divisor $P_D \in \overline{\mathrm{BK}}(X)$;*
  (ii) *the divisor $N_D \geq 0$; either $N_D = 0$, or for $\mathrm{Supp}\, N_D = \cup N_i$, the Beauville–Bogomolov quadratic matrix $(q(N_i, N_j))_{i,j}$ is negative definite; and*
  (iii) *$q(P_D, N_i) = 0$ for all i.*

  The above Zariski q-decomposition is unique. Moreover, if $D \geq 0$ then $P_D \geq 0$; if D is a $\mathbb{Q}$-divisor then so are P_D and N_D. Finally, for all integers $s \geq 1$, we have the natural isomorphism: $H^0(X, sP_D) \cong H^0(X, sD)$.

(II) *Suppose the condition $(*)$ that there is a bimeromorphic map $\sigma_1 : X_1 \dashrightarrow X$ from a compact hyperkähler manifold X_1 such that $\sigma_1^* P_D$ is nef* (i.e., $\sigma_1^* P_D \in \bar{K}(X_1)$; see Theorem 1.2). *Then there is a bimeromorphic morphism $\sigma_2 : X_2 \to X$ such that for each $k \in \{1, 2\}$ there is a* Zariski–Fujita decomposition (*or* Zariski F-decomposition *for short*) $D_k = P_k + N_k$ *for $D_k := \sigma_k^* D$, in the sense of Fujita* [7] (cf. [10, Def. 7-3-2, 7-3-5]). *Thus the following hold.*
  (i) *the divisor P_k is nef, i.e., $P_k \in \bar{K}(X)$, the nef cone;*
  (ii) *the divisor N_k is effective; $F \geq \tau^* N_k$ holds whenever there are a bimeromorphic morphism $\tau : X' \to X_k$ and divisor $F \geq 0$ with $\tau^* D_k - F$ nef;*

(iii) *the natural isomorphism:* $H^0(X_k, sP_k) \cong H^0(X_k, sD_k)$, *for all integers* $s \geq 1$.

The above Zariski F-decomposition is unique. (iii) follows from (i) and (ii). Moreover, if $D \geq 0$ *then* $P_k \geq 0$*; if* D *is a* $\mathbb{Q}$*-divisor then so are* P_k *and* N_k.

The condition $(*)$ in Theorem 3.1 is always satisfied by projective X, by Theorem 1.2.

The result below is parallel to the surface case.

Corollary 3.2. *With the notation in Theorem* 3.1 (I)*, the Iitaka D-dimension* $\kappa(X, D)$ *equals* $\kappa(X, P_D)$*; we have* $\kappa(X, D) = \dim X$ *(maximal case) if and only if* $q(P_D) > 0$.

Remark 3.3. (1) The Zariski decomposition $D_k = P_k + N_k$ in Theorem 3.1 (II) is also in the sense of Cutkosky–Kawamata–Moriwaki (cf. [10, Def. 7-3-2, 7-3-5]).

(2) By the proof, the Zariski–Fujita decomposition in Theorem 3.1 (II) for D_1 on X_1, coincides with the Zariski q-decomposition in Theorem 3.1 (I) for D_1.

(3) The N_D in Theorem 3.1 (I) is the smallest effective divisor such that $D - N_D \in \overline{\mathrm{BK}}(X)$: if $N' \geq 0$ such that $P' := D - N' \in \overline{\mathrm{BK}}(X)$, then $N' \geq N_D$. Indeed, $q(N' - N_D, N_i) = q(P_D - P', N_i) = -q(P', N_i) \leq 0$ (cf. Lemma 2.2 (1)) for every $N_i \leq N_D$ and hence $N' - N_D \geq 0$ by Zariski's lemma as in the proof of [12, Chapter I, 3.2].

(4) By the above reasoning and Lemma 2.2 (2), the Zariski q-decomposition in Theorem 3.1(I) (for projective X) coincides with both the σ- and ν-decomposition in Nakayama [14, Chapter III, Def. 1.16 and 3.2].

3.4. **Proof of Theorem 3.1**

Part (I) is proved in [4, Section 4], but the intersection-form based arguments in Zariski [17] and Fujita [6] (cf. also Miyanishi [12, Chapter I, 3.1–3.7]) work well for (I) with almost no change, although we use $q(*)$ and Lemma 2.2 instead of the intersection form for surfaces. Nef divisors on a surface correspond to elements in the birational Kähler cone $\overline{\mathrm{BK}}(X)$ (cf. Lemma 2.2 (1)). The only non-intersection based usage of the Riemann–Roch theorem in [12, Lemma 3.5] may be replaced by Lemma 2.2 (5) (3).

