Math. Res. Lett. **20** (2013), *no.* 02, 401–408

POINTWISE BOUNDS ON QUASIMODES OF SEMICLASSICAL SCHRÖDINGER OPERATORS IN DIMENSION TWO

Hart F. Smith and Maciej Zworski

Abstract. We prove sharp pointwise bounds on quasimodes of semiclassical Schrödinger operators with arbitrary smooth real potentials in dimension two. This end-point estimate was left open in the general study of semiclassical L^p bounds conducted by Koch et al. [2]. However, we show that the results of [2] imply the two-dimensional end-point estimate by scaling and localization.

1. Introduction

Let $\mathrm{g}_{ij}(x)$ be a positive-definite Riemannian metric on $\mathbb{R}^2$ with the corresponding Laplace–Beltrami operator,

$$\Delta_{\mathrm{g}} u := \frac{1}{\sqrt{\bar{\mathrm{g}}}} \sum_{i,j} \partial_{x_j} \left(\mathrm{g}^{ij} \sqrt{\bar{\mathrm{g}}}\, \partial_{x_j} u \right), \quad (\mathrm{g}^{ij}) := (\mathrm{g}_{ij})^{-1}, \quad \bar{\mathrm{g}} := \det(\mathrm{g}_{ij}),$$

and let $V \in C^\infty(\mathbb{R}^2)$ be real valued. We prove the following general bound, the analogue of which was already established (under an additional necessary condition) in higher dimensions in [2], but which was open in dimension two:

Theorem 1. *Suppose that $h \leq 1$, and $u \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$. Suppose that u satisfies*

$$\left\| -h^2 \Delta_{\mathrm{g}} u + V u \right\|_{L^2} \leq h, \quad \|u\|_{L^2} \leq 1. \tag{1.1}$$

Then for all $K \Subset \mathbb{R}^2$,

$$\sup_{x \in K} |u(x)| \leq C_K\, h^{-\frac{1}{2}}, \tag{1.2}$$

where the constant C_K depends only on g, *V, and K.*

A function u satisfying (1.1) is sometimes called a weak quasimode. It is a local object in the sense that if u is a weak quasimode then ψu, $\psi \in C_c^\infty(\mathbb{R}^2)$ is also one, so the theorem is easily reformulated with g, V, and u defined on an open subset of $\mathbb{R}^2$. The localization is also valid in phase space: for instance if $\chi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ then $\chi^w(x, hD)u$ is also a weak quasimode — see [1, Chapter 7] or [4, Chapter 4] for the review of the Weyl quantization $\chi \mapsto \chi^w$.

Received by the editors September 16, 2012.

If $\liminf_{|x|\to\infty} V > 0$, then $-h^2\Delta + V$ (defined on $C_c^\infty(\mathbb{R}^2)$) is essentially self-adjoint and the spectrum of $-h^2\Delta + V$ is discrete in a neighborhood of 0 — see for instance [1, Chapter 4]. In this case, weak quasimodes arise as *spectral clusters*:

$$(1.3)\quad u = \sum_{|E_j|\leq Ch} c_j w_j, \quad (-h^2\Delta + V)w_j = E_j w_j, \quad \langle w_j, w_k\rangle_{L^2} = \delta_{jk}, \quad \sum_j |c_j|^2 \leq 1.$$

Then u is a weak quasimode in the sense of (1.1). Since $V(x) \geq c_0 > 0$ for $|x| \geq R$, Agmon estimates (see for instance [1, Chapter 6]) and Sobolev embedding show that $|u(x)| \leq e^{-c_1/h}$, $c_1 > 0$, for $|x| \geq R$. Hence we get global bounds

$$|u(x)| \leq Ch^{-\frac{1}{2}}, \qquad x \in \mathbb{R}^2.$$

It should be stressed however that a weak quasimode is a more general notion than a spectral cluster.

The result also holds when $\mathbb{R}^2$ is replaced by a two-dimensional manifold and, as in the example above, gives global bounds on spectral clusters (1.3) when the manifold is compact. If $V < 0$ this is also a by-product of of the Avakumovic–Levitan–Hörmander bound on the spectral function – see [3], and for a proof of a semiclassical generalization see [2, Section 3] or [4, Section 7.4].

