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ON COMPLEX SPACES WITH PRESCRIBED SINGULARITIES

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To the memory of our unforgettable teacher and shining example Hans Grauert

Abstract. For a given complex space Y we construct a complex space X such that $Sing(X) = Y$.

1. Introduction

For a reduced complex space X we denote by $Sing(X)$ the set of singular points of X. In this paper we are dealing with the following question: given a reduced complex space Y, does there exists a reduced complex space X such that $Sing(X) = Y$. We show that the answer is "yes". Namely we prove the following theorem:

Theorem 1. *Let Y be a reduced complex space. Then there exists a reduced complex space X such that:*
(1) $Sing(X) = Y$, $\dim(X) = \dim(Y) + 2$.
(2) *along $Reg(Y)$, the complex space X has only quadratic singularities, (i.e., the product of a complex manifold of dimension $n = \dim(Y)$ and a surface with an isolated quadratic 2-dimensional singularity).*

Moreover, if Y is normal then X can be chosen to be normal and if Y is locally irreducible then X can be chosen to be locally irreducible.

If Y is a complex manifold the proof is trivial because one can choose $X = Y \times S$ where S has only one singular point. Obviously this argument does not work if $Sing(Y) \neq \emptyset$ because $Sing(Y \times S) = Sing(Y) \times S \bigcup Y \times Sing(S)$. To prove our main theorem we consider a resolution of singularities $\pi : \tilde{Y} \to Y$ (which exists by the results of Bierstone and Milmann [3], and Aroca, et al. [1]) and over $\tilde{Y}$ we consider a rank 2 vector bundle $E \to \tilde{Y}$, which is relatively negative. On each fiber of E we have the equivalence relation $x \sim (-x)$. If we let $F := E/\sim$ we obtain a locally trivial fibration $\tau : F \to \tilde{Y}$ with typical fiber $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = z_3^2\}$, which has a quadratic two-dimensional isolated singularity. From F we get the desired complex space X by applying the relative Remmert quotient theorem (see [11]) and Wiegmann quotient theorem [15].

In the embedded case, i.e., if Y is a complex subspace of a complex manifold Z, we give another construction of X using only Wiegmann quotient theorem. In this particular case, we obtain:

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Theorem 2. *Suppose that Z is a complex manifold and Y is a closed subspace of Z. Then there exists a complex space X with the following properties:*

(1) *$Sing(X) = Y$ and $\dim(X) = \dim(Z) + 1$.*
(2) *X is locally irreducible.*
(3) *The normalization of X is smooth and therefore X is not normal at any point of Y.*
(4) *If Z is connected then X is irreducible.*

2. Preliminaries

Throughout this paper all complex spaces are assumed to be reduced.

We recall that a complex space X is called holomorphically convex if the holomorphically convex hull of every compact subset is compact.

Definition 1. A holomorphic map of complex spaces $\pi : X \to S$ is called holomorphically convex if for any point $s \in S$ there exists an open neighborhood U of s such that $X(U) := \pi^{-1}(U)$ is holomorphically convex. If for any point s we can find U such that $X(U)$ is Stein then π is called a Stein morphism.

Knorr and Schneider in [11] proved the following result:

Theorem 3. *Suppose that $\pi : X \to S$ is a holomorphically convex map between two complex spaces. Then there exists a complex spaces R and a holomorphic map $\rho : X \to R$, called the relative Remmert reduction of π, such that $\rho_* \mathcal{O}_X = \mathcal{O}_R$ (so ρ is proper, surjective, and has connected fibers) and a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\rho} & R \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\sigma} & \\ & S & \end{array}$$

with σ being a Stein morphism.

Throughout this paper a complex space X is called 1-convex if there exists a smooth exhaustion function $\phi : X \to \mathbb{R}$ which is strictly plurisubharmonic outside a compact subset $K \subset X$.

Definition 2. A holomorphic map $\pi : X \to S$ is called 1-convex if for any $s \in S$ there exists an open neighborhood U of s, a $\mathcal{C}^\infty$ function $\phi : X(U) \to \mathbb{R}$ and a real number $c_0 \in \mathbb{R}$ such that:

(1) $\phi_{|\{x \in X(U) : \phi(x) > c_0\}}$ is 1-convex,
(2) for every $c \in \mathbb{R}$ we have that $\pi_{|\{x \in X(U) : \phi(x) \leq c\}}$ is a proper map.

The following Theorem is Satz. 3.4 in [11], see also [14].

Theorem 4. *Every 1-convex map is holomorphically convex.*

We recall the definition of a relatively exceptional set given in [11].

Definition 3. Suppose that $\pi : X \to S$ is a holomorphic map between two complex spaces and $A \subset X$ is a closed analytic subset such that $\pi_{|A}$ is proper and has nowhere discrete fibers. A is called relatively exceptional with respect to π if there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ & \searrow^{\pi} \quad \swarrow_{\pi'} & \\ & S & \end{array}$$

where Y is a complex space and π' and Φ are holomorphic maps, such that:

(i) $\pi'_{|\Phi(A)}$ has discrete fibers,
(ii) Φ induces a biholomorphism $X \setminus A \to Y \setminus \Phi(A)$,
(iii) $\Phi_*(\mathcal{O}_X) = \mathcal{O}_Y$.

Definition 4. If $\pi : X \to S$ is a holomorphic map between two complex spaces and A is a closed analytic subset of X, then A is called maximally proper over S if $\pi_{|A}$ is proper, has nowhere discrete fibers and for any closed analytic subset A' of X with these two properties we have $A' \subset A$.

The following result is Satz 5.4 of [11].