The sketch of a constructive proof for (I): Let E_i $(1 \leq i \leq t_1)$ be all prime divisors such that $q(D, E_i) < 0$. Then $(q(E_i, E_j))_{i,j}$ is a negative definite matrix [ibid. Proof of 3.6]. Let F_1 be a (non-negative, by Zariski's lemma) combination of E_i such that $D_1 := D - F_1$ satisfies $q(D_1, E_i) = 0$ $(1 \leq i \leq t_1)$. Then D_1 is pseudo-effective (and effective when D is effective); cf. Lemma 2.2 (3) and [ibid. 3.3]. Let E_j $(t_1 + 1 \leq j \leq t_1 + t_2)$ be all prime divisors satisfying $q(D_1, E_j) < 0$. Then $(q(E_i, E_j))_{1 \leq i,j \leq t_1+t_2}$ is negative definite; cf. [ibid. 3.5] and Lemma 2.2 (7). Let F_2 be a (non-negative) combination of E_k $(1 \leq k \leq t_1 + t_2)$ such that $D_2 := D - F_2$ satisfies $q(D_2, E_k) = 0$. Then D_2 is pseudo-effective (and effective when D is effective); cf. Lemma 2.2 (3) and [ibid. 3.3]. Since X has finite Picard number and by Lemma 2.2 (3), this process will terminate at step r, and $P_D := D_r$ and $N_P := \sum_{i=1}^{r} F_i$ satisfy Theorem 3.1 (I); cf. [ibid. 3.7] and Lemma 2.2 (1).

Next we prove Theorem 3.1 (II). Let $D = P_D + N_D$ be as in (I). Set

$$D_1 := \sigma_1^* D, \quad P_1 := \sigma_1^* P_D, \quad N_1 := \sigma_1^* N_D.$$

Then $D_1 = P_1 + N_1$ is the Zariski q-decomposition for D_1, by the uniqueness in (I) and since σ_1^* is compatible with $q(*)$ (cf. Lemma 2.2 (6)). Let $\pi : X_2 \to X_1$ be a blowup such that the composition $\sigma_2 = \sigma_1 \circ \pi : X_2 \to X$ is holomorphic. Set

$$D_2 := \sigma_2^* D, \quad P_2 := \pi^* P_1, \quad N_2 := D_2 - P_2.$$

Note that P_2 is nef and $N_2 = \sigma_2^* N_D + E$ for some σ_2- (and hence π-) exceptional divisor E, since σ_1 is isomorphic in codimension one (cf. Lemma 2.2 (6)).

We claim that $E \geq 0$ and hence $N_2 \geq 0$. Indeed, $E = \sigma_2^* D - \pi^* P_1 - \sigma_2^* N_D$ and $-E$ is σ_2-nef. Furthermore, we have $\sigma_{2*} E = 0$, since E is σ_2-exceptional. Hence, $E \geq 0$ by the negativity lemma [11, Lemma 3.39].

Now we show that $D_k = P_k + N_k$ $(k = 1, 2)$ is the Zariski F-decomposition as in (II). The condition (II-i) is of course true. For a direct proof of the condition (II-iii), by the projection formula (for the first and last equalities below), since σ_1 is isomorphic in codimenion one, and applying (I) to D_1, for every integer $s \geq 1$, we have:

$$H^0(X_2, sD_2) = H^0(X, sD) = H^0(X_1, sD_1) = H^0(X_1, sP_1) = H^0(X_2, sP_2).$$

To show (II-ii), we consider D_2 only (because D_1 is similar and easier), and replacing π by a further blowup, we have only to show the assertion $(**)$: if $P' := D_2 - F$ is nef for an effective $\mathbb{R}$-divisor F then $F \geq N_2$. Note that $\sigma_{2*} P' \in \overline{\mathrm{BK}}(X)$ (cf. Lemma 2.2 (1)(2))). So $\sigma_{2*} F \geq N_D$ by applying Remark 3.3 to $\sigma_{2*}(P') = \sigma_{2*}(D_2 - F) = D - \sigma_{2*} F$. Now $-(F - N_2) = P' - \pi^* P_1$ is π-nef, and

$$\sigma_{1*}\pi_*(F - N_2) = \sigma_{2*}(F - \sigma_2^* N_D - E) = \sigma_{2*} F - N_D \geq 0$$

so $\pi_*(F - N_2) \geq 0$. Hence, $F - N_2 \geq 0$ by [ibid.]. This proves $(**)$ and also (II-ii).