In higher dimensions the theorem requires an additional phase space localization assumption and is a special case of [2, Theorem 6]: Suppose $p(x,\xi)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\partial_x^\alpha \partial_\xi^\beta p(x,\xi) = \mathcal{O}(\langle\xi\rangle^m)$ for some m. Suppose that $K \Subset \mathbb{R}^n$ and $\chi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and that for $(x,\xi) \in \operatorname{supp}\chi$

$$p(x,\xi) = 0, \quad d_\xi p(x,\xi) = 0 \implies d_\xi^2 p(x,\xi) \text{ is non-degenerate.}$$

Then for $u(h)$ such that

$$(1.4)\qquad \operatorname{supp} u(h) \subset K, \quad u(h) = \chi^w(x, hD)u + \mathcal{O}_{\mathscr{S}}(h^\infty),$$

we have

$$(1.5)\qquad \|u(h)\|_\infty \leq C\, h^{-\frac{n-1}{2}} \Big(\|u(h)\|_{L^2} + \frac{1}{h}\|p^w(x, hD)u\|_{L^2}\Big), \qquad n \geq 3.$$

When $n = 2$ the bound holds with $(\log(1/h)/h)^{\frac{1}{2}}$, which is optimal in general if $d_\xi^2 p$ is not positive definite — see [2, Sections 3 and 6] and Section 3 below for examples.

A small bonus for Schrödinger operators in dimension two is the fact that the frequency localization condition in (1.4) required for (1.5) is not necessary — see (2.5) below. Furthermore, as noted already, in all dimensions the compact support condition on u is easily dropped when working with local estimates on u.

The proof of Theorem 1 is reduced to a local result presented in Proposition 3. That result follows in turn from a rescaling argument involving several cases, some of which use the following result that forms part of [2, Corollary 1].

Theorem 2. *Suppose that* $u = u(h)$ *satisfies* (1.1), *and that* (1.4) *holds. If* $V(x) \neq 0$ *for* $x \in \operatorname{supp} u$, *or if* g^{ij} *is positive definite and* $dV(x) \neq 0$ *for* $x \in \operatorname{supp} u$, *then*

$$\|u\|_{L^\infty} = \mathcal{O}\big(h^{-\frac{n-1}{2}}\big), \quad n \geq 2.$$

This result is the basis for Propositions 4 and 5 used in our proof. The case of Theorem 2 with $dV \neq 0$ is the most technically involved result in the paper [2]. We do not know of any simpler way to obtain the bound (1.2).

2. Proof of Theorem 1

By compactness of K, it suffices to prove uniform L^∞ bounds on u over a small ball about each point in K, where in our case the diameter of the ball can be taken to depend only on $\mathcal{C}^N$ estimates for g and V over a unit sized neighborhood of K, for some large N. Without loss of generality we consider a ball centered at the origin in $\mathbb{R}^2$. Let

$$B = \{x \in \mathbb{R}^2 \,:\, |x| < 1\}, \quad B^* = \{x \in \mathbb{R}^2 \,:\, |x| < 2\}.$$

After a linear change of coordinates, we may assume that

$$\mathrm{g}^{ij}(0) = \delta^{ij}. \tag{2.1}$$

Next, by replacing $V(x)$ by $cV(cx)$ and $\mathrm{g}^{ij}(x)$ by $\mathrm{g}^{ij}(cx)$, for some constant $c \leq 1$ depending on the $\mathcal{C}^2$ norm of g and V over a unit neighborhood of K, we may assume that

$$\sup_{x\in B^*} |V(x)| + |dV(x)| \leq 2, \quad \sup_{x\in B^*} |d^2V(x)| + \sum_{i,j=1}^{2} |d\mathrm{g}^{ij}(x)| \leq .01. \tag{2.2}$$

This has the effect of multiplying h by a constant in the equation (1.1), which can be absorbed into the constant C_K in (1.2).

In general, we let

$$C_N = \sup_{x\in B^*} \sup_{|\alpha|\leq N} \Big(|\partial^\alpha V(x)| + \sum_{i,j=1}^{2} |\partial^\alpha \mathrm{g}^{ij}(x)| \Big), \tag{2.3}$$

and will deduce Theorem 1 as a corollary of the following:

Proposition 3. *Suppose* $h \leq 1$, *that* g, V *satisfy* (2.1) *and* (2.2), *and that* u *satisfies*

$$\big\|{-h^2}\Delta_{\mathrm{g}} u + Vu\big\|_{L^2(B^*)} \leq h, \quad \|u\|_{L^2(B^*)} \leq 1. \tag{2.4}$$

Then

$$\|u\|_{L^\infty(B)} \leq C\, h^{-\frac{1}{2}},$$

where the constant C *depends only on* C_N *in* (2.3) *for some fixed* N.

