Math. Res. Lett. **20** (2013), *no.* 02, 367–383

QUATERNIONS AND KUDLA'S MATCHING PRINCIPLE

TUOPING DU AND TONGHAI YANG

ABSTRACT. In this paper, we prove some interesting identities, among average representation numbers (associated to definite quaternion algebras) and degrees of Hecke correspondences on Shimura curves (associated to indefinite quaternion algebras).

1. Introduction

Kudla observed in [Ku2] a simple but striking identity between two genus theta series from two different quadratic spaces as both are special values of the same Eisenstein series, which he called the matching principle. It is striking as different quadratic spaces give different arithmetics and geometry. The simple identity connects them. In this paper, we use this principle to prove some new identities on quaternion algebras, relating the representation numbers for lattices in definite quaternions and degrees of Hecke operators in indefinite quaternions.

Let D be a square-free positive integer, and let $B = B(D)$ be the unique quaternion algebra of discriminant D over $\mathbb{Q}$, i.e., B is ramified at a finite prime p if and only if $p|D$. The reduced norm, denoted by det in this paper, gives a canonical quadratic form Q on B and makes it a quadratic space $V = (B, \det)$. For a positive integer N prime to D, let $\mathcal{O}_D(N)$ be an Eichler order in B of conductor N. We can view $L = (\mathcal{O}_D(N), det)$ as an even integral lattice in V. The quaternion B is definite if and only if D has odd number of prime factors. When B is definite, it is a very interesting and hard question to compute the representation number (for a positive integer m)

$$r_L(m) = |\{x \in \mathcal{O}_D(N) : \det x = m\}|.$$

On the other hand, the average over the genus $\mathrm{gen}(L)$, which we denote by

$$r_{D,N}(m) = r_{\mathrm{gen}(L)}(m) = \left(\sum_{L_1 \in \mathrm{gen}(L)} \frac{1}{|\mathrm{Aut}(L_1)|}\right)^{-1} \sum_{L_1 \in \mathrm{gen}(L)} \frac{r_{L_1}(m)}{|\mathrm{Aut}(L_1)|},$$

is a product of so-called local densities, thanks to Siegel's seminal work in 1930s [Si]. These densities are computable (see, for example, [Ya1]). We remark that $\mathrm{gen}(L)$ consists of (equivalence classes) right ideals of all maximal orders when $N = 1$. Note that $r_{D,N}$ depends only on D and N, and is independent of the choice of Eichler order $\mathcal{O}_D(N)$. Using Kudla's matching principle ([Ku2, Section 4], see also Section 2), we will prove the following theorem in Section 4.

Received by the editors August 6, 2012.

2000 *Mathematics Subject Classification.* 11G15, 11F41, 14K22.

Theorem 1.1. *Let D be a square-free positive integer with even number of prime factors, let $p \neq q$ be two different primes not dividing D, and let N be a positive integer prime to Dpq. Then*

$$-\frac{2}{q-1}r_{Dp,N}(m) + \frac{q+1}{q-1}r_{Dp,Nq}(m) = -\frac{2}{p-1}r_{Dq,N}(m) + \frac{p+1}{p-1}r_{Dq,Np}(m)$$

for every positive integer m.

We remark that $r_{p,N}(m)$ has a geometric interpretation (see Proposition 4.2).

When $B(D)$ is indefinite, the representation number does not make sense anymore as a number can be represented infinitely many times. In this case, the geometry of Shimura curves come in. We fix an embedding of $i : B(D) \hookrightarrow M_2(\mathbb{R})$ such that $B(D)^\times$ is invariant under the automorphism $x \mapsto x^* = {}^t x^{-1}$ of $\mathrm{GL}_2(\mathbb{R})$. Let $\Gamma_0^D(N) = \mathcal{O}_D(N)^1$ be the group of (reduced) norm 1 elements in $\mathcal{O}_D(N)$ and let $X_0^D(N) = \Gamma_0^D(N)\backslash\mathbb{H}$ be the associated Shimura curve. For a positive integer m, let $T_{D,N}(m)$ be the Hecke correspondence on $X_0^D(N)$ defined by

$$T_{D,N}(m) = \{([z_1],[z_2]) \in X_0^D(N) \times X_0^D(N) : z_1 = i(x)z_2, \text{ for some } x \in \mathcal{O}_D(N), \det x = m\}. \tag{1.1}$$

Define $\deg T_{D,N}(m) = \deg(T_{D,N}(m) \to X_0^D(N))$ under the projection $([z_1],[z_2]) \mapsto [z_1]$.

Let $\Omega_0 = \frac{1}{2\pi}y^{-2}dx \wedge dy$ be the normalized differential on $X_0^D(N)$, and let

$$\mathrm{vol}(X_0^D(N), \Omega_0) = \int_{X_0^D(N)} \Omega_0$$

be the volume of $X_0^D(N)$ with respect to Ω_0, which is a positive rational number (see (5.3)).

Finally, we define the normalized degree (when D has even number of prime factors)

$$r'_{D,N}(m) = -\frac{2}{\mathrm{vol}(X_0^D(N), \Omega_0)} \deg T_{D,N}(m). \tag{1.2}$$

Using Kudla's matching principle and reinterpretation of the Fourier coefficients of some theta series associated to indefinite quaternion algebras, we will prove the following analogues of Theorem 1.1 in Section 5.

Theorem 1.2. *Let D be a square-free positive integer with odd number of prime factors, let $p \neq q$ be two different primes not dividing D, and let N be a positive integer prime to Dpq. Then*

$$-\frac{2}{q-1}r'_{Dp,N}(m) + \frac{q+1}{q-1}r'_{Dp,Nq}(m) = -\frac{2}{p-1}r'_{Dq,N}(m) + \frac{p+1}{p-1}r'_{Dq,Np}(m)$$

for every positive integer m.

Theorem 1.3. *Let D be a square-free positive integer with odd number of prime factors, let $p \nmid D$ be a prime, and let N be a positive integer prime to Dp. Then*

$$r'_{Dp,N}(m) = -\frac{2}{p-1}r_{D,N}(m) + \frac{p+1}{p-1}r_{D,Np}(m).$$

Theorem 1.4. *Let $D > 1$ be a square-free positive integer with even number of prime factors, let $p \nmid D$ be a prime, and let N be a positive integer prime to Dp. Then*

$$r_{Dp,N}(m) = -\frac{2}{p-1} r'_{D,N}(m) + \frac{p+1}{p-1} r'_{D,Np}(m).$$

The last theorem should also hold for $D = 1$ although our proof does not go through (see [Du]). We would need regularized Siegel–Weil formula for $M_2(\mathbb{Q})$, which is not known unfortunately despite a lot of general results on regularized Siegel–Weil formulas pioneered by Kudla and Rallis in 1990s ([KR3]). Clearly, Theorems 1.1 and 1.2 are consequences of Theorems 1.3 and 1.4. We stated Theorems 1.1 and 1.2 because they look really nice and symmetric and their proofs are a little simpler in the sense they do not need matching at the infinity place.

Relations between different quaternions have been extensively studied. For example, the well-known Jacquet–Langlands correspondence gives correspondence between irreducible automorphic representations of different quaternions. In some sense, above relations can be viewed as examples of explicit Jacquet–Langlands correspondence on modular forms (instead of representations). In his famous work on level-lowering work [Ri2], Ribet discovered some deep relations between the pq-new part of Hecke ring of $X_0(pqN)$ modulo q and the Hecke ring of $X_0^{pd}(N)$ modulo q. We will briefly mention its relation to our representation numbers in Section 4 and refer to [Ri2] for detail. As far as we know, relations in this paper are new, which is a little surprising.

