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ANOTHER PROOF OF ZAGIER'S EVALUATION FORMULA OF THE MULTIPLE ZETA VALUES $\zeta(2,\ldots,2,3,2,\ldots,2)$

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Abstract. Using some transformation formulas of the generalized hypergeometric series ${}_3F_2$, we give another proof of D. Zagier's evaluation formula of the multiple zeta values $\zeta(2,\ldots,2,3,2,\ldots,2)$.

In a recent paper [5], Zagier found and proved an evaluation formula

$$\zeta(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) = 2\sum_{r=1}^{a+b+1}(-1)^r c_{a,b}^r \zeta(2r+1)\zeta(\underbrace{2,\ldots,2}_{a+b+1-r}), \tag{1}$$

where a,b are nonnegative integers and

$$c_{a,b}^r = \binom{2r}{2a+2} - \left(1-\frac{1}{2^{2r}}\right)\binom{2r}{2b+1}.$$

Here for positive integers $k_1,\ldots,k_n$ with $k_n \geqslant 2$, the multiple zeta value $\zeta(k_1,\ldots,k_n)$ is defined by the following infinite series:

$$\zeta(k_1,\ldots,k_n) = \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

The evaluation formula (1) plays an important role in the work of Brown [2], who showed that all multiple zeta values can be represented as $\mathbb{Q}$-linear combinations of multiple zeta values of the same weight with all arguments are 2's and 3's and all periods of mixed Tate motives over $\mathbb{Z}$ are $\mathbb{Q}[(2\pi i)^{\pm 1}]$-linear combinations of multiple zeta values.

As in [5], let $H(a,b)$ (resp. $\widehat{H}(a,b)$) denote the left-hand side (resp. the right-hand side) of (1). One considers the following two generating functions:

$$F(x,y) = \sum_{a,b=0}^{\infty}(-1)^{a+b+1}H(a,b)x^{2a+2}y^{2b+1},$$

$$\widehat{F}(x,y) = \sum_{a,b=0}^{\infty}(-1)^{a+b+1}\widehat{H}(a,b)x^{2a+2}y^{2b+1}.$$

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Zagier expressed these two functions by classical special functions. For the function F, by [5, Proposition 1], we know that

$$\frac{\pi}{\sin \pi y} \cdot F(x,y) = \frac{d}{dz}\bigg|_{z=0} {}_3F_2\left(\begin{matrix} x, -x, z \\ 1+y, 1-y \end{matrix}; 1\right). \tag{2}$$

On the other hand, by (the proof of) [5, Proposition 2], we have

$$\begin{aligned} &\frac{\pi}{\sin \pi y} \cdot \widehat{F}(x,y) \\ =&\psi(1+y)+\psi(1-y)-\frac{1}{2}[\psi(1+x+y)+\psi(1-x-y)+\psi(1+x-y) \\ &+\psi(1-x+y)]-\frac{\sin \pi x}{2\sin \pi y}\cdot[\psi(1+(x+y)/2)+\psi(1-(x+y)/2) \\ &-\psi(1+(x-y)/2)-\psi(1-(x-y)/2)-\psi(1+x+y) \\ &-\psi(1-x-y)+\psi(1+x-y)+\psi(1-x+y)]. \end{aligned} \tag{3}$$

Here, the generalized hypergeometric series ${}_3F_2$ is defined as (see [1])

$${}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{n!(\beta_1)_n(\beta_2)_n} z^n,$$

with the ascending Pochhammer symbol

$$(\alpha)_n = \begin{cases} 1, & \text{if } n=0, \\ \alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n>0. \end{cases}$$

And $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Zagier proved indirectly that $F = \widehat{F}$. The purpose of this short note is to give a direct proof of the equality of the right-hand sides of (2) and (3). Our proof uses some transformation formulas of the ${}_3F_2$-series. To save space, below we will denote the special value ${}_3F_2\left(\begin{smallmatrix} \alpha_1,\alpha_2,\alpha_3 \\ \beta_1,\beta_2 \end{smallmatrix}; 1\right)$ by ${}_3F_2\left(\begin{smallmatrix} \alpha_1,\alpha_2,\alpha_3 \\ \beta_1,\beta_2 \end{smallmatrix}\right)$. As in [4], we need two transformation formulas. The first one is (see [1, Section 3.8, Equation (1), p. 21])

$$\begin{aligned} {}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}\right) =& \frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)} {}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \beta_2-\alpha_3 \\ \alpha_1+\alpha_2-\beta_1+1, \beta_2 \end{matrix}\right) \\ &+ \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+\alpha_2-\beta_1)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)} \\ &\times {}_3F_2\left(\begin{matrix} \beta_1-\alpha_1, \beta_1-\alpha_2, \beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3 \\ \beta_1-\alpha_1-\alpha_2+1, \beta_1+\beta_2-\alpha_1-\alpha_2 \end{matrix}\right), \end{aligned} \tag{4}$$

provided that $\Re(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3) > 0$ and $\Re(\alpha_3-\beta_1+1) > 0$. The second one is (see [1, Example 7, p. 98])

$${}_3F_2\left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}\right) = \frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)} {}_3F_2\left(\begin{matrix} \beta_1-\alpha_1, \beta_1-\alpha_2, \alpha_3 \\ \beta_1, \beta_1+\beta_2-\alpha_1-\alpha_2 \end{matrix}\right), \tag{5}$$

provided that $\Re(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3) > 0$ and $\Re(\beta_2-\alpha_3) > 0$.