We start the proof of Proposition 3 by recording the following two propositions, which are consequences of Theorem 2.

Proposition 4. *Suppose that both* (2.1)–(2.2) *hold, and that* $\frac{1}{2} \leq |V(x)| \leq 2$ *for* $|x| \leq 2$. *If the following holds, and* $h \leq 1$,

$$\left\| -h^2 \Delta_{\mathrm{g}} u + V u \right\|_{L^2(B^*)} \leq h, \quad \|u\|_{L^2(B^*)} \leq 1,$$

then $\|u\|_{L^\infty(B)} \leq C\, h^{-\frac{1}{2}}$, *where* C *depends only on* C_N *in* (2.3) *for some fixed* N.

Proposition 5. *Suppose that both* (2.1)–(2.2) *hold, and that* $V(0) = 0$ *and* $|dV(0)| = 1$. *If the following holds, and* $h \leq 1$,

$$\left\| -h^2 \Delta_{\mathrm{g}} u + V u \right\|_{L^2(B^*)} \leq h, \quad \|u\|_{L^2(B^*)} \leq 1,$$

then $\|u\|_{L^\infty(B)} \leq C\, h^{-\frac{1}{2}}$, *where* C *depends only on* C_N *in* (2.3) *for some fixed* N.

To see that these follow from Theorem 2, we first may assume that u is compactly supported in $|x| < \frac{3}{2}$. Indeed, the assumptions imply $\|du\|_{L^2(|x|<3/2)} \lesssim h^{-1}$, so that one may cut off u by a smooth function which is supported in $|x| < \frac{3}{2}$ and equals 1 for $|x| < 1$ without affecting the hypotheses. We may then modify g and V outside B^* so that (2.2)–(2.3) are global bounds.

In Proposition 5 above, since $|d^2 V| \leq .01$, we have $.98 \leq |dV(x)| \leq 1.02$ for $|x| \leq 2$, so since g is positive definite the conditions on g and V in Theorem 2 are met. We remark that the conditions of Proposition 5 guarantee that the zero set of V is a nearly-flat curve through the origin, although this is not strictly needed to apply the results of [2]. That the resulting constant C depends only on C_N for some fixed finite N follows from the proofs in [2].

Finally, the condition in (1.4) that $u - \chi^w(x, hD)u = \mathcal{O}_{\mathscr{S}}(h^\infty)$ for some $\chi \in \mathcal{C}_c^\infty$ is not needed for Theorem 2 to hold for positive definite g^{ij} in dimension two. To see this, we note that if $|V| < 2$ and $|\mathrm{g}^{ij}(x) - \delta_{ij}| \leq 0.02$ on the ball $|x| < 2$, then if u is supported in $|x| < \frac{3}{2}$ and $\varphi(\xi) = 1$ for $|\xi| < 4$, condition (1.1) implies that

$$\|(hD)^2 (u - \varphi(hD)u)\|_{L^2} = \mathcal{O}(h).$$

This follows by the semiclassical pseudodifferential calculus (see [4, Theorem 4.29]), since for $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp} \varphi_0 \subset B^*$, $\varphi_0(x)(1 - \varphi(\xi))|\xi|^2/(|\xi|_{\mathrm{g}}^2 + V(x)) \in S(\mathbb{R}^2 \times \mathbb{R}^2)$.

Hence, writing $\hat{u}(\xi)$ for the standard Fourier transform of u,

$$\begin{aligned} \|u - \varphi(hD)u\|_{L^\infty} &\leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |1 - \varphi(h\xi)||\hat{u}(\xi)|\, d\xi \\ &\leq C \int |h\xi|^2 |1 - \varphi(h\xi)||\hat{u}(\xi)|(1 + |h\xi|^2)^{-1}\, d\xi \\ &\leq C \|(hD)^2(u - \varphi(hD)u)\|_{L^2} \left(\int_{\mathbb{R}^2} (1 + |h\xi|^2)^{-2}\, d\xi \right)^{\frac{1}{2}} \\ &\leq C h\, h^{-1} = C, \end{aligned} \tag{2.5}$$

an even better estimate than required. Hence, we are reduced to proving estimates on $\varphi(hD)u$, which by compact support of u satisfies (1.4).

We supplement Propositions 4 and 5 with the following two lemmas.