Proposition 1. *Suppose that $\pi : X \to S$ is a holomorphic map and $A \subset X$ is a closed analytic subspace of X. We assume that A has a neighborhood W such that $\pi_{|W}$ is 1-convex and A is maximally proper over S in W. Then A is relatively exceptional with respect to S.*

We identify a vector bundle with the sheaf of germs of local sections in the bundle. Suppose that X is a compact complex space and $p : E \to X$ is a holomorphic vector bundle of rank r. We let $\pi : \mathbb{P}(E) \to X$ be the holomorphic fiber bundle for which $\pi^{-1}(x)$ is the space of all $(r-1)$-dimensional linear subspaces of $p^{-1}(x)$. In general, for a coherent sheaf $\mathcal{F}$ on X one ca associate a projective variety over X, $\mathbb{P}(\mathcal{F})$, obtaining in this way a contravariant functor. For details we refer to [8] and [5], Chapter 1. For the proof of the following theorem see [7] and [10].

Theorem 5. *The following statements are equivalent:*

(a) *$L = \mathcal{O}_{\mathbb{P}(E)}(1)$ is ample.*
(b) *For every coherent sheaf $\mathcal{F}$ on X there exists a positive integer m_0 such that $H^q(X, \mathcal{F} \otimes S^m(E)) = 0$ for every $q \geq 1$, $m \geq m_0$ ($S^m(E)$ denotes the m-th symmetric power of E).*
(c) *For every coherent sheaf $\mathcal{F}$ on X there exists a positive integer m_0 such that $\mathcal{F} \otimes S^m(E)$ is spanned by its global sections.*
(d) *The zero section of E^* is exceptional.*
(e) *The zero section of E^* has a strongly pseudoconvex neighborhood.*

A vector bundle is called ample if the above equivalent conditions are satisfied. A vector bundle is called negative if its dual is ample.

We will need the following generalization in the relative case. Suppose that $\pi : X \to S$ is a proper holomorphic map and $p : E \to X$ is a holomorphic vector bundle.

Definition 5. (a) E is called relatively negative if its restriction to every fiber of $\pi^{-1}(s)$ is negative in the sense of Grauert, i.e., the null-section has a strictly pseudoconvex neighborhood.

(b) E is called relatively ample if its dual E^* is relatively negative.
(c) $\pi : X \to S$ is called relatively ample if there exists a relatively ample line bundle $p : L \to X$.

For the next Lemma see Corollary 2.7 in [13]

Lemma 1. *Suppose that s_0 is a point in S and $E_{|\pi^{-1}(s_0)}$ is negative. Then there exists a neighborhood U of s_0 such that $\pi \circ p$ is a 1-convex morphism on $p^{-1}(\pi^{-1}(U))$.*

Corollary 1. *If π has nowhere discrete fibers then E is relatively negative iff its null-section is relatively exceptional.*

Remark: For more general results concerning the relative blowing down of complex spaces, see [6].

Suppose now that X and Y are complex spaces, $f : X \to Y$ is a proper holomorphic map, and $L \to X$ a holomorphic line bundle. It was proved in [13], Theorem 3.6, (using the results on 1-convex morphisms obtained in [11]) that L is relatively ample with respect to f if and only if for every coherent sheaf $\mathcal{F}$ on X and every compact set $K \subset Y$ there exists a positive integer $n_0 = n_0(K, \mathcal{F})$ such that $R^q f_*(\mathcal{F}(n)) = 0$ on K for every $n \geq n_0$ and every $q \geq 1$ ($\mathcal{F}(n)$ stands for $\mathcal{F} \otimes L^n$). At the same time in [2], chapter 4, Théorème 4.1, it was shown that this last property implies that for every point $y \in Y$ there exists a neighborhood V of y and a large enough positive integer n such that, on $f^{-1}(V)$, the cannonical morphism $f^{-1}(V) \to \mathbb{P}(f_*(L^n)$ is an embedding. Moreover, in the proof of this theorem of [2] (page 179) it was shown that by further increasing n we obtain that for every relatively compact open subset U of Y the cannonical morphism $f^{-1}(U) \to \mathbb{P}(f_*(L^n))$ is an embedding for n large enough (n depending on U). Therefore putting together Theorem 3.6 in [13] and Theorem 4.1, chapter 4 in [2], when X and Y are compact, we have:

Theorem 6. *If X and Y are compact complex spaces, $f : X \to Y$ is a holomorphic map, and $L \to X$ a holomorphic line bundle, the following are equivalent:*

(a) *L is relatively ample with respect to f.*
(b) *There exists n_0 such that $R^q f_*(\mathcal{F}(n)) = 0$ for every $n \geq n_0$ and every $q \geq 1$.*
(c) *There exists n_0 such that the canonical morphism $f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for every $n \geq n_0$.*
(d) *There exists n_1 such that $X \to \mathbb{P}(f_*(L^n))$ is an embedding for $n \geq n_1$.*

Remark. From *(c)* we have an embedding $X \hookrightarrow \mathbb{P}(f^* f_* L^n) = \mathbb{P}(f_* L^n) \times_Y X$, hence a map $X \to \mathbb{P}(f_* L^n)$. Condition *(d)* means that increasing n this map becomes an embedding.

The following Lemma is a folklore result (see e.g. [9] Exercise 5.12). For reader's convenience we provide a proof.