Since

$$(x)_n(-x)_n = \frac{1}{2}(x)_n(1-x)_n + \frac{1}{2}(1+x)_n(-x)_n,$$

we have

$$(6)\qquad {}_3F_2\left(\begin{matrix} x,-x,z \\ 1+y,1-y \end{matrix}\right) = \frac{1}{2}{}_3F_2\left(\begin{matrix} x,1-x,z \\ 1+y,1-y \end{matrix}\right) + \frac{1}{2}{}_3F_2\left(\begin{matrix} 1+x,-x,z \\ 1+y,1-y \end{matrix}\right).$$

Note that the right-hand side of (6) is symmetric about $x \leftrightarrow -x$. Hence, we only need to consider the first ${}_3F_2$-series of the right-hand side of (6). Applying the transformation formula (4) with $\alpha_1 = x, \alpha_2 = z, \alpha_3 = 1-x, \beta_1 = 1+y, \beta_2 = 1-y$, we get

(7)

$$\begin{aligned} {}_3F_2\left(\begin{matrix} x,1-x,z \\ 1+y,1-y \end{matrix}\right) &= \frac{\Gamma(1+y)\Gamma(1-x+y-z)}{\Gamma(1-x+y)\Gamma(1+y-z)}{}_3F_2\left(\begin{matrix} x,x-y,z \\ x-y+z,1-y \end{matrix}\right) \\ &+ \frac{\Gamma(1+y)\Gamma(1-y)\Gamma(x-y+z-1)\Gamma(1-z)}{\Gamma(x)\Gamma(z)\Gamma(x-y)\Gamma(2-x-z)}{}_3F_2\left(\begin{matrix} 1-x+y,1+y-z,1-z \\ 2-x+y-z,2-x-z \end{matrix}\right). \end{aligned}$$

Applying the transformation formula (5) to the first ${}_3F_2$-series of the right-hand side of (7) with $\alpha_1 = x, \alpha_2 = x-y, \alpha_3 = z, \beta_1 = x-y+z, \beta_2 = 1-y$, we get

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} x,1-x,z \\ 1+y,1-y \end{matrix}\right) \\ &= \frac{\Gamma(1+y)}{\Gamma(1+y-z)}\frac{\Gamma(1-y)}{\Gamma(1-y-z)}\frac{\Gamma(1-x-y)}{\Gamma(1-x-y+z)}\frac{\Gamma(1-x+y-z)}{\Gamma(1-x+y)} \\ &\quad \times {}_3F_2\left(\begin{matrix} -y+z,z,z \\ x-y+z,1-x-y+z \end{matrix}\right) \\ &\quad + \frac{\Gamma(1+y)\Gamma(1-y)}{\Gamma(x)\Gamma(2-x-z)}\frac{\Gamma(x-y-1+z)}{\Gamma(x-y)}\frac{\Gamma(1-z)}{\Gamma(z)}{}_3F_2\left(\begin{matrix} 1-x+y,1+y-z,1-z \\ 2-x+y-z,2-x-z \end{matrix}\right) \\ &\equiv [1+\psi(1+y)z][1+\psi(1-y)z][1-\psi(1-x-y)z][1-\psi(1-x+y)z][1+0z] \\ &\quad + \frac{y}{1-x}\frac{\sin\pi x}{\sin\pi y}\frac{z}{x-y-1}{}_3F_2\left(\begin{matrix} 1-x+y,1+y,1 \\ 2-x+y,2-x \end{matrix}\right) \qquad \pmod{z^2} \end{aligned}$$

as $z \to 0$. Here, we have used the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}.$$

Now using the summation formula

$$(8)\qquad \begin{aligned} {}_3F_2\left(\begin{matrix} 1,\alpha,\beta \\ 1+\alpha,2+\alpha-\beta \end{matrix}\right) = \frac{\alpha(1+\alpha-\beta)}{\beta-1}&[\psi(\alpha)-\psi(2+\alpha-2\beta) \\ &-\psi((\alpha+1)/2)+\psi((\alpha+3)/2-\beta)] \end{aligned}$$

provided that $\Re(\alpha-2\beta) > -2$ with $\alpha = 1-x+y, \beta = 1+y$, we get

(9)

$$\begin{aligned} &\left.\frac{d}{dz}\right|_{z=0} {}_3F_2\left(\begin{matrix} x,1-x,z \\ 1+y,1-y \end{matrix}\right) \\ &= \psi(1+y)+\psi(1-y)-\psi(1-x+y)-\psi(1-x-y) \\ &\quad - \frac{\sin\pi x}{\sin\pi y}\cdot[\psi(1-x+y)-\psi(1-x-y)-\psi(1-(x-y)/2)+\psi(1-(x+y)/2)]. \end{aligned}$$

Combining (6) and (9) gives the desired equality between (2) and (3).

For a proof of the summation formula (8), one may apply [3, Equation (4)] with $m = 1$ and $k = l = 0$ to get

$$
{}_3F_2\left(\begin{matrix}1, \alpha, \beta \\ 1+\alpha, 2+\alpha-\beta\end{matrix}\right) = \frac{-\alpha(1+\alpha-\beta)}{2(\beta-1)}[\psi((\alpha+1)/2) - \psi(\alpha/2) + \psi(\alpha/2-\beta+1) - \psi((\alpha+1)/2-\beta+1)].
$$

The above equation with the help of the formula

$$
\psi(2z) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi(z+1/2) + \log 2
$$

implies (8). This completes the proof.

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