Katsurada and Schulze-Pillot studied in [KSP] the action of Hecke operator $T(p)$ on genus theta functions and discovered some very interesting identities between different genus theta functions. These identities are also examples of Kudla's matching principle as they noted in [KSP, Section 6]. Our $r_{D,N}(m)$'s are Fourier coefficients of some genus theta functions of weight 2. However, our formulae are different from theirs.

This paper is organized as below. In Section 2, we review the Weil representation and Kudla's matching principle in general case. In Section 3, we prove some explicit local matchings between Schwartz functions on the division algebra and the matrix algebra over local fields. As a consequence, we prove a global identity between theta integrals on two different quaternions (global matching). In Section 4, we look at the definite quaternion case and unwind the theta integral. We show that $r_{D,N}(m)$ is the m-Fourier coefficient of some theta integrals considered in Section 3 and prove Theorem 1.1. We also give a geometric interpretation of $r_{p,N}(m)$ at the end of this section and its connection with the 'singular set' studied by Ribet in [Ri2] and [Ri1]. In Section 5, we associate the product of two (identical) Shimura curves to the quadratic space coming from an indefinite quaternion and interpret the Fourier coefficients of the theta integral via 'the degree of Hecke correspondences'. As a result, we prove Theorems 1.2, 1.3 and 1.4.

2. Preliminaries and Kudla's matching principle

Let $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ be the canonical unramified additive character such that $\psi_\infty(x) = e^{2\pi i x}$. Let (V, Q) be a non-degenerate quadratic space over $\mathbb{Q}$ of even dimension m with the quadratic form Q, and let

$$\chi_V(x) = \left(x, (-1)^{\frac{m(m-1)}{2}} \det V\right)_{\mathbb{A}}$$

be the associated quadratic character. Let $\omega = \omega_{\psi,V}$ be the associated Weil representation of $O(V)(\mathbb{A}) \times SL_2(\mathbb{A})$ on $S(V(\mathbb{A}))$, where $S(V(\mathbb{A}))$ is the Schwartz–Bruhat function space. One has a Weil representation $\omega_p = \omega_{\psi,V_p}$ of $O(V)(\mathbb{Q}_p) \times SL_2(\mathbb{Q}_p)$ on $S(V_p)$ for each prime p, and $\omega = \otimes\omega_p$. Concretely, the orthogonal group $O(V)(\mathbb{A})$ on $S(V(\mathbb{A}))$ linearly,

$$\omega(h)\varphi(x) = \varphi(h^{-1}x).$$

The $\mathrm{SL}_2(\mathbb{A})$-action is determined by (see, for example, [Ku1])

$$\begin{aligned} &\omega(n(b))\varphi(x) = \psi(bQ(x))\varphi(x), \\ &\omega(m(a))\varphi(x) = \chi_V(x)|a|^{\frac{m}{2}}\varphi(ax), \\ &\omega(w)\varphi = \gamma(V)\widehat{\varphi} = \gamma(V)\int_{V(\mathbb{A})}\varphi(y)\psi((x,y))dy, \end{aligned} \tag{2.1}$$

where for $a \in \mathbb{A}^\times$, $b \in \mathbb{A}$

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

dy is the Haar measure on $V(\mathbb{A})$ self-dual with respect to $\psi((x,y))$, and $\gamma(V) = \prod_p \gamma(V_p) = 1$, where $\gamma(V_p)$ is a eighth root of unity associated to the local Weil representation at p (local Weil index). Let $P = NM$ be the standard Borel subgroup of SL_2, where N and M are subgroups of $n(b)$ and $m(a)$, respectively. It is well known that the theta kernel ([We])

$$\theta(g,h,\varphi) = \sum_{x\in V(\mathbb{Q})}\omega(g)\varphi(h^{-1}x), \quad \varphi \in S(V(\mathbb{A})) \tag{2.2}$$

is left $O(V)(\mathbb{Q}) \times \mathrm{SL}_2(\mathbb{Q})$-invariant, and thus an automorphic form on $[O(V)] \times [\mathrm{SL}_2]$. For an algebraic group G over $\mathbb{Q}$, we write $[G] = G(\mathbb{Q})\backslash G(\mathbb{A})$. The theta integral

$$I(g,\varphi) = \frac{1}{\mathrm{vol}([O(V)])}\int_{[O(V)]}\theta(g,h,\varphi)\,dh \tag{2.3}$$

is an automophic form on $[\mathrm{SL}_2]$ if the integral is absolutely convergent, which is the case precisely when V is anisotropic or $m - r > 2$, where r is the Witt index of V. There is another way to construct automorphic forms from $\varphi \in S(V(\mathbb{A}))$ via Eisenstein series.

For $s \in \mathbb{C}$, let $I(s,\chi_V)$ be the principal series representation of $\mathrm{SL}_2(\mathbb{A})$ consisting of smooth functions $\Phi(s)$ on $\mathrm{SL}_2(\mathbb{A})$ such that

$$\Phi(nm(a)g,s) = \chi_V(a)|a|^{s+1}\Phi(g,s). \tag{2.4}$$

There is a $\mathrm{SL}_2(\mathbb{A})$-intertwining map ($s_0 = \frac{m}{2} - 1$)

$$\lambda = \lambda_V : S(V(\mathbb{A})) \to I(s_0,\chi_V), \quad \lambda(\varphi)(g) = \omega(g)(0). \tag{2.5}$$

Let $K_\infty K$ be the subgroup $SO_2(\mathbb{R}) \times \mathrm{SL}_2(\hat{\mathbb{Z}})$ of $\mathrm{SL}_2(\mathbb{A})$. A section $\Phi(s) \in I(s,\chi)$ is called standard if its restriction to $K_\infty K$ is independent of s. By the Iwasawa decomposition $G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K_\infty K$, the function $\lambda(\varphi) \in I(s_0,\chi)$ has a unique extension to a standard section $\Phi(s) \in I(s, \chi)$ with $\Phi(s_0) = \lambda(\varphi)$. The Eisenstein series is given by

$$E(g, s, \varphi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q})} \Phi(\gamma g, s). \tag{2.6}$$

When V is anisotropic or $m - r > 2$, Kudla and Rallis ([KR1] [KR2]) proved that the Eisenstein series is holomorphic at $s = s_0$ and produces an automorphic form on $[\mathrm{SL}_2]$. The well-known Siegel–Weil formula, as extended by Kudla and Rallis ([KR1], [KR2]), asserts that the two automorphic forms $I(g, \varphi)$ and $E(g, s_0, \varphi)$ are the same:

Theorem 2.1. *(Siegel-Weil formula) Assume that V is anisotropic or $\dim V - r > 2$. Then for every $\varphi \in S(V(\mathbb{A}))$, the Eisenstein series $E(g, s; \varphi)$ is holomorphic at $s_0 = m/2 - 1$, and*

$$E(g, s_0, \varphi) = \kappa I(g, \varphi),$$

where $\kappa = 2$ when $m \leq 2$ and $\kappa = 1$ otherwise.

Let $V^{(1)}, V^{(2)}$ be two quadratic spaces with the same dimension and the same quadratic character χ. There is a following diagram:

$$\begin{array}{ccc} S(V^{(1)}(\mathbb{A})) & \xrightarrow{\lambda_{V^{(1)}}} & \\ & & I(s_0, \chi) \\ S(V^{(2)}(\mathbb{A})) & \xrightarrow{\lambda_{V^{(2)}}} & \end{array} . \tag{2.7}$$

Following Kudla [Ku2], we make the following definition.

