A REMARK ON A CONJECTURE OF BUZZARD–GEE AND THE COHOMOLOGY OF SHIMURA VARIETIES

CHRISTIAN JOHANSSON

ABSTRACT. We compare the conjecture of Buzzard–Gee on the association of Galois representations to C-algebraic automorphic representations with the conjectural description of the cohomology of Shimura varieties due to Kottwitz, and the reciprocity law at infinity due to Arthur. This is done by extending Langlands's representation of the L-group associated with a Shimura datum to a representation of the C-group of Buzzard–Gee. The approach offers an explanation of the explicit Tate twist appearing in Kottwitz's description.

1. Introduction

A central theme in modern algebraic number theory is the connection between automorphic forms and Galois representations. This link has its origins in class field theory, and evolved from the work of Weil et al. among others before its scope was vastly expanded by Langlands in the late 1960s and 1970s, linking the emerging theory of automorphic representations to Grothendieck's conjectural theory of motives, where Galois representations appear as realizations. For automorphic representations of GL_n, the first precise conjectures were given by Clozel in [Clo] where he defined a notion of algebraicity and conjectured that there should be motives (hence Galois representations) associated with algebraic automorphic representations.

In [BG], Buzzard and Gee generalize the notion of algebraicity and (a weak form of) Clozel's conjectures to an arbitrary connected reductive group G over a number field F. Perhaps somewhat surprisingly, they define two notions of algebraicity. On the one hand they define a notion of L-algebraicity, and given an L-algebraic automorphic representation π of G they conjecture the existence of an ℓ-adic Galois representation

$$\rho_\pi^L \,:\, Gal(\overline{F}/F) \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

They also define a notion of C-algebraicity which generalizes Clozel's definition of algebraicity. Buzzard and Gee (following an idea of Deligne) define a canonical $\mathbb{G}_m$-extension $\widetilde{G}$ of G and given a C-algebraic automorphic representation π of G, they canonically construct an L-algebraic automorphic representation $\widetilde{\pi}$ of $\widetilde{G}$ and thus conjecturally get an associated Galois representation

$$\rho_\pi = \rho_{\widetilde{\pi}}^L \,:\, Gal(\overline{F}/F) \to {}^L \widetilde{G}(\overline{\mathbb{Q}}_\ell) = {}^C G(\overline{\mathbb{Q}}_\ell)$$

Since the beginnings of the theory, an important test case for any conjecture on the association of Galois representation to automorphic representations has been the case of Shimura varieties, starting with the modular curve. Langlands initiated a program to compute the (ℓ-adic) cohomology of arbitrary Shimura varieties (with coefficients in

---

certain local systems) and later on Kottwitz gave a very precise conjectural description of these cohomology groups [Kot]. The automorphic representations that contribute to this description are called cohomological and they are the one of most important classes of C-algebraic automorphic representations.

In this short paper, we wish to test the Buzzard–Gee conjecture on the association of Galois representations with C-algebraic automorphic forms in the case of cohomological automorphic representations on groups admitting a Shimura variety. Namely, given a cohomological automorphic representation π, we show that Kottwitz's description implies (or rather is equivalent to) that the composition of ρ_π with a certain fixed algebraic representation r_C of ^CG occurs in the cohomology (for the precise statement, see Conjecture 8). The representation r_C is an extension to ^CG of a representation of ^LG (originally defined by Langlands in [Lan]) that is used in Kottwitz's description. Our minor reformulation has the slight advantage that it explains (or perhaps internalizes) a somewhat mysterious Tate twist by half the dimension of the Shimura variety that occurs in Kottwitz's description.

The structure of this note is as follows: In Section 2 we define the representation r_C and in Section 3 we prove Corollary 3 which is the main result needed to relate the Buzzard-Gee conjecture and Kottwitz's description of the cohomology of Shimura varieties. This uses only some basic Lie theory and the definition of a Shimura datum. Finally in Section 4, we recall Kottwitz's description together with a related result of Arthur [Art] and make the comparison.

