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A SPLITTING THEOREM ON TORIC MANIFOLDS

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Abstract. Using the Calabi flow, we prove that any extremal Kähler metric ω_E on a product toric variety $X_1 \times X_2$ is a product extremal Kähler metric.

1. Introduction

In [3], the authors considered the following problem:

Problem 1.1. *Let $X_i, i = 1, 2$ be two Kähler manifolds with Kähler classes $[\omega_i]$. Suppose ω_E is an extremal Kähler metric in the Kähler class $[\omega_1 + \omega_2]$. Can we conclude that ω_E is a product metric, i.e., $\omega_E = \omega_{E,1} + \omega_{E,2}$ where $\omega_{E,i}$ is an extremal Kähler metric in $[\omega_i]$.*

In this short paper, we solve the problem in the case of toric manifolds.

Theorem 1.2. *If X_i are toric manifolds, then ω_E is a product metric.*

2. Motivations and setup

Let X be a n-dimensional Kähler manifold with Kähler class $[\omega]$. The set of relative Kähler potentials is

$$\mathcal{H} = \{\varphi \in C^\infty(X) \mid \omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0\}.$$

An extremal Kähler metric ω_φ in the sense of Calabi is defined by the condition that its scalar curvature R_φ is a potential of a Killing vector field of (X, ω_φ). In [5], Calabi proved that any extremal metric is invariant under a maximal compact subgroup of the reduced automoprhism group of X. As any such groups are conjugated, one can fix the isometry group of an extremal Kähler metric. In particular, if we further assume that X is a toric manifold, then without loss of generality we can assume that if an extremal metric ω_E exists then it is invariant under the (real) torus $\mathbb{T}^n$. We thus focus on the space of $\mathbb{T}^n$-invariant relative Kähler potentials:

$$\mathcal{H}_{\mathbb{T}^n} = \{\varphi \in C^\infty(X) \mid \varphi \text{ is invariant under } \mathbb{T}^n,\ \omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0\},$$

where ω is a $\mathbb{T}^n$-invariant Kähler metric on X. By the equivariant Moser lemma, the space of $\mathbb{T}^n$-invariant Kähler metrics ω_φ given by elements in $\mathcal{H}_{\mathbb{T}^n}$ can be alternatively realized as the space $\mathcal{J}^\omega_{\mathbb{T}^n}$ of $\mathbb{T}^n$-invariant ω-compatible complex structures on the toric symplectic manifold (X, ω), see Abreu [2], Guillemin [19, 20] and Donaldson [9]. The latter is identified, via the momentum map, to the corresponding Delzant polytope $P \subset \mathbb{R}^n$ (see Delzant [16]). Recall that P is a compact convex polytope satisfying

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the following conditions:

- For any facet P_i of P, there exists an inward normal vector $\vec{v}_i$ corresponding to P_i.
- For any vertex v of P, there are exactly n facets $P_1, \dots, P_n$ meeting at v and the inward normal vectors $\vec{v}_1, \dots, \vec{v}_n$ form a basis of $\mathbb{Z}^n$.

Suppose that P has d-facets. For every facet P_i, we choose c_i such that $l_i(x) = \langle x, \vec{v}_i \rangle + c_i$ vanishes on P_i.

Definition 2.1. A smooth strictly convex function u on the interior of P is called a symplectic potential if

- u extends as a continuous function over ∂P and its restriction to the interior of each face of P is smooth and strictly convex;
- $$u(x) = \sum_{i=1}^{d} \frac{1}{2} l_i(x) \ln l_i(x) + f(x),$$
  where $f(x)$ is a smooth function up to the boundary of P.

The main point of this definition is that, for appropriate choice of angular coordinates $(t_1, \dots, t_n)$, the almost complex structure

$$J_u = \begin{pmatrix} 0 & \vdots & -(D^2u)^{-1} \\ \cdots & \cdots & \cdots \\ (D^2u) & \vdots & 0 \end{pmatrix}$$

is an element of $\mathcal{J}^{\omega}_{\mathbb{T}^n}$ and the elements of $\mathcal{H}_{\mathbb{T}^n}$ are in one to one correspondence with symplectic potentials u as above, see [2, 9, 19, 20].

Furthermore, Abreu [1] wrote down the expression of the scalar curvature R_u of the Kähler metric $g_u(\cdot,\cdot) = \omega(\cdot, J_u \cdot)$. It follows that a symplectic potential u_E corresponds to an extremal Kähler metric if and only if the corresponding scalar curvature R_E is an affine function. Note that in this case, R_E is a priori determined by the Delzant polytope P, by the property that for any affine function f on P, we have (see [9])

$$\mathcal{L}(f) = 2\int f \, d\sigma - \int_P f R_E \, d\mu = 0,$$

where $d\mu$ is the standard Lebesgue measure on P and $d\sigma$ is the induced boundary Lebesgue measure on ∂P: on every facet P_i, we require that $dl_i \wedge d\sigma$ is $d\mu$ up to a sign.