Definition 2.2. For a prime $p \leq \infty$, $\varphi_p^{(i)} \in S(V_p^{(i)})$, $i = 1, 2$, are said to be matching if

$$\lambda_{V_p^{(1)}}(\varphi_p^{(1)}) = \lambda_{V_p^{(2)}}(\varphi_p^{(2)}).$$

$\varphi^{(i)} = \prod_p \varphi_p^{(i)} \in S(V^{(i)}(\mathbb{A}))$ are said to be matching if they match at each prime p.

By the Siegel–Weil formula, we have the following Kudla's matching principle ([Ku2, Section 4]): Under the assumption of Theorem 2.1 for both $V^{(1)}$ and $V^{(2)}$, one has, for a matching pair $(\varphi^{(1)}, \varphi^{(2)})$, the following identity:

$$I(g, \varphi^{(1)}) = I(g, \varphi^{(2)}). \tag{2.8}$$

This implies that their Fourier coefficients are equal, which we use in this paper.

3. Matchings on quaternions

There are two quaternion algebras over a local field $\mathbb{Q}_p$, the matrix algebra $B^{sp} = M_2(\mathbb{Q}_p)$ (split quaternion) and the division quaternion B^{ra} (ramified quaternion). Let $V = V^{sp}$ or V^{ra} be the associated quadratic space with reduced norm as the quadratic form $\det(x) = xx^\iota$, where ι is the main involution on the quaternion algebra B. Both spaces have trivial quadratic character χ_{trivial}. So we have $\mathrm{SL}_2(\mathbb{Q}_p)$-intertwining operators

$$\lambda : S(V) \to I(1), \quad \varphi \mapsto \lambda(\varphi) = \omega_V(g)\varphi(0).$$

Here $I(s) = I(s, \chi_{\text{trivial}})$. We will use superscript ra and sp to indicate the association with division or matrix quaternion algebra. It is known ([Ku2]) that λ^{sp} is surjective while the image of λ^{ra} is of codimension 1. So every function φ^{ra} has some matching in $S(V^{sp})$. The purpose of this section is to give some explicit matchings and obtain some interesting global identities.

3.1. The finite prime case $p < \infty$. We assume that $p < \infty$ in this subsection. Let $L^{ra} = \mathcal{O}_{B^{ra}}$ be the maximal order in B^{ra}, which consists of all elements of B whose reduced norm is in $\mathbb{Z}_p$. We do not use the subscript p for simplicity in this subsection. Let $L_0^{sp} = M_2(\mathbb{Z}_p)$ and

$$L_1^{sp} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : c \equiv 0 \pmod{p} \right\}.$$

Then dual lattices are given by

$$L^{ra,\sharp} = \pi^{-1} L^{ra}, \quad L_1^{sp,\sharp} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}_p) : a, c, d \in \mathbb{Z}_p, b \in \frac{1}{p}\mathbb{Z}_p \right\}.$$

Here $\pi \in B^{ra}$ is a "uniformizer" satisfying $\pi^\iota = -\pi$ and $\pi^2 = p$, and

$$L^\sharp = \{x \in V : (x, L) \subset \mathbb{Z}_p\}.$$

We denote

$$\varphi^{ra} = \text{char}(L^{ra}), \quad \varphi^{ra,\sharp} = \text{char}(L^{ra,\sharp}),$$

and

$$\varphi_i^{sp} = \text{char}(L_i^{sp}), \quad i = 0, 1, \quad \text{and } \varphi_2^{sp} = \text{char}(L_1^{sp,\sharp}). \tag{3.1}$$

Proposition 3.1. *Let the notation be as above. Then*

(1) $\varphi^{ra} \in S(V^{ra})$ *matches with* $\frac{-2}{p-1}\varphi_0^{sp} + \frac{p+1}{p-1}\varphi_1^{sp} \in S(V^{sp})$.

(2) $\varphi^{ra,\sharp} \in S(V^{ra})$ *matches with* $\frac{2p}{p-1}\varphi_0^{sp} - \frac{p+1}{p-1}\varphi_2^{sp} \in S(V^{sp})$.

Proof. (1) Since

$$\text{SL}_2(\mathbb{Z}_p) = K_0(p) \cup N(\mathbb{Z}_p) w K_0(p), \quad K_0(p) = L_1^{sp} \cap \text{SL}_2(\mathbb{Z}_p),$$

$I(1)^{K_0(p)}$ has dimension 2, and $\Phi \in I(1)^{K_0(p)}$ is determined by $\Phi(1)$ and $\Phi(w)$. Note that $K_0(p)$ is generated by $n(b)$ and $n_-(c) = w^{-1}n(-c)w$, $b \in \mathbb{Z}_p$ and $c \in p\mathbb{Z}_p$. Using this, one can check that φ^{ra}, φ_i^{sp} are all $K_0(p)$-invariant for the respective Weil representations. We check $\omega(n_-(-c))\varphi^{ra} = \varphi^{ra}$ and leave others to the reader. One has

$$\omega^{ra}(w)\varphi^{ra}(x) = \gamma(V^{ra})\varphi^{ra,\sharp}(x)\text{vol}(L^{ra}).$$

So

$$\begin{aligned} \omega^{ra}(n(-c)w)\varphi^{ra}(x) &= \gamma(V^{ra})\text{vol}(L^{ra})\psi_p(-c\det(x))\varphi^{ra,\sharp}(x) \\ &= \gamma(V^{ra})\varphi^{ra,\sharp}(x)\text{vol}(L^{ra}), \end{aligned}$$

i.e.,

$$\omega^{ra}(n(-c)w)\varphi^{ra} = \omega^{ra}(w)\varphi^{ra}.$$

So

$$\omega^{ra}(n_-(c))\varphi^{ra} = \omega^{ra}(w^{-1})\omega^{ra}(n(-c)w)\varphi^{ra} = \varphi^{ra}$$

as claimed.

Now we have $\lambda^{ra}(\varphi^{ra}), \lambda^{sp}(\varphi_i^{sp}) \in I(1)^{K_0(p)}$. Direct calculation gives

$$\begin{aligned} \lambda^{ra}(\varphi^{ra})(1) = 1, &\quad \lambda^{ra}(\varphi^{ra})(w) = \gamma(V^{ra})p^{-1} \\ \lambda^{sp}(\varphi_0^{sp})(1) = 1, &\quad \lambda^{sp}(\varphi_0^{sp})(w) = \gamma(V^{sp}), \\ \lambda^{sp}(\varphi_1^{sp})(1) = 1, &\quad \lambda^{sp}(\varphi_0^{sp})(w) = \gamma(V^{sp})p^{-1}. \end{aligned}$$

Since $\gamma(V^{sp}) = -\gamma(V^{ra}) = 1$, one has

$$\lambda^{ra}(\varphi^{ra}) = \frac{-2}{p-1}\varphi_0^{sp} + \frac{p+1}{p-1}\varphi_1^{sp}.$$

This proves (1). Claim (2) is similar and is left to the reader. One just needs to replace $K_0(p)$ by

$$K_0^+(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : \ b \equiv 0 \pmod p \right\}.$$