2. The representation r_C

In this section, we will set up some notation, recall the notion of a C-group from [BG] and define a representation r_C which will be an extension of a representation of Langlands that we will denote r_L (whose definition we will recall below). For simplicity, all our dual groups and L-groups will be complex algebraic groups and complex pro-algebraic groups, respectively, and will be identified with their $\mathbb{C}$-points. In Section 4 we will fix an isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, where ℓ is a fixed rational prime. Given a connective reductive group G over a field F with L-group $^LG = Gal(\overline{F}/F) \ltimes \widehat{G}$ (we will always use the Galois form of the L-group), we may use ι to consider the associated pro-algebraic group $^LG(\overline{\mathbb{Q}}_\ell)$ over $\overline{\mathbb{Q}}_\ell$, identified with its $\overline{\mathbb{Q}}_\ell$-points, and by abuse of notation we will denote by ι the map $^LG \to {}^LG(\overline{\mathbb{Q}}_\ell)$ induced by ι.

Let G be a connected reductive group over $\mathbb{Q}$ and let X be a $G(\mathbb{R})$-conjugacy class of homomorphisms

$$h : \mathbb{S} \to G_\mathbb{R}$$

of real algebraic groups, where $\mathbb{S}$ denotes Deligne's torus $\mathrm{Res}^\mathbb{C}_\mathbb{R}\mathrm{GL}_{1/\mathbb{C}}$. The pair (G, X) is required to satisfy Deligne's axioms for a Shimura datum (see [Del] Section 2.1.1). We will let E denote the reflex field of (G, X) and we denote by d the complex dimension of X. Given $h \in X$, we let μ_h denote the corresponding cocharacter of $G_\mathbb{C}$ defined by

$$\mu_h(z) = h_\mathbb{C}(z, 1)$$

where we have fixed an isomorphism $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$ (and $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ embeds by $z \mapsto (z, \bar{z})$). We will follow the conventions of [Del] for associating real Hodge

structures with representations of $\mathbb{S}$ (i.e., the (p,q)-space is the subspace where $\mathbb{S}(\mathbb{C})$ acts by $(z,w) \mapsto z^{-p}w^{-q}$) and the normalization of the reciprocity map of local class field theory (uniformizers go to geometric Frobenii). Fix a pinning (T,B) of G with corresponding pinning $(\widehat{T}, \widehat{B})$ of $\widehat{G}$ that is fixed by the Galois action (we suppress the choice of elements in the root spaces for the simple roots). This gives us the notion of dominant weights and coweights and positive roots and coroots for G and $\widehat{G}$. We let μ denote an element of the conjugacy class of $(-\mu_h)_{h \in X}$ which is antidominant. The representation r_L is defined in two steps. μ defines an antidominant integral weight $\widehat{\mu}$ of $\widehat{G}$ and hence a unique irreducible representation

$$r : \widehat{G} \to \mathrm{Aut}(V_\mu)$$

that has $\widehat{\mu}$ as an extreme weight. We extend this to our desired representation

$$r_L : {}^L G_E = \Gamma_E \ltimes \widehat{G} \to \mathrm{Aut}(V_\mu)$$

by letting $\Gamma_E = Gal(\overline{\mathbb{Q}}/E)$ act trivially on the weight space of $\widehat{\mu}$.

Buzzard and Gee have defined the notion of a C-group ${}^C G$ for G ([BG] Definition 5.3.2), which is defined to be the L-group ${}^L \widetilde{G}$ of a canonical extension

$$1 \to \mathrm{GL}_1 \to \widetilde{G} \to G \to 1$$

of G. Proposition 5.3.3 of [BG] gives a central isogeny

$$\widehat{G} \times \mathbb{C}^\times \to \widehat{\widetilde{G}}$$

which is Galois-equivariant and whose kernel is generated by the element $(e,-1)$ of order 2. Here, $e = \widehat{\chi}(-1)$, where χ is the sum of positive roots of G and $\widehat{\chi}$ is the sum of the positive coroots of $\widehat{G}$. We wish to extend r_L to a representation r_C of ${}^C G_E = \Gamma_E \ltimes \widehat{\widetilde{G}}$. Let $\langle -, - \rangle$ denote the pairing between the character and cocharacter lattices of G and $\widehat{G}$ to $\mathbb{Z}$. Part 1) of the following Lemma is well known.

Lemma 1.   (1) $\langle \chi, \mu \rangle = -d$.
   (2) e acts as $(-1)^{-d}$ on V_μ.