Lemma 2. *Suppose that X and Y are compact complex spaces, $f : X \to Y$ a holomorphic map, $G \to Y$ an ample line bundle and $L \to X$ a relatively ample line bundle with respect to f. Then $L \otimes f^* G$ is ample on X.*

Proof. Using Theorem 6, we choose a positive integer n such that we have an embedding j over Y:

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \mathbb{P}(f_* L^n) \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

such that $L^n = j^*(\mathcal{O}(1))$. By [8], Proposition 1.5, if $\mathcal{F}_1 \to \mathcal{F}_2$ is a sheaf epimorphism then one has an embedding $\mathbb{P}(\mathcal{F}_2) \hookrightarrow \mathbb{P}(\mathcal{F}_1)$ over Y, which is linear over each fiber. Since G is ample it follows that, for ν large enough, $f_* L^n \otimes G^\nu$ is generated by global sections. Hence we have an epimorhism $\mathcal{O}_Y^k \longrightarrow f_* L^n \otimes G^\nu$ for some k. Because G is a line bundle we have that $\mathbb{P}(f_* L^n \otimes G^\nu) = \mathbb{P}(f_* L^n)$. Passing to the associated projective spaces, we get an embedding $h : \mathbb{P}(f_* L^n) \hookrightarrow Y \times \mathbb{P}^{k-1}$ over Y such that $\mathcal{O}(1)$ over $\mathbb{P}(f_* L^n)$ is the pull-back by h of the hypersection bundle of $\mathbb{P}^{k-1}$. Composing with j and using again the ampleness of G we get that $L^n \otimes f^* G^\mu$ is ample for every μ. In particular it is ample for $\mu = n$ and this in turn implies that $L \otimes f^* G$ is ample. □

We will briefly recall some facts about desingularization of complex spaces (see [3]).

Let X be a complex space and $Z \subset X$ a smooth closed complex subspace. For any point $x_0 \in X$ we choose U an open neighborhood of x_0 together with a closed embedding $U \hookrightarrow B \Subset \mathbb{C}^N$ where B is an open ball in $\mathbb{C}^N$. Then Z corresponds to a complex submanifold W of B and we consider the blow-up of B with center W. In this blow-up we consider the proper transform of U and in this way we obtain the blow-up of U with center $U \cap Z$. This construction does not depend on the local embedding and the local blow-ups patch-up to get the blow-up of X with (smooth) center Z.

The following result (Theorem 13.4 of [3]) is the fundamental theorem of global desingularization of complex spaces.

Theorem 7. *Any complex space X admits a desingularization $\pi : \tilde{X} \to X$ such that π is the composition of a locally finite sequence of blow-ups with smooth centers and $\pi^{-1}(Sing(X))$ is a divisor with normal crossings in $\tilde{X}$.*

In this theorem locally finite means that on compact sets all but finitely many blow-ups are trivial.

Corollary 2. *The desingularization $\pi : \tilde{X} \to X$ given by Theorem 7 is relatively ample, the relatively ample line bundle $p : L \to \tilde{X}$ corresponding to the exceptional divisor of π.*

Proof. Let

$$\cdots \to X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X$$

be the sequence of blow-ups given by Theorem 7 and $L_j \to X_j$ the line bundle corresponding to the exceptional divisor of π_j. Each L_j is relatively ample with respect to π_j.

Suppose that x is a point in X. We consider the restrictions of L_1 and L_2 to $\pi_1^{-1}(x)$ and, respectively, $(\pi_1 \circ \pi_2)^{-1}(x)$ and we denote them by $L_1 \to \pi_1^{-1}(x)$ and $L_2 \to (\pi_1 \circ \pi_2)^{-1}(x)$. We have that $L_1 \to \pi_1^{-1}(x)$ is ample and $L_2 \to (\pi_1 \circ \pi_2)^{-1}(x)$

is relatively ample with respect to π_2. We apply Lemma 2 and we deduce that $L_2 \otimes \pi_2^*(L_1) \to (\pi_1 \circ \pi_2)^{-1}(x)$ is ample.

We conclude that $L_2 \otimes \pi_2^*(L_1) \to X$ is relatively ample with respect to $\pi_1 \circ \pi_2$. We continue inductively this procedure and we obtain that the line bundle L defined, by abuse of notation, by $L = \otimes_{i \in \mathbb{N}} L_i \to \tilde{X}$ is relatively ample with respect to π.

The infinite tensor product of line bundles (and the entire construction) makes sense since the sequence of blow-ups is locally finite. □

Definition 6. ([15]) Suppose that $(X, \mathcal{O}_X)$ is a complex space, F is a subset of $\mathcal{O}_X(X)$ and let $\phi_F : X \to \mathbb{C}^F$, $\phi_F(x) = (f(x))_{f \in F}$.

(a) $(X, \mathcal{O}_X)$ is called F-separable if ϕ_F is injective.

(b) $(X, \mathcal{O}_X)$ is called F-convex if ϕ_F is proper.

F-separable means that functions in F separate the points of X and F-convex means that for every discrete sequence $\{x_n\}$ in X there exists a function $f \in F$ such that $\{|f(x_n)|\}$ is unbounded.

The following theorem, generalizing a result of Remmert, was proved by Wiegmann [15].

Theorem 8. *Suppose that $(X, \mathcal{O}_X)$ is a reduced complex space and F is a subalgebra of $\mathcal{O}_X(X)$ such that $(X, \mathcal{O}_X)$ is F-convex. Then there exists an F-convex and F-separable reduced Stein space $(Y, \mathcal{O}_Y)$ together with a proper surjective holomorphic mapping $p : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that if $\pi : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is the induced morphisms of $\mathbb{C}$-algebras then $\pi(\mathcal{O}_Y(Y)) \supset F$. Moreover, $(Y, \mathcal{O}_Y)$ is unique, up to isomorphism, with these properties, if F is closed in $\mathcal{O}_X(X)$ then $\pi(\mathcal{O}_Y(Y)) = F$ and if $F = \mathcal{O}_X(X)$ then π is an isomorphism.*

The complex space $(Y, \mathcal{O}_Y)$ is called the Remmert reduction of $(X, \mathcal{O}_X)$ with respect to F and is denoted by $R_F(X, \mathcal{O}_X)$. Note that Remmert's theorem corresponds to the case $F = \mathcal{O}_X(X)$.