□

To find matching pairs of coset functions $\varphi_\mu^{ra} = \mathrm{char}(\mu + L^{ra})$, we first label them. Let k be the unique unramified quadratic field extension of $\mathbb{Q}_p$, and let $\mathcal{O}_k = \mathbb{Z}_p + \mathbb{Z}_p u$ be the ring of integers of k with $u \in \mathcal{O}_k^\times$. Fix one optimal embedding $k \hookrightarrow B^{ra}$, then there is a uniformizer π of B^{ra} such that $\pi r = \bar{r}\pi$ for $r \in k$ and $\pi^2 = p$. One has then

$$L^{ra} = \mathcal{O}_{B^{ra}} = \mathcal{O}_k + \mathcal{O}_k\pi = \mathbb{Z}_p + \mathbb{Z}_p u + \mathbb{Z}_p \pi + \mathbb{Z}_p up.$$

So one has an isomorphism

$$(\mathbb{Z}/p\mathbb{Z})^2 \cong L^{ra,\sharp}/L^{ra}, \quad (i,j) \mapsto \mu_{i,j}^{ra} = \frac{i + ju}{\pi}.$$

Using the identification, we write $\varphi_{i,j}^{ra}$ for $\mathrm{char}(\mu_{i,j}^{ra} + L^{ra})$. Similarly, we denote $\varphi_{i,j}^{sp}$ for $\mathrm{char}(\mu_{i,j}^{sp} + L_1^{sp})$, where $\mu_{i,j}^{sp} = \begin{pmatrix} 0 & \frac{j}{p} \\ i & 0 \end{pmatrix}$. Let

$$K(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : \ a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \pmod p \right\}.$$

Since

(3.2) $$N(\mathbb{Z}_p)M(\mathbb{Z}_p)\backslash \mathrm{SL}_2(\mathbb{Z}_p)/K(p) = \{1, wn(j), 0 \le j \le p-1\},$$

one has $\dim I(1)^{K(p)} = p + 1$.

Lemma 3.2. *Let the notation be as above. Then*

1. *When* $ab \equiv cd \pmod p$ *and* $(a,b), (c,d) \ne (0,0)$, *one has* $\lambda^{sp}(\varphi_{a,b}^{sp}) = \lambda^{sp}(\varphi_{c,d}^{sp})$.
2. *The set* $\{\lambda^{sp}(\varphi_0^{sp}), \lambda^{sp}(\varphi_{1,j}^{sp}), 0 \le j \le p-1\}$ *gives a basis of* $I(1)^{K(p)}$.

Proof. (1) By (3.2), it suffices to check the values at $\{1, wn(i), 0 \le i \le p-1\}$.

$$\begin{aligned} \lambda^{sp}(\varphi_{a,b}^{sp})(wn(i)) &= \int_{\mu_{a,b}+\mathcal{O}_p} \psi_p(i \det(x))dx \\ &= \frac{1}{p} e(abi/p) \end{aligned}$$

and

$$\lambda^{sp}(\varphi^{sp}_{a,b})(1) = 0,$$

where $e(x) = e^{2\pi\sqrt{-1}x}$. The result follows.

(2) By (3.2), we see that $\Phi \in I(1)^{K(p)}$ is determined by the values at

$$\{1, wn(i), 0 \le i \le p-1\}.$$

Suppose

$$a\lambda^{sp}(\varphi^{sp}_0) + \sum_{0\le j\le p-1} a_j \lambda^{sp}(\varphi^{sp}_{1,j}) = 0,$$

where $a, a_j \in \mathbb{C}$.

Taking the value at $g = 1$:

$$a\lambda^{sp}(\varphi^{sp}_0)(1) + \sum_{0\le j\le p-1} a_j \lambda^{sp}(\varphi^{sp}_{1,j})(1) = 0,$$

we get $a = 0$. Next take values at $wn(i)$, $1 \le i \le p-1$:

$$\sum_{0\le j\le p-1} a_j \lambda^{sp}(\varphi^{sp}_{1,j})(wn(i)) = 0.$$

So $\vec{a} = (a_0, \dots, a_{p-1})^t$ is a solution of the linear equation system

$$\mathbf{A}\vec{a} = 0,$$

where

$$\mathbf{A} := p\big(\lambda^{sp}(\varphi^{sp}_{1,j})(wn(i))\big)_{0\le i,j\le p-1} = \left(e\left(\frac{ij}{p}\right)\right)_{0\le i,j\le p-1}.$$

Since $\det(\mathbf{A}) \neq 0$, one has $a_j = 0, 0 \le j \le p-1$. The result follows. □

Proposition 3.3. *Assume that $(k,l) \neq 0$. Let $\mathbf{A} = (e(\frac{ij}{p}))_{0\le i,j\le p-1}$ be the matrix in the proof of Lemma 3.2, and let $\mathbf{A}_j$ be the matrix obtained by replacing jth column of $\mathbf{A}$ by column $\{-e\left(\frac{-id_{k,l}}{p}\right), 0 \le i \le p-1\}$, where*

$$d_{k,l} = k^2 + kl\,Tr_{k/\mathbb{Q}_p}(u) + l^2 \mathrm{N}_{k/\mathbb{Q}_p}(u).$$

Then $\sum_{j=0}^{p-1} c_{k,l}(j)\varphi^{sp}_{1,j} \in S(V^{sp})$ matches with $\varphi^{ra}_{k,l}$, where $c_{k,l}(j) = \frac{\det \mathbf{A}_j}{\det \mathbf{A}}$.

Proof. Since $\lambda^{ra}(\varphi^{ra}_{k,l}) \in I(1)^{K(p)}$, it is a linear combination of the basis of $I(1)^{K(p)}$ given in last lemma. Write

$$\lambda^{ra}(\varphi^{ra}_{k,l}) = b_{k,l}\lambda^{sp}(\varphi^{sp}_0) + \sum_{0\le j\le p-1} c_{k,l}(j)\lambda^{sp}(\varphi^{sp}_{1,j}),$$

where $b_{k,l}, c_{k,l}(j) \in \mathbb{C}$. Taking the value at 1, one gets $b_{k,l} = 0$.

Taking the value at $\{wn(i),\ 0 \le i \le p-1\}$, one has

$$\lambda^{ra}(\varphi^{ra}_{k,l})(wn(i)) = \sum_{0\le j\le p-1} c_{k,l}(j)\lambda^{sp}(\varphi^{sp}_{1,j})(wn(i)).$$

It is easy to check that

$$\lambda^{ra}(\varphi^{ra}_{k,l})(wn(i)) = -\frac{1}{p}e\left(-\frac{i}{p}(k^2 + klTr(u) + l^2N(u))\right) = -\frac{1}{p}e\left(-\frac{id_{k,l}}{p}\right).$$

From the proof of Lemma 3.2, it is known that $\lambda^{sp}(\varphi_{1,j}^{sp})(wn(i)) = \frac{1}{p}e(ij/p)$. So we get $c_{k,l}(j) = \frac{\det \mathbf{A}_j}{\det \mathbf{A}}$. □

3.2. The case $p = \infty$. In this subsection we consider the case $\mathbb{Q}_p = \mathbb{R}$ and recall a matching pair given in [Ku2]. Note that B^{ra} in this case is the Hamilton division algebra, and V^{ra} has signature $(4, 0)$. Let $\varphi_\infty^{ra}(x) = e^{-2\pi \det(x)} \in S(V^{ra})$, then φ_∞^{ra} is of weight 2 in the sense

$$\omega^{ra}(k_\theta)\varphi_\infty^{ra} = e^{2i\theta}\varphi_\infty^{ra}, \quad k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right).$$

On the other hand, Kudla constructed a family of weight 2 Schwartz function $\varphi_\infty^{sp} \in S(V^{sp})$ as follows [Ku2, Section 4.8]. Recall $V^{sp} = M_2(\mathbb{R})$. Given an orthogonal decomposition

$$V^{sp} = V^+ \oplus V^-, \quad x = x^+ + x^-, \tag{3.3}$$

with V^+ of signature $(2, 0)$ and V^- of signature $(0, 2)$. One defines (Kudla used the notation $\tilde{\phi}(x, z)$)

$$\varphi_\infty^{sp}(x, V^-) = (4\pi(x^+, x^+) - 1)e^{-\pi(x^+,x^+)+\pi(x^-,x^-)}.$$

Kudla proved the following proposition [Ku2, Section 4.8].