Proof.   (1) By axiom (2.1.1.1) for a Shimura datum (in the notation of [Del] Section 2.1.1) and the choice of μ, $\mu(z)$ acts as z^{-1} or 1 on the positive root spaces (i.e., $\langle \alpha, \mu \rangle = -1$ or 0 for each positive root α) and as z or 1 on the negative root spaces. The result follows since there are d roots on which $\mu(z)$ acts as z^{-1}, by axiom (2.1.1.2) for a Shimura datum ([Del] Section 2.1.1).
   (2) Since e is central it acts on V_μ by a scalar and hence it is enough to compute the action on the weight space of $\widehat{\mu}$. By part 1), the action is by

$$\widehat{\mu}(e) = \widehat{\mu}(\widehat{\chi}(-1)) = (-1)^{\langle \widehat{\mu}, \widehat{\chi} \rangle} = (-1)^{\langle \chi, \mu \rangle} = (-1)^{-d}.$$

$\square$

Let us now extend r to a representation of $\widehat{G} \times \mathrm{GL}_1$ by

$$(g,z) \mapsto r(g)z^{-d}.$$

By Lemma 1 $(e,-1)$ acts trivially so this defines a representation of $\widehat{\widetilde{G}}$. We may extend this to our desired representation r_C of ${}^C G_E$ on V_μ by letting Γ_E act trivially on the

weight space of $\widehat{\mu}$; by the Γ_E-equivariance of the isogeny $\widehat{G} \times \mathrm{GL}_1 \to \widehat{\widetilde{G}}$ this defines an extension of r_L.

3. The result

Let τ be an irreducible algebraic representation of G, and let π be a discrete automorphic representation of G which is cohomological for τ, i.e., there is an integer i such that $H^i(\mathfrak{g}_\infty, K_\infty, \pi_\infty \otimes \tau) \neq 0$, where $\mathfrak{g}_\infty$ is the Lie algebra of $G_{\mathbb{R}}$, $K_\infty \subseteq G(\mathbb{R})$ is the stabilizer of some fixed $x \in X$ and $H^i(\mathfrak{g}_\infty, K_\infty, -)$ is the relative Lie algebra cohomology. Lemma 7.2.2 of [BG] says that π is C-algebraic ([BG], Definition 3.1.2).

Remark. π is also C-arithmetic ([BG], Definition 3.1.4). This follows from the argument in Section 2.3 of [BR] taking coefficients in $\mathcal{F}_\tau$ (defined near the end of the first paragraph of Section 4 of this paper) instead of the trivial local system.

Let π be a C-algebraic automorphic representation on G. We will briefly review the construction of an L-algebraic ([BG], Definition 3.1.1) automorphic representation $\widetilde{\pi}$ of $\widetilde{G}$, canonically associated to π (see [BG], discussion before Conjecture 5.3.4). Given π, one uses the canonical map $\widetilde{G} \to G$ to pull back π to a C-algebraic automorphic representation π' ([BG] Lemma 5.1.2). The central isogeny $\widehat{G} \times \mathrm{GL}_1 \to \widehat{\widetilde{G}}$ mentioned in the previous section is dual to an isogeny $(c, \xi) : \widetilde{G} \to G \times \mathrm{GL}_1$ defined over $\mathbb{Q}$. Then

$$g \mapsto |\xi(g)|^{1/2}$$

is a character $\widetilde{G}(\mathbb{A}) \to \mathbb{C}^\times$, and we define $\widetilde{\pi}$ to be the twist $\pi' \otimes |\xi(-)|^{1/2}$. It is an L-algebraic automorphic representation of $\widetilde{G}$.

For a local or global field F of characteristic 0 we let $\mathcal{L}_F$ denote the Langlands group of F. It carries a canonical surjection $\mathcal{L}_F \twoheadrightarrow W_F$, where W_F is the Weil group of F. When F is local, we have $\mathcal{L}_F = W_F$ if F is archimedean (with the canonical map being the identity) and we take $\mathcal{L}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ if F is non-archimedean (the canonical map being projection onto the first factor). When F is global, this group only exists conjecturally. We will use it only for motivation (in particular to make the comparison with [Kot]); in the end all conjectures and results may be stated using only $\mathcal{L}_F$ for local fields.

Let $|-|$ denote the composition of $\mathcal{L}_F \twoheadrightarrow W_F$ with the norm character $W_F \to \mathbb{C}^\times$.

Lemma 2. *1) Let p be a finite prime where π and G are unramified and let $\phi_p : \mathcal{L}_{\mathbb{Q}_p} \to {}^L G$ be the L-parameter (Satake parameter) associated with π_p. Then the L-parameter $\widetilde{\phi}_p : \mathcal{L}_{\mathbb{Q}_p} \to {}^C G$ attached to $\widetilde{\pi}_p$ is given by*

$$\widetilde{\phi}_p(w) = (\phi_p(w), |w|^{1/2})$$

where we abuse notation and denote by $(\phi_p(w), |w|^{1/2})$ the image of $(\phi_p(w), |w|^{1/2}) \in {}^L G \times \mathbb{C}^\times$ in ${}^C G$.