Lemma 6. *Suppose that (2.1)–(2.2) hold, and that $|V(x)| \le 99\,h$ for $|x| \le 2h^{\frac12}$. If the following holds, and $h \le 1$,*

$$\big\|-h^2\Delta_{\mathrm g} u + Vu\big\|_{L^2(|x|<2h^{1/2})} \le h, \quad \|u\|_{L^2(|x|<2h^{1/2})} \le 1,$$

then $\|u\|_{L^\infty(|x|<h^{1/2})} \le C\,h^{-\frac12}$, where C depends only on C_N in (2.3) for some fixed N.

Proof. Consider the function $\tilde u(x) = h^{\frac12}u(h^{\frac12}x)$, and $\tilde{\mathrm g}^{ij}(x) = \mathrm g^{ij}(h^{\frac12}x)$. Then, since $\|Vu\|_{L^2(|x|<2h^{1/2})} \le 99h$, we have

$$\|\Delta_{\tilde{\mathrm g}}\tilde u\|_{L^2(|x|<2)} \le 100, \quad \|\tilde u\|_{L^2(|x|<2)} \le 1.$$

Since the spatial dimension is 2, interior Sobolev estimates yield $\|\tilde u\|_{L^\infty(|x|<1)} \le C$, where we note that the conditions (2.1) and (2.2) hold for $\tilde{\mathrm g}$ since $h^{\frac12} \le 1$. □

Lemma 7. *Suppose that both (2.1)–(2.2) hold, and that $\frac12 c \le |V(x)| \le 2c$ for $|x| \le 2c^{\frac12}$. If the following holds, and $h \le c \le 1$,*

$$\big\|-h^2\Delta_{\mathrm g} u + Vu\big\|_{L^2(|x|<2c^{1/2})} \le h, \quad \|u\|_{L^2(|x|<2c^{1/2})} \le 1,$$

then $\|u\|_{L^\infty(|x|<c^{1/2})} \le C\,h^{-\frac12}$, where C depends only C_N in (2.3) for some fixed N.

Proof. Let $\tilde u(x) = c^{\frac12}u(c^{\frac12}x)$, $\tilde{\mathrm g}^{ij}(x) = \mathrm g^{ij}(c^{\frac12}x)$, and $\tilde V(x) = c^{-1}V(c^{\frac12}x)$. Note that the assumptions on $V(x)$ in the statement and in (2.2) imply that $|dV(x)| \le c^{\frac12}$ for $|x| < 2c^{1/2}$, so that $\tilde V$ satisfies (2.2), and the constants C_N in (2.3) can only decrease for $c \le 1$. Then with $\tilde h = c^{-1}h \le 1$,

$$\| - \tilde h^2\Delta_{\tilde{\mathrm g}}\tilde u + \tilde V\tilde u\|_{L^2(|x|<2)} \le \tilde h, \quad \|\tilde u\|_{L^2(|x|<2)} \le 1.$$

By Proposition 4, we have $\|\tilde u\|_{L^\infty(|x|<1)} \le C\tilde h^{-\frac12}$, giving the desired result. □

Proof of Proposition 3. It suffices to prove that for each $|x_0| < 1$ there is some $\frac12 \ge r > 0$, so that $\|u\|_{L^\infty(|x-x_0|<r)} \le C\,h^{-\frac12}$, with a global constant C. Without loss of generality we take $x_0 = 0$.

We will split consideration up into four cases, depending on the relative size of $|V(0)|$ and $|dV(0)|$. Since for h bounded away from 0 the result follows by elliptic estimates, we will assume $h \le \frac14$ so that $h^{\frac12}$ below is at most $\frac12$.

Case 1: $|V(0)| \le h$, $|dV(0)| \le 8h^{\frac12}$. Since $|d^2V(x)| \le 0.01$, then Lemma 6 applies to give the result with $r = h^{\frac12}$.