For a complex space $(Z, \mathcal{O}_Z)$ we let $T(Z, \mathcal{O}_Z)$ be the underlying topological space Z and, for an open subset U of Z, $\Gamma_U(Z, \mathcal{O}_Z) = \mathcal{O}_Z(U)$. We recall briefly Wiegmann's construction. The topological space $T(R_F(X, \mathcal{O}_X))$ is defined as $T(R_F(X, \mathcal{O}_X)) = X/\sim$ and p is the quotient map, where, for $x_1, x_2 \in X$, $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$ for every $f \in F$. The structure sheaf is defined as follows. For $y \in T(R_F(X, \mathcal{O}_X))$ let m_y by the ideal of F that contains all function $f \in F$ that vanish on $p^{-1}(y)$. For every open subset U of $T(R_F(X, \mathcal{O}_X))$, $\Gamma_U(R_F(X, \mathcal{O}_X))$ is the algebra of all functions $g \in \mathcal{O}_X(p^{-1}(U))$ such that for every point $y \in U$ there exists a positive integer k, a convergent power series $\sum_{i_1,\ldots,i_k}^{\infty} c_{i_1,\ldots,i_k} T_1^{i_1} \cdots T_k^{i_k} \in \mathbb{C}[\langle T_1, \ldots, T_k \rangle]$ and $f_1, \ldots, f_k \in m_y$ such that $\sum_{i_1,\ldots,i_k}^{\infty} c_{i_1,\ldots,i_k} f_1^{i_1} \cdots f_k^{i_k}$ converges uniformly to g on a neighborhood of $p^{-1}(y)$.

Lemma 3. *Suppose that $(X, \mathcal{O}_X)$ is a reduced complex space, F and G are two subalgebras of $\mathcal{O}_X(X)$ such that $(X, \mathcal{O}_X)$ is F-convex, $F \subset G$ and F is dense in G. Then the canonical morphism $R_F(X, \mathcal{O}_X) \to R_G(X, \mathcal{O}_X)$ is an isomorphism.*

Proof. It follows from the discussion after Theorem 8 that if F is a dense subset of G then $T(R_F(X, \mathcal{O}_X)) = T(R_G(X, \mathcal{O}_X))$ and that for every open subset U of $T(R_F(X, \mathcal{O}_X))$ we have $\Gamma_U(R_F(X, \mathcal{O}_X)) \subset \Gamma_U(R_G(X, \mathcal{O}_X))$.

Let $\overline{F}$ be the closure of F (hence $G \subset \overline{F}$) and let $Y := T(R_F(X, \mathcal{O}_X))$. We have then $F \subset \Gamma_Y(R_F(X, \mathcal{O}_X)) \subset \Gamma_Y(R_G(X, \mathcal{O}_X)) \subset \Gamma_Y(R_{\overline{F}}(X, \mathcal{O}_X)) = \overline{F}$. As $\Gamma_Y(R_F(X, \mathcal{O}_X))$ and $\Gamma_Y(R_G(X, \mathcal{O}_X))$ are closed in $\mathcal{C}(Y)$ (the algebra of continuous functions on Y) and $\overline{F}$ is the smallest closed subset containing F, it follows that the map $\Gamma_Y(R_F(X, \mathcal{O}_X)) \to \Gamma_Y(R_G(X, \mathcal{O}_X))$ is bijective. As both $R_F(X, \mathcal{O}_X)$ and $R_G(X, \mathcal{O}_X)$ are reduced Stein spaces it follows that the canonical morphism $R_F(X, \mathcal{O}_X) \to R_G(X, \mathcal{O}_X)$ is an isomorphism. □

In Wiegmann's theorem one needs X to be F-convex. In particular, X has to be $\mathcal{O}_X(X)$-convex which is a strong global condition. On the other hand, it may happen that $\mathcal{O}_X(X) = \mathbb{C}$ (e.g., if X is compact) and then the Remmert reduction is just a point. For our purpose we need to apply Wiegmann's theorem *locally*. To be able do this, we need a "patching" result. This is the purpose of the following proposition.

Proposition 2. *Suppose that $(X, \mathcal{O}_X)$ is a reduced complex space and $\{V_i\}_{i\in\mathbb{N}}$ is a locally finite open covering of X. Let F_i be a closed subalgebra of $\mathcal{O}_X(V_i)$, $\sim_i$ be the equivalence relation on V_i induced by F_i ($x_1 \sim_i x_2$ iff $f(x_1) = f(x_2)$ $\forall f \in F_i$), and $F_{ij} = F_{ji}$ be a closed subalgebra of $\mathcal{O}_X(V_i \cap V_j)$. We assume that:*

(a) *$\mathcal{O}_X|V_i$ is F_i-convex,*

(b) *$F_{i|V_i\cap V_j}$ is a dense subset of F_{ij} for every $i, j \in \mathbb{N}$,*

(c) *$V_i \cap V_j$ is saturated with respect to $\sim_i$ for every $i, j \in \mathbb{N}$.*

Then there exists a reduced complex space $(Y, \mathcal{O}_Y)$, a proper holomorphic map $p : X \to Y$ and an open covering $\{U_i\}_i$ of Y such that $(U_i, \mathcal{O}_{Y|U_i})$ is isomorphic to $R_{F_i}(V_i, \mathcal{O}_X|V_i)$ and $p_{|U_i}$ is the canonical morphism given by Theorem 8.