2) Let $\phi_{\mathbb{C}} : \mathcal{L}_{\mathbb{C}} \to {}^L G$ be the restriction of the L-parameter associated with π_∞. Then the restriction of the L-parameter of $\widetilde{\pi}_\infty$ to $\mathcal{L}_{\mathbb{C}}$ is given by

$$\widetilde{\phi}_{\mathbb{C}}(w) = (\phi_{\mathbb{C}}(w), |w|^{1/2}).$$

Proof.

(1) This follows from the construction of the unramified Local Langlands corre-
spondence as described in Section 10.4 of [Bor] and is implicit in [BG] (in the
derivation of Conjecture 5.3.4 from Conjecture 3.2.1); we will content our-
selves with giving a brief sketch of the proof. Let $\widetilde{T}$ resp. $\widetilde{B}$ be the inverse
images of T resp. B under $\widetilde{G} \to G$. π_p occurs as a subquotient of some unram-
ified principal series $\mathrm{Ind}_B^G \chi$ (normalized induction), where $\chi : T(\mathbb{Q}_p) \to \mathbb{C}^\times$ is
an unramified character. Since parabolic induction behaves well with respect
to pullback $\widetilde{G}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$, π' occurs as a subquotient of $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \chi'$, where
χ' is the composition of χ with $\widetilde{T} \to T$. This implies that the L-parameter
of π' is $w \mapsto (\phi_p(w), 1)$ since the correspondence for unramified characters
of unramified tori is functorial. To get the L-parameter of $\widetilde{\pi}$ we use that
the unramified Local Langlands correspondence behaves well with respect to
unramified twists; see e.g., Remark 2.2.1 of [BG] (although it is perhaps sim-
pler to deduce this from the construction in [Bor] since parabolic induction
behaves well under twists, rather than using the Satake isomorphism directly
as is done in [BG]).

(2) This is proved in exactly the same way as (1), although the details are sim-
pler, using the construction of the complex Local Langlands correspondence
(see [Bor] Section 11.4). Here one deduces the behavior with respect to twists
from same property for parabolic induction as indicated in the end of the
proof of (1).

□

The following simple consequence is our main technical result.

Corollary 3. *With notation as in Lemma 2, we have*

$$(r_L \circ \phi_v) \otimes |-|^{-d/2} = r_C \circ \widetilde{\phi}_v$$

for v finite where π and G are unramified or $v = \mathbb{C}$.

Proof. By Lemma 2 and the definition of r_C we have

$$r_C(\widetilde{\phi}_v(w)) = r_C(\phi_v(w), |w|^{1/2}) = r_L(\phi_v(w))|w|^{-d/2}$$

as desired.

□

4. Comparison with Kottwitz's Conjecture and Arthur's Result

Let us try to briefly describe the conjecture of Kottwitz and the result of Arthur.
For simplicity, we will follow Arthur's presentation in Section 9 of [Art]. Recall our
Shimura datum (G, X). If $K \subseteq G(\mathbb{A}^\infty)$ is a compact open subgroup we will let Sh_K
denote the canonical model over E of the corresponding Shimura variety and $\overline{Sh}_K$
its minimal compactification [Pin]. We write Sh for the inverse system $(Sh_K)_K$ and
$\overline{Sh}$ for $(\overline{Sh}_K)_K$. The reflex field E comes with an embedding into $\mathbb{C}$ and we let $\overline{\mathbb{Q}}$
denote the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. Let ℓ be a fixed rational prime; we fix
an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, thus we may view $\overline{\mathbb{Q}}$ as a subfield of $\overline{\mathbb{Q}}_\ell$. Let τ be an
algebraic representation of G. Then we may associate to τ a sheaf $\mathcal{F}_\tau$ which is either a
constructible local system of $\mathbb{Q}$-vector spaces on $Sh_K(\mathbb{C})^{an}$, a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on Sh_K