The uniqueness of the Zariski F-decomposition is due to the condition (II-ii). The rest of (II) is from the construction and (I). This completes the proof of Theorem 3.1.

3.5. **Proof of Corollary 3.2**

By Theorem 3.1 (I), $D = P_D + N_D$ and $\kappa(X, D) = \kappa(X, P_D)$. If $q(P_D) > 0$ then P_D is big (cf. Lemma 2.2 (5)) and hence $\kappa(X, P_D) = \dim X$. Conversely, suppose that $\kappa(X, D) = \dim X$. Then X is Moishezon and Kähler and hence projective, and P_D (and also D) are big. So $P_D = A + E$ for an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor E. By Lemma 2.2 (1), $q(P_D) \geq q(P_D, A) \geq q(A, A) > 0$. This proves Corollary 3.2.

3.6. **Proof of Theorem 1.2**

Replacing P by a small multiple, we may assume that (X, P) is terminal (cf. [11, Cor. 2.35]). By the minimal model program (MMP), [3, Cor. 1.4.1] and the termination of P-flops on hyperkähler projective manifolds (to be proved in Theorem 4.1), there is a surjective-in-codimension-one birational map $\sigma : X \dashrightarrow X'$ such that (X', P') (with $P' := \sigma_* P$) is $\mathbb{Q}$-factorial and terminal, and $K_{X'} + P'$ is nef. Furthermore, $\sigma = \sigma_r \circ \cdots \circ \sigma_1$, where each $\sigma_i : X_i \dashrightarrow X_{i+1}$ is either divisorial or a flip. Let $P_i \subset X_i$ be the image of P. If $\sigma_1 : X = X_1 \to X_2$ is a $(K_{X_1} + P_1)$-negative divisorial contraction with E_1 the exceptional (necessarily irreducible) divisor, then $\ell_1.(K_{X_1} + P_1) < 0$ for a general curve ℓ_1 in a general fibre of the restriction map $\sigma_1 | E_1$; on the other hand, one has $\ell_1.P_1 \geq 0$ since $P_1 \in \overline{\mathrm{BK}}(X_1) \cap \mathrm{N}^1(X_1) = \overline{\mathrm{Mov}}(X_1)$, the closed movable cone; see [14, Chapter III, Prop. 1.14] and Remark 3.3, or [4, Prop. 4.4, Lemma 4.9] and Lemma 2.2. This is a contradiction, noting that $K_{X_1} = 0$. Thus for $i = 1$, the σ_i is a $(K_{X_i} + P_i)$-flip and hence a P_i-flop as defined in 4.2, so X_{i+1} is a projective hyperkähler manifold by [15, Cor. 1], and $P_{i+1} \in \overline{\mathrm{BK}}(X_{i+1})$ by Lemma 2.2 (6); this is true for all i, by the same reasoning. Letting $\tau = \sigma^{-1}$, this proves Theorem 1.2.

Remark 3.7. Let X be a projective hyperkähler manifold. Using Theorem 1.2, we can show that the following are equivalent.

(1) X is bimeromorphic to a projective hyperkähler manifold X' with a (holomorphic) Lagrangian fibration.
(2) X is bimeromorphic to a projective hyperkähler manifold X' admitting a nef, non-big $\mathbb{Q}$-divisor L' with $\kappa(X', L') \geq \dim X/2$.
(3) There is a $\mathbb{Q}$-divisor L with $\dim X/2 \leq \kappa(X, L) < \dim X$.
(4) There is a dominant meromorphic map $g : X \dashrightarrow B'$ with general fibre of non-general type and $\dim X/2 \leq \dim B' < \dim X$.

4. Terminations of flops between projective hyperkähler manifolds

In this section, we prove that any sequence of D-flops between projective hyperkähler manifolds terminates after finitely many steps.

Theorem 4.1. *Let X be a projective hyperkähler manifold and D an effective $\mathbb{R}$-divisor on X. Assume that (X, D) has at worst log canonical singularities. Then there exist no sequences of D-flops which have infinite length.*

We start with the definition of D-flops.