Proof. We define the following relation on X: $x \sim y$ if and only if there exists $i \in \mathbb{N}$ such that $x, y \in V_i$ and $x \sim_i y$. Note that if $x \in V_i$, $y \in V_i \cap V_j$ and $x \sim_i y$ then using (c) we get that $x \in V_i \cap V_j$ and by (b) and Lemma 3 we get that $x \sim_j y$. This shows that $\sim$ is an equivalence relation. Moreover, each V_i is saturated with respect to $\sim$. Let $Y = X/\sim$, endowed with the quotient topology, and $p : X \to Y$ be the quotient map. We set $U_i = p(V_i)$ which is an open subset of Y. By Wiegmann's construction of $R_{F_i}(V_i, \mathcal{O}_X|V_i)$ explained above we have that $T(R_{F_i}(V_i, \mathcal{O}_X|V_i)) = U_i$. We define the structure sheaf $\mathcal{O}_Y$ as follows: if Ω is an open subset of Y and $f \in \mathcal{C}(\Omega)$ then $f \in \mathcal{O}_Y(\Omega)$ if and only if for every point $y \in U_i$ for some $i \in I$ there exists D an open subset of Y such that $D \subset \Omega \cap U_i$ and $f_{|D} \in \Gamma_D(R_{F_i}(V_i, \mathcal{O}_X|V_i))$. By Lemma 3 this definition does not depend on the choice of i. The fact that $(U_i, \mathcal{O}_{Y|U_i})$ is isomorphic to $R_{F_i}(V_i, \mathcal{O}_X|V_i)$ follows from the construction of the relative Remmert reduction. □

Example. Suppose that $X = \mathbb{P}^1$. Let B_1, B_2, B_3 be three balls (in local coordinate charts) such that $B_1 \cup B_2 \cup B_3 = \mathbb{P}^1$ and $B_i \cap B_j$ is Runge in B_i for every $i, j \in \{1, 2, 3\}$. We assume that $a := [0 : 1] \in B_1 \setminus (\overline{B}_2 \cup \overline{B}_3)$. Let $F_2 = F_{22} = \mathcal{O}(B_2)$, $F_3 = F_{33} = \mathcal{O}(B_3)$, $F_1 = F_{11} = \{f \in \mathcal{O}(B_1) : f'(a) = 0\}$ and, for $i \neq j$, $F_{i,j} = \mathcal{O}(B_i \cap B_j)$. Then we are in the hypothesis of Proposition 2. Note that a holomorphic function f, defined in a neighborhood of the origin $0 \in \mathbb{C}$, satisfies $f'(0) = 0$ if and only if there exists a holomorphic function F of two variables, defined in a neighborhood of the origin in $\mathbb{C}^2$, such that $f(z) = F(z^3, z^2)$ and the map $z \to (z^3, z^2)$ is a parameterization of the cusp singularity $\{(x, y) \in \mathbb{C}^2 : x^2 = y^3\}$.

We deduce that the complex space that we obtain by applying Proposition 2 is $Y = \{[z_0 : z_1 : z_2] \in \mathbb{P}^2 : z_0^2 z_2 = z_1^3\}$ and $p : \mathbb{P}^1 \to Y$ is given by $p([x_0 : x_1]) = [x_0^3 : x_0^2 x_1 : x_1^3]$.

3. The results

Lemma 4. *If X is a complex space then any open covering has a locally finite open refinement $\{\Omega_m\}_{m\in\mathbb{N}}$ such that Ω_m is Stein for every $m \in \mathbb{N}$ and the pair $(\Omega_{m_1}, \Omega_{m_1} \cap \Omega_{m_2})$ is Runge for every $m_1, m_2 \in \mathbb{N}$.*

Proof. We consider $\{W_j\}_{j\in\mathbb{N}}$, $\{V_j\}_{j\in\mathbb{N}}$, $\{U_j\}_{j\in\mathbb{N}}$ locally finite countable open covering of X such that $\{U_j\}_{j\in\mathbb{N}}$ is a refinement of the given covering, $W_j \Subset V_j \Subset U_j$ and U_j is Stein for every $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ and each $x \in \overline{W}_j$ we choose $\phi_{j,x} : U_j \to [0,\infty)$ a plurisubharmonic function such that:

(a) $\phi_{j,x}(x) = 0$ and $\{z \in U_j : \phi_{j,x}(z) < 1\} \subset V_j$,
(b) if, for some $k \in \mathbb{N}$, $\{z \in U_j : \phi_{j,x}(z) < 1\} \cap \overline{V}_k \neq \emptyset$ then $\{z \in U_j : \phi_{j,x}(z) < 1\} \subset U_k$.

Then $\{z \in U_j : \phi_{j,x}(z) < 1\}_{x\in\overline{W}_j}$ is an open covering of $\overline{W}_j$. We extract a finite subcovering $\{z \in U_j : \phi_{j,s}(z) < 1\}_{s\in A_j}$ where A_j is a finite set and we set $\Omega_{j,s} := \{z \in U_j : \phi_{j,s}(z) < 1\}$. The $\{\Omega_{j,s}\}_{j,s}$ is a locally finite open covering of X. Since $\phi_{j,x}$ is plurisubharmonic on U_j each $\Omega_{j,s}$ is Stein. On the other hand, if $\Omega_{j,s} \cap \Omega_{k,l} \neq \emptyset$, as $\Omega_{k,l} \subset V_k$, we have that $\Omega_{j,s} \cap V_k \neq \emptyset$ and hence by property (b) above we have that $\Omega_{j,s} \subset U_k$ This implies that $\Omega_{j,s} \cap \Omega_{k,l} = \{z \in \Omega_{j,s} : \phi_{k,l}(z) < 1\}$ which is Runge in $\Omega_{j,s}$, see [12]. If we choose a bijection $\chi : \mathbb{N} \to \{(j,s) : j \in \mathbb{N},\ s \in A_j\}$ and we set $\Omega_m := \Omega_{\chi(m)}$ we get the desired family. □

Proof of Theorem 1. Let $\nu : Y_1 \to Y$ be the normalization map and $\tau : Z \to Y_1$ be a desingularization map which is relatively ample. Let $p : L \to Z$ be a relatively negative line bundle (which exists by Corollary 2)and set $E := L \oplus L$.