CHRISTIAN JOHANSSON

284

or a vector bundle with flat connection on Sh_K. They satisfy various compatibilities with respect to transition maps in Sh and comparison theorems for the relevant cohomology theories; we will abuse notation and denote them all by $\mathcal{F}_\tau$, as well as the canonical extension of the smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_\tau$ on Sh_K to $\overline{Sh}_K$ (defined via the theory of perverse sheaves). There is a geometric action of $G(\mathbb{A}^\infty)$ on Sh and $\overline{Sh}$. For any suitable cohomology theory H we write

$$H^*(Sh_K, \mathcal{F}_\tau) = \bigoplus_i H^i(Sh_K, \mathcal{F}_\tau)$$

$$H^*(Sh, \mathcal{F}_\tau) = \varinjlim H^*(Sh_K, \mathcal{F}_\tau) \in Mod(G(\mathbb{A}^\infty) \times ?)$$

where the ? signifies that the cohomology theory may carry extra structure. We use analogous notation for $\overline{Sh}$.

The starting point for the analysis of the cohomology of Shimura varieties is the result

$$H^*_{(2)}(Sh, \mathcal{F}_\tau) = \bigoplus_\pi m(\pi) \pi^\infty \otimes H^*(\mathfrak{g}_\infty, K_\infty, \pi_\infty \otimes \tau) \in Mod(G(\mathbb{A}^\infty))$$

of Borel and Casselman [BC], where $H_{(2)}$ denotes L^2-cohomology, π runs through the irreducible admissible representations of $G(\mathbb{A})$, $m(\pi)$ is the multiplicity of π in the discrete spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ and $H^*(\mathfrak{g}_\infty, K_\infty, -) = \bigoplus_i H^i(\mathfrak{g}_\infty, K_\infty, -)$ is total relative Lie algebra cohomology. To π such that $m(\pi) \neq 0$ one should conjecturally be able to attach a discrete A-parameter ψ and we may, assuming Conjecture 8.1 of [Art], rewrite the above as

$$(4.1) \qquad H^*_{(2)}(Sh, \mathcal{F}_\tau) = \bigoplus_\psi \bigoplus_{\pi \in \Pi_\psi} m_\psi(\pi) \pi^\infty \otimes H^*(\mathfrak{g}_\infty, K_\infty, \pi_\infty \otimes \tau)$$

where ψ runs through the set $\Psi_0(G)$ of discrete A-parameters ([Art] p. 56), Π_ψ is the A-packet attached to ψ and $m_\psi(\pi)$ is a certain multiplicity. Put

$$V_\psi = \bigoplus_{\pi_\infty \in \Pi_{\psi_\infty}} H^*(\mathfrak{g}_\infty, K_\infty, \pi_\infty \otimes \tau).$$

Associated with ψ is a group S_ψ defined on p. 52 of [Art] (as well as local versions S_{ψ_v} for any place v of $\mathbb{Q}$). V_ψ carries a representation of S_{ψ_∞}, a Hodge structure and a Lefschetz decomposition ([Art] pp. 59–61). Arthur then defines, for each π, a vector space U_π that only depends on π^∞ and carries an action of S_ψ, an action of S_ψ on $U_\pi \otimes V_\psi$ and rewrites (4.1) as

$$(4.2) \qquad H^*_{(2)}(Sh, \mathcal{F}_\tau) = \bigoplus_\psi \bigoplus_{\pi^\infty \in \Pi_{\psi_\infty}} \pi^\infty \otimes (U_\pi \otimes V_\psi)_{\epsilon_\psi}$$

where ϵ_ψ is a certain sign character of S_ψ and $(-)_{\epsilon_\psi}$ denotes the subspace where S_ψ acts as ϵ_ψ. This uses equation (8.5) of [Art] for the multiplicity $m_\psi(\pi)$. By various comparison theorems together with Zucker's conjecture equation (4.2) holds for ℓ-adic intersection cohomology of the minimal compactification

$$H^*_{et,\ell}(\overline{Sh}, \mathcal{F}_\tau) = \bigoplus_\psi \bigoplus_{\pi^\infty \in \Pi_{\psi_\infty}} \pi^\infty \otimes (U_\pi \otimes V_\psi)_{\epsilon_\psi}$$

as $G(\mathbb{A}^\infty)$-modules, after applying ι (as ι is fixed, we will omit it from the notation). $(U_\pi \otimes V_\psi)_{\epsilon_\psi}$ carries an action of S_ψ (by ϵ_ψ), and for L^2-cohomology it carries a Hodge structure and a Lefschetz decomposition, whereas for ℓ-adic intersection cohomology it carries a representation of $Gal(\overline{\mathbb{Q}}/E)$ and a Lefschetz decomposition. We wish to describe this extra structure.