Case 2: $|V(0)| \le h$, $|dV(0)| \ge 8h^{\frac12}$. Since we may add a constant of size h to V without affecting (2.4), we may assume $V(0) = 0$. By rotating we may then assume

$$V(x) = \beta x_1 + f_{ij}(x)x_ix_j,$$

where $\beta = |dV(0)| \geq 8h^{\frac{1}{2}}$. Dividing V by 4 if necessary we may assume $\beta \leq \frac{1}{2}$. Let $\tilde{u} = \beta u(\beta x)$, $\tilde{\mathrm{g}}^{ij}(x) = \mathrm{g}^{ij}(\beta x)$, and

$$\tilde{V}(x) = \beta^{-2} V(\beta x) = x_1 + f_{ij}(\beta x) x_i x_j.$$

With $\tilde{h} = \beta^{-2} h < 1$ we have

$$\| - \tilde{h}^2 \Delta_{\tilde{\mathrm{g}}} \tilde{u} + \tilde{V} \tilde{u} \|_{L^2(|x|<2)} \leq \tilde{h}, \quad \|\tilde{u}\|_{L^2(|x|<2)} \leq 1.$$

Proposition 5 applies, since $\tilde{g}$ and $\tilde{V}$ satisfy (2.1)–(2.2), and the constants C_N in (2.3) for $\tilde{g}$ and $\tilde{V}$ are bounded by those for g and V. Thus, $\|\tilde{u}\|_{L^\infty(|x|<1)} \leq C\tilde{h}^{-\frac{1}{2}}$, giving the desired result on u with $r = |dV(0)|$.

Case 3: $|V(0)| \geq h$, $|dV(0)| \leq 9|V(0)|^{\frac{1}{2}}$. In this case, with $c = |V(0)|$, it follows that $\frac{1}{2}c \leq |V(x)| \leq 2c$ for $|x| \leq \frac{1}{20}c^{\frac{1}{2}}$. We may apply Lemma 7 with V replaced by $\frac{1}{1600}V$ to get the desired result with $r = \frac{1}{40}|V(0)|^{\frac{1}{2}}$.

Case 4: $|V(0)| \geq h$, $|dV(0)| \geq 9|V(0)|^{\frac{1}{2}}$. Since $|d^2V(x)| \leq .01$, it follows that there is a point x_0 with $|x_0| \leq \frac{1}{8}|V(0)|^{\frac{1}{2}}$ where $V(x_0) = 0$. Since $|dV(x_0)| \geq 8|V(0)|^{\frac{1}{2}} \geq 8h^{\frac{1}{2}}$, we may translate and apply Case 2 to get L^∞ bounds on u over a neighborhood of radius $|dV(x_0)|$ about x_0. This neighborhood contains the neighborhood about 0 of radius $r = 0.9998\,|dV(0)|$. □

3. A counter-example for indefinite g

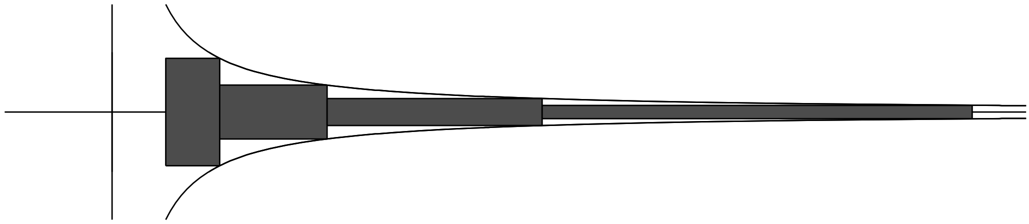
In [2, Section 6], it was shown that there exist u_h for which

$$\| - h^2(\partial_{x_1}^2 - \partial_{x_2}^2) u_h + (x_1^2 - x_2^2) u_h \|_{L^2} \leq h, \quad \|u_h\|_{L^2} \leq 1, \tag{3.1}$$

for which $\|u_h\|_{L^\infty} \approx |\log h|^{\frac{1}{2}} h^{-\frac{1}{2}}$, showing that the assumption of definiteness of g cannot be relaxed to non-degeneracy in the main theorem. In [2, Theorem 6] the positive result was established showing that this growth of $\|u_h\|_{L^\infty}$ for indefinite, non-degenerate g in two dimensions is in fact worst case.

The example of [2] was produced using harmonic oscillator eigenstates. Here we present a different construction of such a u_h with similar L^∞ growth to help illustrate the role played by the degeneracy of g. The idea is to produce a collection $u_{h,j}$ of functions satisfying (3.1) (or equivalent), for which $u_{h,j}(0) = h^{-\frac{1}{2}}$, and where j runs over $\approx |\log h|$ different values. The examples will have disjoint frequency support, hence are orthogonal in L^2. Upon summation over j the L^2 norm then grows as $|\log h|^{\frac{1}{2}}$, whereas the L^∞ norm grows as $|\log h|\, h^{-\frac{1}{2}}$, yielding an example with worst case growth after normalization.