Proposition 3.4. *For any orthogonal decomposition (3.3), $(\varphi_\infty^{ra}, \varphi_\infty^{sp}(x, V^-))$ is a matching pair, and their (same) image in $I(1)$ is the unique weight 2 section Φ_∞^2 given by*

$$\Phi_\infty^2(n(b)m(a)k_\theta) = |a|^2 e^{2i\theta}.$$

Because of this proposition, we will simply write φ_∞^{sp} for $\varphi_\infty^{sp}(\,, V^-)$.

3.3. Global matching. The following matching proposition is clear from Kudla's matching principle (2.8) and Propositions 3.1, 3.3, and 3.4.

Proposition 3.5. *Let $D_1, D_2 > 1$ be two square-free positive integers, and let $V(D_i)$ be the quadratic spaces associated to the quaternion algebras $B(D_i)$ over $\mathbb{Q}$ (with reduced norm as the quadratic form), $i = 1, 2$. Assume that $\varphi^{(i)} = \prod_p \varphi_p^{(i)} \in S(V(D_i)(\mathbb{A}))$ satisfy the following conditions:*

1. *When $p = \infty$, $\varphi_\infty^{(i)}$ is φ_∞^{sp} or φ_∞^{ra} depending on whether $B(D_i)_\infty$ is split or non-split.*
2. *When $p \nmid D_1D_2\infty$ or $p | gcd(D_1, D_2)$, we identify $V(D_1)_p = V(D_2)_p$ and take any $\varphi_p^{(1)} = \varphi_p^{(2)} \in S(V(D_1)_p)$.*
3. *When $p | lcm(D_1, D_2)$ but $p \nmid gcd(D_1, D_2)$, one of $B(D_i)$ is B_p^{sp} and the other one is B_p^{ra}, we take $(\varphi_p^{(1)}, \varphi_p^{(2)})$ to be a matching pair in Propositions 3.1 and 3.3.*

Then $(\varphi^{(1)}, \varphi^{(2)})$ is a matching pair, and

$$I(g, \varphi^{(1)}) = I(g, \varphi^{(2)}), \quad g \in \mathrm{SL}_2(\mathbb{A}).$$

In next two sections, we will give arithmetic and geometric interpretations of the theta integrals in some special cases and prove the theorems in the introduction.

4. Definite quaternions, representations numbers, and supersingular elliptic curves

We first review a general fact about positive-definite quadratic forms for the convenience of the reader. Let (V, Q) be a positive-definite quadratic space of even dimension m. Define

$$\varphi_\infty(x) = e^{-2\pi Q(x)} \in S(V_\infty).$$

Then we have

$$\varphi_\infty(hx) = \varphi_\infty(x), \quad \omega(k_\theta)\varphi_\infty = e^{\frac{m}{2}i\theta}\varphi_\infty$$

for $h \in O(V)(\mathbb{R})$ and $k_\theta \in \mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R})$. For any $\varphi_f \in S(\hat{V})$, where $\hat{V} = V \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, the theta kernel

$$\theta(\tau, h, \varphi_f\varphi_\infty) = v^{-\frac{m}{4}}\theta(g_\tau, h, \varphi_f\varphi_\infty)$$

is a holomorphic modular form of weight $\frac{m}{2}$ for some congruence subgroup. Here $g_\tau = n(u)m(\sqrt{v})$ for $\tau = u + iv \in \mathbb{H}$. So

$$I(\tau, \varphi_f\varphi_\infty) = v^{-\frac{m}{4}}I(g_\tau, \varphi_f\varphi_\infty)$$

is also a modular form of weight $\frac{m}{2}$.

For an even integral lattice L in V, we let

$$\theta(\tau, L) = \theta(\tau, \mathrm{char}(\hat{L})\varphi_\infty), \quad I(\tau, L) = I(\tau, \mathrm{char}(\hat{L})\varphi_\infty). \tag{4.1}$$

Recall that two lattices L_1 and L_2 in V are equivalent if there is $h \in O(V)(\mathbb{Q})$ such that $hL_1 = L_2$. Two lattices L_1 and L_2 are in the same genus if they are equivalent locally everywhere, i.e., there is $h \in O(V)(\hat{\mathbb{Q}})$ such that $hL_1 = L_2$. We recall that $O(V)(\mathbb{A})$ acts on the set of lattices as follows: $hL = (h_f\hat{L}) \cap V$ where h_f is the finite part of $h = h_f h_\infty$. Let $\mathrm{gen}(L)$ be the genus of L — the set of equivalence classes of lattices in the same genus of L. Then the above discussion implies that

$$O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})/K(L)O(V)(\mathbb{R}) \cong \mathrm{gen}(L), \quad [h] \mapsto hL,$$

where $K(L)$ is the stabilizer of $\hat{L}$ in $O(V)(\hat{\mathbb{Q}})$.

Proposition 4.1. *Let*

$$r_L(n) = |\{x \in L : Q(x) = n\}|, \quad r_{\mathrm{gen}(L)}(n) = \left(\sum_{L' \in \mathrm{gen}(L)} \frac{1}{|O(L')|}\right)^{-1} \sum_{L' \in \mathrm{gen}(L)} \frac{r_{L'}(n)}{|O(L')|},$$

where $O(L)$ is the stabilizer of L in $O(V)$. Then, for $q = e(\tau)$,

$$\theta(\tau, h, L) = \sum_{n=0}^{\infty} r_{hL}(n)q^n,$$

$$I(\tau, L) = \sum_{m=0}^{\infty} r_{\mathrm{gen}(L)}(n)q^n.$$

In particular, the modular form $I(\tau, L)$ is a genus theta function.