To that end, let $\psi : \mathcal{L}_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ be the A-parameter attached to π. We write ψ_E for the restriction to $\mathcal{L}_E \times \mathrm{SL}_2(\mathbb{C})$ and let ϕ_E denote the associated L-parameter $\mathcal{L}_E \to {}^L G$, defined by

$$\phi_E(w) = \psi_E \left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right).$$

Kottwitz composes ϕ_E with r_L to obtain an L-parameter ϕ_{E,r_L} and conjectures that there should exist a motive M_ψ over E whose ℓ-adic étale realization $H_{et,\ell}(M_\psi)$ satisfies

$$WD_\iota(H_{et,\ell}(M_\psi)|_{Gal(\overline{E}_v/E_v)}) = (\phi_{E,r_L}|-|^{-d/2})|_{\mathcal{L}_{E_v}}$$

for each place $v \nmid \ell\infty$ of E (where WD_ι means take the associated complex Weil-Deligne representation, using ι). Moreover, its de Rham realization $H_{dR,v}(M_\psi)$ with respect to a place $v \mid \infty$ of E should satisfy

$$H_{dR,v}(M_\psi) = (\phi_{E,r_L}|-|^{-d/2})|_{\mathcal{L}_{\overline{E}_v}}.$$

Here $H_{dR,v}(M_\psi)$, as a Hodge structure, carries an action of $\mathbb{C}^\times \cong W_{\overline{E}_v}$. In both cases the Lefschetz decomposition on $H_{et}(M_\psi)$ and $H_{dR}(M_\psi)$ should be given by $r_L \circ \psi_E|_{\mathrm{SL}_2(\mathbb{C})}$. Kottwitz verifies ([Kot] p. 200) that these actions of $r_L \circ \psi_E$ (hence of ϕ_{E,r_L} and its twist) commute with the action of S_ψ on V_ψ, and hence that we get induced structures on $(U_\pi \otimes V_\psi)_{\epsilon_\psi}$. For the finite places we then have

Conjecture 4 (Kottwitz, [Kot] p. 201). *Assume that the derived group G^{der} of G is simply connected and that the maximal $\mathbb{R}$-split torus of the center $Z(G)$ of G is $\mathbb{Q}$-split. Then the $Gal(\overline{\mathbb{Q}}/E)$-representation*

$$W_{et,\ell}^*(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi^\infty, H_{et,\ell}^*(\overline{Sh}, \mathcal{F}_\tau))$$

is isomorphic to

$$\bigoplus_{\psi \,:\, \pi^\infty \in \Pi_{\psi^\infty}} (U_\pi \otimes H_{et}(M_\psi))_{\epsilon_\psi}.$$

Remark.

(1) Kottwitz does much more than stating the conjecture. Assuming Arthur's conjectures on the discrete spectrum (i.e., Conjectures 8.1 and 8.5 of [Art]), some conjectures on transfer and a formula for the number of points modulo primes of good reduction for Sh, Kottwitz computes the contribution of the Euler characteristic $\sum(-1)^i H_{c,et,\ell}^i(Sh, \mathcal{F}_\tau)$ of compact support cohomology to the Euler characteristic $\sum(-1)^i H_{et,\ell}^i(\overline{Sh}, \mathcal{F}_\tau)$. Since Kottwitz has to work with $\sum(-1)^i H_{et,\ell}^i(\overline{Sh}, \mathcal{F}_\tau)$ (the fundamental technique used being a comparison between geometric sides of trace formulas), his conjecture looks slightly different to what we have written above.

(2) Of course, Kottwitz's conjecture has been (partially) proven in many cases of PEL type A or C, and for groups related to inner forms of GL_2 over a totally real field.

At infinity we have the following:

Theorem 5 (Arthur, [Art] Proposition 9.1). *As representations of $\mathbb{C}^\times$, $\mathrm{SL}_2(\mathbb{C})$ (i.e., the Hodge structure resp. the Lefschetz decomposition) and S_{ψ_∞} we have*

$$V_\psi \cong H_{dR,v}(M_\psi)$$

for v the place associated to the canonical embedding $E \hookrightarrow \mathbb{C}$.