Let $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$, $\sigma(w) = -w$. Clearly $\sigma \circ \sigma$ is the identity of $\mathbb{C}^2$ and therefore we obtain a linear action of $\mathbb{Z}_2$ on $\mathbb{C}^2$. It is easy to see that $\mathbb{C}^2/\mathbb{Z}_2$ is isomorphic to $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = z_3^2\}$ which is a normal surface with only one singular point of quadratic type. By linearity we obtain an action of $\mathbb{Z}_2$ on any vector bundle and in particular on the vector bundle E defined above. Let $\tilde{E}$ be the quotient space of E through this action. We get then a locally trivial fibration $\tilde{p} : \tilde{E} \to Z$ with typical fiber $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = z_3^2\}$. Note that $Sing(\tilde{E}) = Z$ (the zero section). The composition $f := \tau \circ \tilde{p} : \tilde{E} \to Y_1$ is 1-convex, and hence is a holomorphically convex map. Thus we can consider the relative Remmert quotient associated to f. We obtain a complex space W_1 together with a map $g : \tilde{E} \to W_1$ such that $g_* \mathcal{O}_{\tilde{E}} = \mathcal{O}_{W_1}$. We get then a closed embedding $\sigma : Y_1 \hookrightarrow W_1$. Via this embedding Y_1 is the image through g of the null-section of $\tilde{E}$. Note that g is biholomorphic outside the null-section and hence W_1 has singularities precisely on Y_1. There is a natural holomorphic retraction $r : W_1 \to Y_1$, which is a Stein morphism, corresponding to the projection map $f : \tilde{E} \to Y_1$. Over the regular part of Y_1 the space W_1 has only quadratic singularities.

At this moment we reduced the proof of Theorem 1 to the following Lemma (relative contraction for finite maps), which will be applied to the normalization map.

Lemma 5. *Let A and B be complex spaces and $m : A \to B$ be a finite surjective holomorphic map. We assume that A is a closed complex space of a complex space S and m admits a holomorphic extension $\tilde{m} : S \to B$ which is a Stein morphism. Then there exists a complex space T and a holomorphic map $\alpha : S \to T$ such that T contains B as a closed complex subspace, $\alpha_{|A} = m$ and, outside B, α is a biholomorphism between $S \setminus A$ and $T \setminus B$.*

Proof. Using Lemma 4 we choose a locally finite Stein covering $\{D_i\}_{i\in\mathbb{N}}$ of B such that $D_i \cap D_j$ is Runge in D_i and in D_j for every $i, j \in \mathbb{N}$ and $\tilde{m}^{-1}(D_i) \subset S$ is Stein. Therefore $\tilde{m}^{-1}(D_i \cap D_j)$ is Runge in $\tilde{m}^{-1}(D_i)$ and in $\tilde{m}^{-1}(D_j)$ for every $i, j \in \mathbb{N}$. On $\tilde{m}^{-1}(D_i)$ we consider the set F_i of all holomorphic functions $f \in \mathcal{O}(\tilde{m}^{-1}(D_i))$ such that $f_{|A\cap\tilde{m}^{-1}(D_i)}$ comes from a holomorphic function on D_i, i.e., there exists a holomorphic function $g \in \mathcal{O}(D_i)$ with $f_{|A\cap\tilde{m}^{-1}(D_i)} = g \circ m$. Then F_i is a subalgebra of $\mathcal{O}(\tilde{m}^{-1}(D_i))$ and $\tilde{m}^{-1}(D_i)$ is F_i-holomorphically convex. Similarly, we define the set F_{ij} of all holomorphic functions $f \in \mathcal{O}(\tilde{m}^{-1}(D_i \cap D_j))$ such that $f_{|A\cap\tilde{m}^{-1}(D_i\cap D_j)}$ comes from a holomorphic function on $D_i \cap D_j$. Applying Wiegmann quotient theorem to the subalgebras F_i we get a Stein complex space T_i containing D_i as a closed complex subspace. Using Proposition 2, these complex spaces $\{T_i\}_{i\in\mathbb{N}}$ can be glued together and we get the desired complex space T. This concludes the proof of Lemma 5 and of Theorem 1. □

Proof of Theorem 2. Suppose that Ω is a Stein manifold and A is a closed analytic subset of Ω. We denote by $\pi : \Omega \times \mathbb{C} \to \Omega$ the standard projection and we identify a holomorphic function $f \in \mathcal{O}(\Omega)$ with $f \circ \pi$. Hence we have $\mathcal{O}(\Omega) \subset \mathcal{O}(\Omega \times \mathbb{C})$. Let λ be the coordinate function on $\mathbb{C}$ and $F := \{f \in \mathcal{O}(\Omega \times \mathbb{C}) : \frac{\partial f}{\partial \lambda} \equiv 0 \text{ on } A \times \{0\}\}$. Then:

- F is a closed subalgebra of $\mathcal{O}(\Omega \times \mathbb{C})$ and $F \supset \mathcal{O}(\Omega)$,
- if $f \in \mathcal{O}(\Omega \times \mathbb{C})$ and $f_{|A\times\{0\}} \equiv 0$ then $f^2 \in F$.