Proof. (sketch) This is well known and we sketch main steps for the convenience of the reader. The formula for θ follows directly from the definition. For theta integral, note that $\mathrm{char}(\hat{L})$ is $K(L)$-invariant. Write

$$O(V)(\mathbb{A}) = \cup_{j=1}^{r} O(V)(\mathbb{Q}) h_j K(L) O(V)(\mathbb{R}).$$

Then $\mathrm{gen}(L) = \{h_j L : j = 1, \ldots, r\}$, and

$$\begin{aligned}\mathrm{vol}([O(V)]) I(\tau, L) &= \sum_j \theta(\tau, h_j, L) \int_{(h_j^{-1} O(V)(\mathbb{Q}) h_j) \cap K(L) \backslash K(L) O(V)(\mathbb{R})} 1 dh \\ &= \mathrm{vol}(K(L) O(V)(\mathbb{R})) \sum_j \frac{\theta(\tau, h_j, L)}{|O(h_j L)|} \\ &= \mathrm{vol}(K(L) O(V)(\mathbb{R})) \sum_{n=0}^{\infty} \left(\sum_{L' \in \mathrm{gen}(L)} \frac{r_{L'}(n)}{|O(L')|} \right) q^n.\end{aligned}$$

On the other hand, the same calculation gives

$$\mathrm{vol}([O(V)]) = \mathrm{vol}(K(L) O(V)(\mathbb{R})) \sum_{L' \in \mathrm{gen}(L)} \frac{1}{|O(L')|}.$$

One proves the formula for $I(\tau, L)$. □

Proof of Theorem 1.1: Let $V(D)$ be the quadratic space associated to the quaternion algebra $B(D)$ of discriminant D. Recall that a Eichler order of conductor N, denoted by $\mathcal{O}_D(N)$, is an order O of $B(D)$ such that

(1) When $p|D$, $O_p := O \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is the maximal order of $B(D)_p = B_p^{ra}$.
(2) When $p \nmid D\infty$, there is an identification $B(D)_p \cong M_2(\mathbb{Q}_p)$ under which

$$O_p = \mathcal{O}_p(N)^{sp} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : c \equiv 0 \mod p \right\}.$$

Now let $V^{(1)} = V(Dp)$ and $V^{(2)} = V(Dq)$ with D satisfying the condition of the theorem. Then $V^{(i)}$ are positive definite. Define $\varphi^{(1)} = \prod_l \varphi_l^{(1)} \in S(V^{(1)}(\mathbb{A}))$ as follows.

$$\varphi_l^{(1)} = \begin{cases} \varphi_\infty^{ra} & \text{if } l = \infty, \\ \mathrm{char}(\mathcal{O}_l(N)^{sp}) & \text{if } l \nmid Dpq, \\ \varphi_l^{ra} & \text{if } l | Dp, \\ \frac{-2}{q-1} \varphi_{q,0}^{sp} + \frac{q+1}{q-1} \varphi_{q,1}^{sp} & \text{if } l = q, \end{cases}$$

where $\varphi_{l,i}^{sp}$ and φ_l^{ra} are the functions defined in (3.1) with added subscript l. Then one has

$$\varphi_f^{(1)} = \frac{-2}{q-1} \mathrm{char}(\hat{\mathcal{O}}_{Dp}(N)) + \frac{q+1}{q-1} \mathrm{char}(\hat{\mathcal{O}}_{Dp}(Nq)).$$

So

$$I(\tau, \varphi^{(1)}) = \frac{-2}{q-1} I(\tau, \mathcal{O}_{Dp}(N)) + \frac{q+1}{q-1} I(\tau, \mathcal{O}_{Dp}(Nq)).$$

One defines $\varphi^{(2)}$ the same way with the roles of p and q switched. Then $\varphi^{(1)}$ and $\varphi^{(2)}$ form a matching pair by Proposition 3.3. So Proposition 3.5 implies

$$I(\tau,\varphi^{(1)}) = I(\tau,\varphi^{(2)}),$$

that is

$$\begin{aligned}&\frac{-2}{q-1}I(\tau,\mathcal{O}_{Dp}(N)) + \frac{q+1}{q-1}I(\tau,\mathcal{O}_{Dp}(Nq))\\ &\quad= \frac{-2}{p-1}I(\tau,\mathcal{O}_{Dq}(N)) + \frac{p+1}{p-1}I(\tau,\mathcal{O}_{Dq}(Np)).\end{aligned}$$

Taking mth Fourier coefficients, one proves the theorem.

The case $D = 1$ has special geometric meaning as indicated in the introduction. Let $X_0(N)$ be the moduli stack of pairs (E, C), where E is an elliptic curve and C is a cyclic sub-scheme of order N [KM]. It is regular and flat over $\mathbb{Z}$ and smooth over $\mathbb{Z}[\frac{1}{N}]$. For a prime $p \nmid N$, let $SS_p(N)$ be the supersingular locus of $X_0(N)(\bar{\mathbb{F}}_p)$, consisting of the $\bar{\mathbb{F}}_p$-points (E, C) such that E is supersingular, i.e., $\mathrm{End}(E)$ and $\mathrm{End}(E/C)$ are maximal orders in $B(p)$. In this case, the endomorphism ring $\mathrm{End}(E, C)$ is an Eichler order $\mathcal{O}_p(N)$ of conductor N — the intersection of $\mathrm{End}(E)$ and $\mathrm{End}(E/C)$. Every Eichler order in $B(p)$ arises in this way ([Ri2, Proposition 3.6]). Note that $SS_p(N)$ is the singular locus of $X_0(pN)$ modulo p, and is closely related to the torus part of the connected component of the Jacobian $J_0(pN)$ of $X_0(N)$ modulo p. In [Ri2], Ribet studied the Hecke operations on 'pq-new part of torus part of $J_0(pqN)$ modulo p and on the torus part of the Jacobian $J_0^{pq}(N)$ of the Shimura curve $X_0^{pq}(N)$ and discovered a beautiful relation between them, which enables him to switch the level of a modular form from pN to qN, and then lowers the level to N. He related both with an Eichler order $\mathcal{O}_p(qN)$ of the definite quaternion algebra $B(p)$. We refer to [Ri2, Sections 3 and 4] for details. Ribet also gave a geometric proof of relation between the two Hecke actions in [Ri1] later. It is well known that $X_0(N)(\mathbb{C})$ is the (open) modular curve defined in the next section.

For two points $x_1 = (E_1, C_1), x_2 = (E_2, C_2) \in SS_p(N)$, $\mathrm{Hom}(x_1, x_2)$, which consists of isogenies $(f : E_1 \to E_2)$ with $f(C_1) \subset C_2$, is a quadratic lattice with respect to $\deg f$. Moreover, they form one genus as x_1 and x_2 run over $SS_p(N)$ — the genus of the quadratic lattice $\mathrm{End}((E, C))$ for a (any) $(E, C) \in SS_p(N)$ ([Ya2]). So, we have

Proposition 4.2. *One has*

$$r_{p,N}(m) = \left(\sum_{x_1,x_2\in SS_p(N)} \frac{1}{|\mathrm{Aut}(x_1)||\mathrm{Aut}(x_2)|}\right)^{-1} \sum_{x_1,x_2\in SS_p(N)} \frac{r_{\mathrm{Hom}(x_1,x_2)}(m)}{|\mathrm{Aut}(x_1)||\mathrm{Aut}(x_2)|}.$$

5. Indefinite quaternions and Shimura curves

In this section (until the proof of Theorems 1.2, 1.3, and 1.4), we assume that $D > 0$ has even number prime factors, and let $B = B(D)$ be the associated indefinite quaternion. In this case, $V = (B, \det)$ is of signature $(2, 2)$ and is anisotropic when $D > 1$. According to [Ku2, Theorem 4.23], the theta integral $I(g,\varphi)$ in Proposition 3.5 is a generating function of degrees of some divisors with respect to the tautological line bundle over the Shimura curve associated to V. In our case, the line bundle can be identified with the line bundle of two variable modular forms of weight 1, and the divisors can be identified with Hecke correspondences on a Shimura curve.