Remark.
(1) The way we have described it, it is perhaps not obvious that the above statement makes sense without assuming some conjectures, but it may be formulated entirely without reference to the global A-parameter ψ and the motive M_ψ, using only the local A-parameter ψ_∞. The definition of V_ψ then only depends on Π_{ψ_∞}, which exists non-conjecturally [VZ], and so does $H_{dR,v}(M_\psi)$ if we define it by the desiderata outlined above, rewritten in a local form (i.e., that the Hodge structure is $(\phi_{\infty,r_L}|-|^{-d/2})|_{\mathcal{L}_{\overline{E}_v}}$ where ϕ_∞ is L-parameter associated with ψ_∞; the Lefschetz decomposition is $r_L \circ \psi_\infty|_{\mathrm{SL}_2(\mathbb{C})}$, and the action of S_{ψ_∞} is via r_L).
(2) This identifies V_μ and V_ψ as vector spaces and hence gives a formula for the dimension of V_ψ.

The next conjecture is then a consequence of Conjecture 8.1 of [Art], as described above:

Conjecture 6 (Arthur, [Art]). *As representations of $\mathbb{C}^\times$ and $\mathrm{SL}_2(\mathbb{C})$ (i.e., the Hodge structure and the Lefschetz decomposition)*

$$W_{dR}^*(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi^\infty, H_{(2)}^*(Sh, \mathcal{F}_\tau))$$

is isomorphic to

$$\bigoplus_{\psi\,:\,\pi^\infty \in \Pi_{\psi^\infty}} (U_\pi \otimes H_{dR,v}(M_\psi))_{\epsilon_\psi}$$

for v the place associated to the canonical embedding $E \hookrightarrow \mathbb{C}$.

We now wish to recast this story using the ideas of Buzzard and Gee and the representation r_C. Since our π are C-algebraic, Buzzard and Gee conjecture ([BG] Conjecture 5.3.4) that there exists a Galois representation

$$\rho_{\pi,\ell} = \rho_{\pi,\ell,\iota} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^C G(\overline{\mathbb{Q}}_\ell)$$

satisfying a list of desiderata, the most important for us being that, for finite primes $p \neq \ell$ such that π is unramified, $\rho_{\pi,\ell}|_{W_{\mathbb{Q}_p}}$ is $\widehat{\widetilde{G}}(\overline{\mathbb{Q}}_\ell)$-conjugate to

$$w \mapsto \iota((\phi_{\pi_p}(w), |w|^{1/2})).$$

In other words, $\rho_{\pi,\ell}$ is a Galois representation associated with the L-algebraic automorphic representation $\widetilde{\pi}$ according to Conjecture 3.2.1 of [BG] (since $w \mapsto (\phi_{\pi_p}(w), |w|^{1/2})$ is the Satake parameter of $\widetilde{\pi}_p$). We remark that the set of possible $\rho_{\pi,\ell}$ depends only on π^∞ and the L-packet of π_∞ (see part 2) of the remark below for some

remarks on the question of uniqueness). Let $\rho_{E,\pi,\ell}$ denote the restriction of $\rho_{\pi,\ell}$ to $Gal(\overline{\mathbb{Q}}/E)$. With notation as above, Corollary 3 gives us

Proposition 7. *Assume that M_ψ exists as above. Then*

(1) $H_{et,\ell}(M_\psi) \cong r_C \circ \rho_{E,\pi,\ell}$ *as representations of $Gal(\overline{\mathbb{Q}}/E)$.*

(2) $H_{dR,v}(M_\psi) \cong r_C \circ \widetilde{\phi}_{\mathbb{C}}$ *as representations of $\mathbb{C}^\times$.*

Remark. (1) Assume Langlands functoriality (say in the weak form of Conjecture 6.1.1 of [BG]). Since $\widetilde{\pi}$ is L-algebraic the transfer Π_E of $\widetilde{\pi}$ to $\mathrm{GL}_{N/E}$ (using r_C and base change $E/\mathbb{Q}$; here $N = \dim V_\mu$) is L-algebraic ([BG] Lemma 6.1.2) with L-parameter $r_C \circ \widetilde{\phi}_E$ and can be taken to be isobaric. Thus one sees from Proposition 7 that M_ψ is the motive conjecturally associated with $\Pi_E \otimes |\cdot|^{(1-N)/2}$ by Clozel ([Clo] Conjecture 4.5).

(2) $\rho_{\pi,\ell}$ may not be uniquely determined by π (see [BG] Remark 3.2.4 for a discussion and some examples), which forces one to speak of all possible $\rho_{\pi,\ell}$ satisfying the desiderata. Of course, the composition of any such $\rho_{\pi,\ell}$ with a representation of $^C G$ does not depend on the choice of $\rho_{\pi,\ell}$ and is determined by π at its unramified finite places.