Suppose that K is a compact subset of $\Omega \times \mathbb{C}$. Then $\widehat{K}^F$, the holomorphically convex hull of K with respect to F is a subset of $\widehat{K}^{\mathcal{O}_{\Omega\times\mathbb{C}}} \cup A$. Indeed, if $z \in \Omega\times\mathbb{C}\setminus(\widehat{K}^{\mathcal{O}_{\Omega\times\mathbb{C}}} \cup A)$ then there exists $f \in \mathcal{O}_{\Omega\times\mathbb{C}}$ such that $f_{|A\times\{0\}} \equiv 0$ and $|f(z)| > \|f\|_K$. It follows that $|f^2(z)| > \|f^2\|_K$ and $f^2 \in F$. At the same time from $\mathcal{O}(\Omega) \subset F$ we get that $\widehat{K}^F \subset \pi^{-1}(\widehat{\pi(K)}^{\mathcal{O}_\Omega})$. Hence, $\widehat{K}^F \subset (\widehat{K}^{\mathcal{O}_{\Omega\times\mathbb{C}}} \cup A) \cap \pi^{-1}(\widehat{\pi(K)}^{\mathcal{O}_\Omega})$, which implies that $\widehat{K}^F$ is compact and hence $\Omega \times \mathbb{C}$ is F-convex.

Similarly, we can show that $\Omega \times \mathbb{C}$ is F-separable. Namely, for any two points $x, y \in \Omega \times \mathbb{C}$, if $x, y \in A \times \{0\}$ then we can choose $f \in \mathcal{O}(\Omega)$ with $f(x) \neq f(y)$ and if at least one of them is not in A we can choose $f \in \mathcal{O}(\Omega \times \mathbb{C}))$ such that $f^2(x) \neq f^2(y)$. Let $(Y, \mathcal{O}_Y) = R_F(\Omega \times \mathbb{C}, \mathcal{O}_{\Omega\times\mathbb{C}})$, $p : \Omega \times \mathbb{C} \to Y$ the canonical morphism and $B = p(A \times \{0\})$, which is a closed analytic subset of Y. Since $\Omega \times \mathbb{C}$ is F-separable it follows that p is a homeomorphism.

We want to show next that $p : \Omega \times \mathbb{C} \setminus A \times \{0\} \to Y \setminus B$ is a biholomorphism and hence, in particular $Sing(Y) \subset B$. It suffices to show that for any open subset U of $\Omega \times \mathbb{C} \setminus A \times \{0\}$ and any $x \in U$ we have that every holomorphic function f on U can be approximated, uniformly on a neighborhood of x by functions in F (this will imply that the functions in F give local coordinates outside $A \times \{0\}$). Let $c \in \mathbb{C}$ be such that $f(x) + c \neq 0$. We choose an open neighborhood V of x such that $V \Subset U$,

$\overline{V} \cap A = \emptyset$, $\overline{V}$ is holomorphically convex and there exists a holomorphic function g defined on a neighborhood of $\overline{V}$ such that $g^2 = f + c$. It follows that we can find $\{h_j\}_{j\geq 0}$, $h_j \in \mathcal{O}(\Omega)$ such that $h_{j|A\times\{0\}} \equiv 0$ and $h_j \to g$ uniformly on $\overline{V}$. It remains to notice that $h_j^2 - c \in F$ and $h_j^2 - c \to f$ uniformly on $\overline{V}$.

Note also that $F \supset \mathcal{O}(\Omega)$ implies that $p_{|\Omega\times\{0\}} : \Omega \times \{0\} \to p(\Omega \times \{0\})$ is a biholomorphism and hence $p_{|A} : A \to B$ is a biholomorphism.

We claim now that $B \subset Sing(Y)$. Let $y \in B$ and $x = p^{-1}(y) \in A$. If Y were smooth in y, it would be normal in y, hence it would be normal in a neighborhood of y, and therefore we could find $U \subset X$ an open neighborhood of x and $W \subset Y$ an open neighborhood of y such that $p(U) = W$ and $p : U \to W$ is a biholomorphism. Therefore for every holomorphic function $f : U \to \mathbb{C}$ we would have that $f \circ p^{-1}$ is holomorphic on W. This would imply that we can approximate f, uniformly on a neighborhood of x, with functions from F. However, the coordinate function $\lambda : U \to \mathbb{C}$ does not satisfy this property.

Lemma 6. *Let M be a Stein manifold, $A \subset M$ a closed analytic subset and $U \subset M$ a Runge open subset of M. Then the restriction map $f \to f_{|U\times\mathbb{C}}$ from $\{f \in \mathcal{O}(M \times \mathbb{C}) : \frac{\partial f}{\partial\lambda} \equiv 0$ on $A \times \{0\}\}$ to $\{f \in \mathcal{O}(U \times \mathbb{C}) : \frac{\partial f}{\partial\lambda} \equiv 0$ on $A \cap U \times \{0\}\}$ has dense image in the topology of uniform convergence on compacts. Here, λ is the coordinate function on $\mathbb{C}$.*

Proof. Let $f : U \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $\frac{\partial f}{\partial\lambda} \equiv 0$ on $A \cap U \times \{0\}$. Because $U \times \mathbb{C}$ is Runge in $M \times \mathbb{C}$ there exists a sequence of holomorphic functions $\{g_n\}_{n\geq 1}$, $g_n \in \mathcal{O}(M \times \mathbb{C})$, such that $g_n \equiv 0$ on $A \cap U \times \{0\}$ and $\{g_{n|U\times\mathbb{C}}\}_{n\geq 1}$ converges to $\frac{\partial f}{\partial\lambda}$. At the same time there exists a sequence $\{h_n\}_{n\geq 1}$, $h_n \in \mathcal{O}(M)$ such that $\{h_{n|U}\}_{n\geq 1}$ converges to $f(z,0)$. For each $n \geq 1$ we consider the following primitive with respect to λ of g_n: $f_n(z,\lambda) = \int_\gamma g_n(z,\xi)\mathrm{d}\xi + h_n(z)$, where $\gamma : [0,1] \to \mathbb{C}$ is a path that joins $0 \in \mathbb{C}$ with λ. For $\gamma(t) = tz$ we get $f_n(z,\lambda) = \int_0^1 g_n(z,t\lambda)\lambda\mathrm{d}t + h_n(z)$. We have then $\frac{\partial f_n}{\partial\lambda} = g_n \equiv 0$ on $A \times \{0\}$. At the same time, since both f and $\int_0^1 \frac{\partial f}{\partial\lambda}(z,t\lambda)\lambda\mathrm{d}t$ are primitives for $\frac{\partial f}{\partial\lambda}$, we have $f(z,\lambda) = \int_0^1 \frac{\partial f}{\partial\lambda}(z,t\lambda)\lambda\mathrm{d}t + f(z,0)$. Hence

$$f_n(z,\lambda) - f(z,\lambda) = \int_0^1 \left(g_n(z,t\lambda) - \frac{\partial f}{\partial\lambda}(z,t\lambda)\right)\lambda\mathrm{d}t + (h_n(z) - f(z,0)).$$