Recall that $B^\times \times B^\times$ acts on V via

$$(g_1, g_2)X = g_1 X g_2^{-1}$$

gives an identification of $\mathrm{GSpin}(V)$ with

$$H = \{(g_1, g_2) \in B^\times \times B^\times : \det g_1 = \det g_2\}.$$

The associated spin norm is $\mu(g_1, g_2) = \det g_1$ and there is an exact sequence

$$1 \to \mathbb{G}_m \to H \to \mathrm{SO}(V) \to 1.$$

Let $\mathbb{D}$ be the Hermitian domain of oriented negative 2-planes in $V_\mathbb{R}$, and

$$\mathcal{L} = \{w \in V_\mathbb{C} = M_2(\mathbb{C}) : (w, w) = 0, (w, \bar{w}) < 0\}$$

on both of which $H(\mathbb{R})$ acts naturally. The map

$$f : \mathcal{L}/\mathbb{C}^\times \cong \mathbb{D}, \quad w = u + iv \mapsto \mathbb{R}(-u) + \mathbb{R}v$$

gives an $H(\mathbb{R})$-equivariant isomorphism between $\mathcal{L}/\mathbb{C}^\times$ and $\mathbb{D}$. Thus $\mathcal{L}$ is a (tautological) line bundle over $\mathbb{D}$. The Hermitian domain has also a tube representation, which we need. Indeed, let

$$\mathcal{D} = \{(z_1, z_2) \in \mathbb{C}^2 : \Im(z_1)\Im(z_2) > 0\} = (\mathbb{H} \times \mathbb{H}) \bigsqcup (\mathbb{H}^- \times \mathbb{H}^-).$$

Then the map

$$\mathbf{w} : \mathcal{D} \to \mathcal{L}, \quad \mathbf{w}(z_1, z_2) = \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix},$$

gives an isomorphism

$$\mathcal{D} \cong \mathcal{L}/\mathbb{C}^\times \cong \mathbb{D}.$$

We will identify $\mathbb{D}$ with $\mathcal{D}$ via this isomorphism. The natural action of $B^\times \times B^\times$ on V induces the following action on $\mathcal{D}$:

$$(g_1, g_2)(z_1, z_2) = (i(g_1)z_1, i(g_2)^* z_2) \tag{5.1}$$

where $g^* = {}^t g^{-1}$ for $g \in \mathrm{GL}_2(\mathbb{R})$. One also has

$$(g_1, g_2)\mathbf{w}(z_1, z_2) = \mathbf{w}((g_1, g_2)(z_1, z_2))(c_1 z_1 + d_1)(c_2 z_2 + d_2) \tag{5.2}$$

if we write

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}), \quad g_2^* = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

Associated to a compact open subgroup K of $H(\hat{Q})$ is a Shimura variety X_K over $\mathbb{Q}$ such that

$$X_K(\mathbb{C}) = H(\mathbb{Q})\backslash\mathbb{D} \times H(\hat{\mathbb{Q}})/K.$$

Moreover, $\mathcal{L}$ descends to a line bundle on X_K, which we continue to denote by $\mathcal{L}$. It can be identified with the line bundle of two variable modular forms of weight $(1, 1)$. In this section, we always assume that

$$K = \{(k_1, k_2) \in \hat{\mathcal{O}}_D(N)^\times \times \hat{\mathcal{O}}_D(N)^\times : \det k_1 = \det k_2\} \subset H(\hat{\mathbb{Q}})$$

which preserves the lattice $L = \mathcal{O}_D(N)$. By the Strong Approximation theorem, one has

Lemma 5.1. *Let the notation be as above. Then one has an isomorphism*

$$X_0^D(N) \times X_0^D(N) \cong X_K, ([z_1],[z_2]) \mapsto [z_1, wz_2].$$

Here $X_0^D(N) = \Gamma_0^D(N)\backslash\mathbb{H}$ *is the Shimura curve defined in the introduction (recall* $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$*).*

Let $\Omega_0 = \frac{1}{2\pi} y^{-2} dx \wedge dy$ be the differential on $X_0^D(N)$, and let π_1 and π_2 be two natural projections of $X_K = X_0^D(N) \times X_0^D(N)$ onto $X_0^D(N)$. Then

$$\Omega = -\frac{1}{2}(\pi_1^*(\Omega_0) + \pi_2^*(\Omega_0)).$$

Moreover, one has by [KRY, (2.7)] and [Mi, Lemma 5.3.2]

$$\text{(5.3)} \qquad \begin{aligned} \mathrm{vol}(X_0^D(N), \Omega_0) &:= \int_{X_0^D(N)} \Omega_0 = -2[\mathcal{O}_D^1 : \Gamma_0^D(N)]\zeta_D(-1) \\ &= \frac{DN}{6} \prod_{p|N}(1+p^{-1}) \prod_{p|D}(1-p^{-1}) \in \frac{1}{6}\mathbb{Z}, \end{aligned}$$

where $\zeta_D(s) = \prod_{p\nmid D}(1-p^{-s})^{-1}$ is the partial zeta function, and $\mathcal{O}_D$ is a maximal order of B containing $\mathcal{O}_D(N)$.

Next, we describe the Kudla cycle on X_K and relate it to Hecke correspondence on $X_0^D(N)$. For a $x \in V(\mathbb{Q})$ with $\det(x) > 0$ and $h \in H(\hat{\mathbb{Q}})$, $x^\perp$ is of signature $(1,2)$ and defines a sub-Shimura variety $Z(x)$ of $X_{hKh^{-1}}$, its right translation by h gives a divisor $Z(x,h)$ in X_K. Let $\varphi_f \in S(\hat{V})^K$ and $m \in \mathbb{Q}_{>0}$. If there is $x_0 \in V(\mathbb{Q})$ such that $\det(x_0) = m$, we define the associated Kudla cycle $Z(m, \varphi_f)$ as

$$Z(m, \varphi_f) = \sum_{j=1}^{r} \varphi_f(h_j^{-1}x_0) Z(x_0, h_j)$$

where

$$\mathrm{Supp}(\varphi_f) \cap \{x \in V(\hat{\mathbb{Q}}) : \det x = m\} = \coprod_{j=1}^{r} K h_j^{-1} x_0.$$

Otherwise, we define $Z(m, \varphi_f) = 0$.

Lemma 5.2. *Let* $T_{D,N}(m)$ *be the Hecke operator on* $X_0^D(N)$ *as in the introduction. Then (under the identification* $X_K \cong X_0^D(N) \times X_0^D(N)$ *in Lemma 5.1)*

$$Z(m, \mathrm{char}(\hat{L})) = T_{D,N}(m)$$

where $L = \mathcal{O}_D(N)$*.*

Proof. Let $L_m = \{x \in L : \det x = m\}$. By the Strong Approximation theorem, one has $H(\hat{\mathbb{Q}}) = H(\mathbb{Q})K$. So in the decomposition ($x_0 \in V(\mathbb{Q})$ with $Q(x_0) = m$)

$$\hat{L}_m = \coprod K h_j^{-1} x_0$$

we may assume that $h_j \in H(\mathbb{Q})$. This implies

$$L_m = \coprod \Gamma_K h_j^{-1} x_0 = \coprod \Gamma_K x_j, \quad x_j = h_j^{-1} x_0 \in L,$$

where $\Gamma_K = K \cap H(\mathbb{Q})$, and

$$Z(m, \varphi_f) = \sum_j Z(x_0, h_j) = \sum_j Z(h_j^{-1} x_0) = \sum_j Z(x_j) = \Gamma_K \backslash \mathcal{D}_m.$$

where $\mathcal{D}_m$ is the set of $(z_1, z_2) \in \mathcal{D}$ which satisfying $z_1 = x(z_2)$ for some $x \in L_m$.