We start by considering the form $\xi_1 \xi_2$ with $V = 0$. To have $\|h^2 \partial_{x_1} \partial_{x_2} u_h\|_{L^2} \leq h$, we will take the Fourier transform of u_h to be contained in the set $|\xi_1 \xi_2| \leq 2h^{-1}$, as well as $|\xi| \leq 2h^{-1}$ to satisfy the frequency localization condition [2, (1.4)]. Our example is then based on the fact that one can find $\approx |\log h|$ disjoint rectangles, each of volume h^{-1}, within this region, as illustrated in the diagram. Each $u_{h,j}$ will be an appropriately scaled Schwartz function with Fourier transform localized to one of the rectangles.



We now fix $\psi, \chi \in \mathcal{C}_c^\infty(\mathbb{R})$, with $0 \le \psi(x) \le 2$ and $0 \le \chi(x) \le 1$, with $\int \psi = \int \chi = 1$, and where

$$\operatorname{supp}\psi \subset [1,2], \quad \operatorname{supp}\chi \subset [-1,1].$$

We additionally assume $\chi(0) = 1$.

Let

$$u_{h,j}(x) = h^{\frac{1}{2}} \int e^{ix_1\xi_1 + ix_2\xi_2} \chi(2^j h\, \xi_1)\psi(2^{-j}\xi_2)\, d\xi_1\, d\xi_2 = h^{-\frac{1}{2}} \check{\chi}(2^{-j}h^{-1}x_1)\check{\psi}(2^j x_2).$$

By the Plancherel theorem, $\|u_{h,j}\|_{L^2} \approx 1$ and $\|h^2 D_1 D_2 u_{h,j}\|_{L^2} \lesssim h$. Furthermore, $u_{h,j}(0) = h^{-\frac{1}{2}}$. By disjointness of the Fourier transforms, we have $\langle u_{h,i}, u_{h,j}\rangle = 0$ for $i \neq j$, and similarly $\langle \partial_{x_1}\partial_{x_2} u_{h,i}, \partial_{x_1}\partial_{x_2} u_{h,j}\rangle = 0$.

We then form

$$u_h(x) = |\log h|^{-\frac{1}{2}} \sum_{1 \le 2^j \le h^{-1}} u_{h,j}(x).$$

Since there are $\approx |\log h|$ terms in the sum, and the terms are orthogonal in L^2, it follows that

$$\|u_h\|_{L^2} \approx 1, \quad \|h^2 \partial_{x_1}\partial_{x_2} u_h\|_{L^2(\mathbb{R}^2)} \lesssim h, \quad u_h(0) \approx |\log h|^{\frac{1}{2}} h^{-\frac{1}{2}}.$$

Although the example is not compactly supported, it is rapidly decreasing (uniformly so for $h < 1$), and one may smoothly cutoff to a bounded set without changing the estimates.

We observe that for this example it also holds that

$$\|x_1 x_2 u_h\|_{L^2} \lesssim h.$$

Hence, u_h is also a counterexample for the form $\xi_1\xi_2 \pm x_1x_2$. Rotating by $\pi/4$ gives the form $\xi_1^2 - \xi_2^2 \pm (x_1^2 - x_2^2)$, including in particular the form considered in [2, Section 6].

We also observe that $x_1^2 u_h$ will be $\mathcal{O}_{L^2}(h)$ if one restricts the sum in u_h to $1 \le 2^j \le h^{-\frac{1}{2}}$, which still has $\approx |\log h|$ values of j, and thus exhibits the same L^∞ growth as u_h. This idea does not however work to yield a counterexample for the form $\xi_1\xi_2 + x_1^2 + x_2^2$.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grants DMS-1161283 and DMS-1201417.

References

[1] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the Semiclassical Limit,* Cambridge University Press, 1999

[2] H. Koch, D. Tataru and M. Zworski, *Semiclassical L^p estimates*, Ann. Henri Poincaré **8**(5) (2007), 885–916.

[3] C. Sogge, *Fourier Integrals in Classical Analysis,* Cambridge Tracts in Mathematics **105**, Cambridge University Press, 1993.

[4] M. Zworski, *Semiclassical Analysis,* Graduate Studies in Mathematics **138**, AMS, 2012.

Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail address: hart@math.washington.edu

Department of Mathematics, University of California, Berkeley, CA 94720, USA
E-mail address: zworski@math.berkeley.edu