Now, if $K \subset M \times C$ is a compact set, we choose K_0, a compact subset of M, and $B \subset \mathbb{C}$ a compact disk centered at the origin such that $K \subset K_0 \times B$. Using $\|g_n - \frac{\partial f}{\partial\lambda}\|_{K_0\times B} \to 0$ and $\|h_n - f(z,0)\|_{K_0} \to 0$, we obtain easily that $\|f_n - f\|_K \to 0$. □

Let now Z be a complex manifold and Y a closed complex subspace of Z. We use Lemma 4 and we choose an open Stein covering $\{\Omega_i\}_{i\in\mathbb{N}}$ of Z such that the pair $(\Omega_i, \Omega_i \cap \Omega_j)$ is Runge for every $i, j \in \mathbb{N}$. Let $F_i := \{f \in \mathcal{O}(\Omega_i \times \mathbb{C}) : \frac{\partial f}{\partial\lambda} \equiv 0$ on $Y \times \{0\}\}$ and, similarly, $F_{ij} := \{f \in \mathcal{O}((\Omega_i \cap \Omega_j) \times \mathbb{C}) : \frac{\partial f}{\partial\lambda} \equiv 0$ on $Y \times \{0\}\}$.

We apply Wiegmann's quotient theorem to F_i and we use Proposition 2, to glue together the complex spaces thus obtained and we get the desired complex space X. Note that because a positive codimension analytic subset does not disconnect a complex manifold it follows that X is locally irreducible and, if Z is connected, X is irreducible. At the same time it follows from our proof that the normalization of X is $Z \times \mathbb{C}$. □

Remarks:

(1) In [4] the following result was proved : given a closed analytic subset A of $\mathbb{C}^n$, $codim(A) \geq 2$, there exists an irreducible analytic hypersurface $H \subset \mathbb{C}^n$ such that $Sing(H) = A$. This shows, in particular, that one can prescribe singularities for Stein spaces. However the construction in [4] cannot be used for arbitrary singularities since it is not functorial and the local models cannot be glued together to obtain a complex space with prescribed singularities.

(2) The following problem was raised to the first author by C. Bănică in connection with the duality on complex spaces: could every complex space Z of bounded Zariski dimension be embedded as a closed analytic subset of a complex manifold?

(3) The following problem remains open: suppose that Y is a reduced complex space, not necessarily normal. Is it possible to find a *normal* complex space X such that $Sing(X) = Y$?

(4) If Y is a projective algebraic variety then one can construct a normal projective algebraic variety X such that $Sing(X) = Y$. We would like to thank Iustin Coandă for this remark.

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References

[1] J.M. Aroca, H. Hironaka and J.L. Vicente, *Desingularization theorems*, (Mem. Math. Inst. Jorge Juan, No. 30) Madrid: Consejo Superior de Investigaciones Científicas 1977.

[2] C. Bănică and O. Stănăşilă, Méthodes algébriques dans la théorie globale des espaces complexes. Troisième édition. Gauthier-Villars, Paris, 1977.

[3] E. Bierstone and P.D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, *Invent. Math.* **128**(2) (1997), 207–302.

[4] J.-P. Demailly, *Construction d'hypersurfaces irréductibles avec lieu singulier donné dans* $\boldsymbol{C}^n$, *Ann. Inst. Fourier* **30**(3) (1980), 219–236.

[5] G. Fischer, *Complex analytic geometry.* Lecture Notes in Mathematics, Vol. **538**. Springer-Verlag, Berlin-New York, 1976.

[6] A. Fujiki, *On the blowing down of analytic spaces*, Publ. Res. Inst. Math. Sci. 10 (1974/75), 473–507.

[7] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[8] A. Grothendieck, *Techniques de construction en géométrie analytique V. Fibrés vectoriels, fibrés projectifs, fibrés en drapeaux.* Séminaire Henri Cartan; 13e année: 1960/61. Exposé no. 12, 15 pp.

[9] R. Hartshorne, *Algebraic geometry.* Graduate Texts in Mathematics, No. (52) Springer-Verlag, New York-Heidelberg, 1977.

[10] R. Hartshorne, *Ample vector bundles*, Publ. Math. Inst. Hautes Études Sci. No. 29 (1966), 63–94.

[11] K. Knorr and M. Schneider, *Relativexzeptionelle analytische Mengen*, Math. Ann. **193** (1971), 238–254.

[12] R. Narasimhan, *The Levi problem for complex spaces. II*, Math. Ann. **146** (1962), 195–216.

[13] M. Schneider, *Familien negativer Vektorraumbündel und* 1*-konvexe Abbildungen*, Abh. Math. Sem. Univ. Hamburg **47** (1978), 150–170.

[14] P. Siegfried, *Un théorème de finitude pour les morphismes q-convexes*, Comment. Math. Helv. **49** (1974), 417–459.
[15] K.-W. Wiegmann, *Über Quotienten holomorph-konvexer komplexer Räume*, Math. Z. **97** (1967), 251–258.

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