Note that there is some $(\gamma_1, \gamma_2) \in \Gamma_K$ with $\det \gamma_1 = \det \gamma_2 = -1$. So for any $(z_1, z_2) \in \mathcal{D}$, there is some $(z_1', z_2') \in \mathbb{H}^2$, such that either (z_1', wz_2') or $(\gamma_1 z_1', \gamma_2 w z_2')$ equals (z_1, z_2). So, one has

$$Z(m, \varphi_f) = (\Gamma_0^D(N) \times \Gamma_0^D(N)) \backslash \mathbb{D}_m^+ = T_{D,N}(m).$$

Here

$$\mathcal{D}_m^+ = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} : z_1 = x(wz_2), \quad \text{for some } x \in L_m\}.$$

Then we get the lemma. □

Theorem 5.3. *For $\varphi_f = \mathrm{char}(\hat{\mathcal{O}}_D(N))$, one has*

$$I(\tau, \varphi_f \varphi_\infty^{sp}) = v^{-1} I(g_\tau, \varphi_f \varphi_\infty^{sp}) = \sum_{m=0}^{\infty} r'_{D,N}(m) q^m$$

where $r'_{D,N}(0) = 1$, and for $m > 0$

$$r'_{D,N}(m) = -\frac{2}{\mathrm{vol}(X_0^D(N), \Omega_0)} \deg T_{D,N}(m)$$

as in the introduction.

Proof. Write

$$I(\tau, \varphi_f \varphi_\infty^{sp}) = \sum_{m=0}^{\infty} c(m) q^m.$$

By [Ku2, Section 4.8], one has $c(0) = 1$ and for $m > 0$

$$c(m) = (\mathrm{vol}(X_K, \Omega^2))^{-1} \int_{Z(m, \varphi_f)} \Omega.$$

Clearly,

$$\mathrm{vol}(X_K, \Omega^2) = \frac{1}{2} \frac{1}{4\pi^2} \int_{X_0^D(N) \times X_0^D(N)} \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \frac{dx_2 \wedge dy_2}{y_2^2} = \frac{1}{2} \mathrm{vol}(X_0^D(N), \Omega_0)^2.$$

On the other hand, $\Omega = -\frac{1}{2}(\pi_1^*(\Omega_0) + \pi_2^*(\Omega_0))$. So Lemma 5.2 gives

$$\begin{aligned} c(m) &= -\frac{1}{2} \int_{T_{D,N}(m)} (\pi_1^*(\Omega_0) + \pi_2^*(\Omega_0)) \\ &= -\int_{T_{D,N}(m)} \pi_1^*(\Omega_0) \\ &= -\deg T_{D,N}(m) \int_{X_0^D(N)} \Omega_0. \end{aligned}$$

So $c(m) = r'_{D,N}(m)$ as claimed. □

Proof of Theorems 1.2, 1.3 and 1.4: Now Theorems 1.2, 1.3 and 1.4 follow the same way as Theorem 1.1. We verify Theorem 1.3 and leave others to the reader. Let $V^{(1)} = V(D)$ and $V^{(2)} = V(Dp)$ as in the notation of proof of Theorem 1.1, and let $\varphi^{(i)} = \prod_l \varphi_l^{(i)} \in S(V^{(i)}(\mathbb{A})$ be as follows. For $l \nmid p\infty$, we identify $\mathcal{O}_D(N)_l$ with $\mathcal{O}_{Dp}(N)_l$ and denote $\varphi_l^{(i)} = \mathrm{char}(\mathcal{O}_D(N)_l)$. Let

$$\varphi_\infty^{(1)} = \varphi_\infty^{ra}, \quad \varphi_\infty^{(2)} = \varphi_\infty^{sp}.$$

Finally, let

$$\varphi_p^{(1)} = -\frac{2}{p-1}\varphi_{p,0}^{sp} + \frac{p+1}{p-1}\varphi_{p,1}^{sp}, \quad \varphi_p^{(2)} = \varphi_p^{ra}.$$

Then $\varphi^{(1)}$ and $\varphi^{(2)}$ match by the results in Section 3. So one has by Proposition 3.5

$$I(\tau, \varphi^{(1)}) = I(\tau, \varphi^{(2)}).$$

Comparing mth coefficients of the both sides, one proves Theorem 1.3.

Acknowledgments

This paper was inspired by Kudla's matching principle. We thank him for his influence and help. We thank Xinyi Yuan for his help to make Section 5 cleaner. We thank Schulze-Pillot for informing us his related joint work with Katsurada [KSP]. We thank the referees for their constructive suggestions and careful listing of the typos in an early version to make this exposition much nicer. The first author acknowledges the Department of Mathematics, University of Wisconsin at Madison for providing him excellent working and learning environment during his visit. The second author acknowledges MPIM at Bonn and Tsinghua Math Science Center for providing excellent working conditions during summer 2012, where he did part of this work. T.Y. Yang is partially supported by an NSF grant DMS1200385 and a Qian Ren Visitor Plan (China). Tuoping Du is partially supported by NSFC grants (Nos. 11171141110 and BK2010007).

References

[Du] T. Du, A regularized Siegel-Weil for the matrix algebra, in progress.

[KSP] H. Katsurada and R. Schulze-Pillot, *Genus theta series, Hecke operators and the basis problem for Eisenstein series.* Automorphic forms and zeta functions, 234–261, World Science Publ., Hackensack, NJ, 2006.

[KM] N. Katz, and B. Mazur, *Arithmetic moduli of elliptic curves.* Annals of Mathematics Studies, **108**. Princeton University Press, Princeton, NJ, 1985. xiv+514 pp.

[Ku1] S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math., **87** (1994), 361–401.

[Ku2] S. Kudla, *Integrals of Borcherds forms.* Compos. Math., **137** (2003), 293–349.

[KRY] S. Kudla, M. Rapoport and T.H. Yang, *Derivatives of Eisenstein series and Faltings heights*, Compos. Math., **140** (2004), 887–951.

[KR1] S. Kudla and S. Rallis, *On the Weil–Siegel formula*, J. Reine Angew. Math., **387** (1988), 1–68.

[KR2] S. Kudla and S. Rallis, *On the Weil–Siegel formula II. The isotropic convergent case*, J. Reine Angew. Math., **391** (1988), 65–84.

[KR3] S. Kudla and S. Rallis, *A regularized Siegel–Weil formula: the first term identity*, Ann. Math. (2) **140** (1994), 1–80.

[Mi] T. Miyake, *Modular forms.* Springer, New York, 1989.

[Ri1] K. Ribet, *Bimodules and abelian surfaces.* in 'Algebraic number theory', 359–407, Advanced Studies of Pure Mathematics, **17**, Academic Press, Boston, MA, 1989.

[Ri2] K. Ribet, *On modular representations of* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms*, Invent. Math. **100** (1990), 431–476.

[Si] C.L. Siegel, *Uber die analytische Theorie der quadratischen Formen*, Ann. Math., **36** (1935), 527–606.

[We] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques(French)*, Acta Math., **113** (1965), 1–87.

[Ya1] T.H. Yang, *An explicit formula for local densities of quadratic forms*,. J. Number Theory, **72**(2) (1998), 309–356.

[Ya2] T.H. Yang, *Arithmetic intersection and Faltings height*. Asian J. Math., **17** (2013), 335–382.

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: dtpnju@gmail.com

Department of Mathematics, University of Wisconsin Madison, Van Vleck Hall, Madison, WI 53706, USA

E-mail address: thyang@math.wisc.edu