Definition 4.2. Let X and X' be $\mathbb{Q}$-Gorenstein normal varieties. A birational map $\phi : X \dashrightarrow X'$ is said to be a *D-flop* if there exists a normal variety Z, projective birational morphisms $f : X \to Z$ and $f' : X' \to Z$ and an effective $\mathbb{R}$-Cartier divisor D on X which satisfy the following properties:

(1) The morphisms f and f' are isomorphic in codimension one.
(2) The maps satisfy the following commutative diagram:

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Z & \end{array}$$

(3) The canonical divisors K_X and K'_X are relatively numerically trivial.
(4) The pair (X, D) has only log canonical singularities and $-D$ is f-ample.
(5) The proper transform D' of D is f'-ample.
(6) The relative Picard numbers $\rho(X/Z)$ and $\rho(X'/Z)$ are one.

Let ϕ_i be birational maps which satisfies the following sequence:

$$\begin{array}{ccccccccc} X := X_1 & & \overset{\phi_1}{\dashrightarrow} & & X_2 & \overset{\phi_2}{\dashrightarrow} & X_3 & \overset{\phi_3}{\dashrightarrow} & X_4 \dashrightarrow \cdots \\ & {\scriptstyle f_1}\searrow & & \swarrow{\scriptstyle f_1^+} \; {\scriptstyle f_2}\searrow & & \swarrow{\scriptstyle f_2^+} \; {\scriptstyle f_3}\searrow & & \swarrow{\scriptstyle f_3^+} & \\ & & Z_1 & & Z_2 & & Z_3 & & \end{array}$$

This sequence is said to be *a sequence of D-flops* if there exists an effective $\mathbb{R}$-Cartier divisor D on X such that ϕ_i is $D^{(i)}$-flop, where $D^{(1)}$ is D and $D^{(i)}$ is the proper transform of $D^{(i-1)}$ by ϕ_{i-1}.

Next we define log discrepancies and minimal log discrepancies.

Definition 4.3. Let X be a normal variety and D a Weil divisor on X, such that $K_X + D$ is $\mathbb{R}$-Cartier. For a birational morphism $\mu : X' \to X$ from a normal variety X' and a prime Weil divisor E' on X', we define the *log discrepancy* $a(E'; X, D)$ by

$$a(E'; X, D) := (\text{The coefficient of } E' \text{ in } K_{X'} - \mu^*(K_X + D)) + 1.$$

For a proper closed subset W of X, we define the *minimal log discrepancy of* (X, D) by

$$\mathrm{mld}(W; X, D) := \inf_{\mu(E') \subseteq W} a(E'; X, D).$$

Now we prove Theorem 4.1. We use the same notation as in Theorem 4.1. According to [16, Theorem], to prove Theorem 4.1, it is enough to show the following two statements.

(1) For each i, the function p_i on X_i which is defined by

$$x \in X_i \mapsto \mathrm{mld}(x; X_i, D^{(i)})$$

is lower semi-continuous.

(2) Let $\mathcal{S}$ be the set of the minimal log discrepancies defined by

$$\mathcal{S} := \bigcup_i \mathrm{mld}(W_i; X_i, D^{(i)}),$$

where W_i is the exceptional locus of f_i. The set $\mathcal{S}$ satisfies the ascending chain condition.

First we prove (1). If X is smooth projective and carries a holomorphic symplectic form, then Z_1 is a symplectic variety by [2, Def. 1.1]. Further, X_i has only $\mathbb{Q}$-factorial terminal singularities, by the construction of our $D^{(i)}$-flops. Thus every X_i is smooth by [15, Cor. 1]. Then each p_i is lower semi continuous by [5, Th. 4.4].

Next we prove (2). Since all X_i are smooth, $\mathrm{mld}(W_i, X_i, D^{(i)}) \leq \dim X_i$. On the other hand, all pairs $(X_i, D^{(i)})$ still have only log canonical singularities since each f_i is the contraction of a $(K_{X_i} + D^{(i)})$-negative extremal ray. Hence, $0 \leq \mathrm{mld}(W_i; X_i, D^{(i)})$. Moreover, since all X_i are smooth and the sets of all coefficients of $D^{(i)}$ are stable, the set of log discrepancies of $(X_i, D^{(i)})$ is discrete by [9, Th. 5.2]. Therefore $\mathcal{S}$ is a finite set and we are done. This proves Theorem 4.1.

Remark 4.4. If X is a projective symplectic variety which has only quotient singularities, it has been announced that there exist no sequences of D-flops, which have infinite length in a very recent preprint [13, Cor. 1.4].

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