We can now state the minor variations of Conjectures 4 and 6 and Theorem 5 given by Proposition 7:

Conjecture 8. *The $Gal(\overline{\mathbb{Q}}/E)$-representation*

$$W_{et,\ell}^*(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi^\infty, H_{et,\ell}^*(\overline{Sh}, \mathcal{F}_\tau))$$

is isomorphic to

$$\bigoplus_{\psi\,:\,\pi^\infty \in \Pi_{\psi^\infty}} (U_\pi \otimes (r_C \circ \rho_{E,\pi,\ell}))_{\epsilon_\psi}.$$

Theorem 9. *We have*

$$V_\psi \cong r_C \circ \widetilde{\phi}_{\mathbb{C}}$$

as representations of $\mathbb{C}^\times$.

Conjecture 10. *As representations of $\mathbb{C}^\times$ (i.e., Hodge structures)*

$$W_{dR}^*(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi^\infty, H_{(2)}^*(Sh, \mathcal{F}_\tau))$$

is isomorphic to

$$\bigoplus_{\psi\,:\,\pi^\infty \in \Pi_{\psi^\infty}} (U_\pi \otimes (r_C \circ \widetilde{\phi}_{\mathbb{C}}))_{\epsilon_\psi}.$$

Thus, we see that the conjectures of Buzzard and Gee are consistent in a natural way with the known and conjectural properties of the cohomology of Shimura varieties, and offers an explanation for the twist by $-d/2$.

Remark. We should remark that another possible way of explaining this Tate twist is via the theory of weights for perverse sheaves. Namely, it is natural to consider $\mathcal{F}_\tau[d]$ (shift defined by $-[d]^i = -^{i+d}$) because it is a (pure) perverse sheaf. The shift $[d]$ lowers the weight of the sheaf and its cohomology by d, which has the same effect as a Tate twist by $d/2$ (undoing the Tate twist by $-d/2$ above) in terms of weights.

Acknowledgments

References

[Art]   J. Arthur, *Unipotent automorphic representations I. Conjectures*, Astérisque **171–172** (1989), 13–71.

[BR]   D. Blasius and J. Rogawski, *Zeta-functions of Shimura varieties*, Motives (Seattle, WA, 1991), Proc. Symp. Pure Mathematics 55, AMS, Providence, RI (1994) 525–571.

[Bor]   A. Borel, *Automorphic L-functions. Automorphic Forms. Representations and L-functions*, Proc. Symp. Pure Mathematics **33**, Part II, AMS, Providence, RI (1979) 27–61.

[BC]   A. Borel and W. Casselman, L^2-*cohomology of locally symmetric manifolds of finite volume*, Duke Math. J. **50** (1983), 625–647.

[BG]   K.M. Buzzard and T. Gee, *The conjectural connections between automorphic representations and Galois representations*, To appear in Proc. of the LMS Durham Symp. 2011. Available at `http://www2.imperial.ac.uk/~tsg/`

[Clo]   L. Clozel, Motifs et formes automorphes: applications du principe de fonctorialité. Automorphic forms, Shimura varieties, and L-functions, Proc. of the Ann Arbour Conf., Perspectives on Mathematics **10**, Academic Press, Boston, MA (1990), 77–159.

[Del]   P. Deligne, *Variétés de Shimura : interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions, Proc. Symp. Pure Mathematics **33**, Part II, AMS, Providence, RI (1979), 247–290.

[Kot]   R. Kottwitz, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties and L-functions, Proceedings of the Ann Arbour Conf., Perspectives on Mathematics **10**, Academic Press, Boston, MA (1990), 161–209.

[Lan]   R.P. Langlands, *Automorphic representations, motives and Shimura varieties*, Ein Märchen. Automorphic Forms. Representations and L-functions. Proc. Symp. Pure Mathematics **33**, Part II, AMS, Providence, RI (1979), 205–246.

[Pin]   R. Pink, *Arithmetical compactification of mixed Shimura varieties*, PhD thesis, Bonner Mathematische Schriften, 209, 1989.

[VZ]   D. Vogan and G. Zuckerman, *Unitary representations with non-zero cohomology*, Compos. Math. **53** (1984), 51–90.

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, ST GILES 24-29, OXFORD OX1 3LB, UK
E-mail address: `johansson@maths.ox.ac.uk`