In the case when (X, L) is a compact polarized manifold, Yau [29], Tian [28] and Donaldson [9] conjectured

Conjecture 2.2. *(X, L) admits constant scalar curvature Kähler (cscK) metrics in $c_1(L)$ if and only if it is K-stable.*

It is known by Donaldson [13], Stoppa [26], Mabuchi [23,24], Stoppa and Székelyhidi [27], Chen and Tian [8] that if (X, L) admits cscK metrics in $c_1(L)$, then (X, L) must be K-stable. In the product case $X = X_1 \times X_2$, $L = L_1 \otimes L_2$, one easily infers that each (X_i, L_i) must be K-stable. Thus, Problem (1.1) would follow from Conjecture (2.2) by using the uniqueness of cscK metrics [8, 14].

In [9], Donaldson considers the toric case and finds that the K-stability is related to the following condition:

Definition 2.3. A rational Delzant polytope is (relative) K-stable if for any convex continuous rational piecewise linear function f one has $\mathcal{L}(f) \geq 0$. And the equality holds if and only if f is an affine function.

He thus conjectures [9]:

Conjecture 2.4 (Donaldson). *A compact toric Kähler manifold admits a compatible extremal metric if and only if $\mathcal{L}(f) \geq 0$ for any convex continuous piecewise linear function f with equality if and only if f is an affine function.*

Once again, it is straightforward to see that if a product of two Delzant polytopes $P = P_1 \times P_2$ is K-stable, such is then each factor P_i. However, as far as Conjecture (2.4) stays open, we must find an alternative argument to establish our Theorem (1.2). To this end, we will use the Calabi flow [4], which was initially introduces as a flow on the space $\mathcal{H}$ defined by

$$\frac{\partial \varphi}{\partial t} = R_\varphi - \underline{R},$$

where R_φ is the scalar curvature of the Kähler metric ω_φ and $\underline{R}$ is a topological constant on X defined by

$$\underline{R} = \frac{2n\pi c_1(X) \wedge [\omega]^{n-1}}{[\omega]^n}.$$

In the toric case, this flow can be rewritten in terms of symplectic potentials as ([9])

$$\frac{\partial u}{\partial t} = \underline{R} - R_u.$$

We shall rather consider the modified version [21]

$$\frac{\partial u}{\partial t} = R_E - R_u.$$

Note that by Chen and He [7], the Calabi flow exists for a short time starting from any $C^{3,\alpha}$ relative Kähler potential. Thus for a smooth symplectic potential u, the Calabi flow starting from u also exists for a short time.

Guan [18] has shown that in the toric setting, for any two symplectic potentials u_1 and u_2, the geodesic in the sense of Mabuchi [22], Semmes [25] and Donaldson [15] connecting them is given by $(1-t)u_1 + tu_2, t \in [0,1]$. The length of this geodesic is

$$\sqrt{\int_P (u_1 - u_2)^2 \; d\mu}$$

Suppose that $u_1(t), u_2(t), t \in [0,1]$ are two modified Calabi flows, we want to show that the geodesic distance between $u_1(t)$ and $u_2(t)$ decreases as t increases. This is essentially known by the work of Calabi and Chen [6]. In fact, we have the following lemma:

Lemma 2.5.

$$\frac{\partial}{\partial t} \int_P (u_1(t) - u_2(t))^2 \, d\mu \leq 0.$$

Proof.
$$\frac{\partial}{\partial t}\int_P (u_1(t)-u_2(t))^2\,d\mu = 2\int_P (u_1(t)-u_2(t))(R_{u_2(t)}-R_{u_1(t)})\,d\mu$$
$$= 2\int_P (u_1(t)_{ij}-u_2(t)_{ij})(u_1(t)^{ij}-u_2(t)^{ij})\,d\mu \quad (*).$$
In the last step, we have used integration by parts as in Lemma 3.3.5 of [9]. For any $x \in P$, let $A = (D^2u_1(t))(x), B = (D^2u_2(t))(x)$, then
$$(u_1(t)_{ij}-u_2(t)_{ij})(u_1(t)^{ij}-u_2(t)^{ij})(x) = \mathrm{Trace}((A-B)(A^{-1}-B^{-1})).$$
Note that A, B are positive-definite matrices, thus there exists an orthonormal matrix O_1 such that $O_1AO_1^t$ is a diagonal matrix $\mathrm{diag}(\lambda_1,\dots,\lambda_n)$. Let $O_2 = \mathrm{diag}(\sqrt{\lambda_1},\dots, \sqrt{\lambda_n})$. Then $O_2^{-1}O_1AO_1^tO_2^{-1}$ is the identity matrix and $\tilde{B} = O_2^{-1}O_1BO_1^tO_2^{-1}$ is still a positive-definite matrix. Note that
$$\mathrm{Trace}((A-B)(A^{-1}-B^{-1})) = \mathrm{Trace}((I_n-\tilde{B})(I_n-\tilde{B}^{-1})).$$
We can again choose an orthonormal matrix O_3 such that $O_3\tilde{B}O_3^t$ is a diagonal matrix $\mathrm{diag}(\bar{\lambda}_1,\dots,\bar{\lambda}_n)$. Then
$$\mathrm{Trace}((A-B)(A^{-1}-B^{-1})) = \sum_{i=1}^n (1-\bar{\lambda}_i)(1-\bar{\lambda}_i^{-1}) \le 0.$$
Thus, $(*) \le 0$. □

3. Proof of Theorem (1.2)

Let (X_i, ω_i) be toric symplectic manifolds with Delzant polytopes P_i. The product manifold $X = X_1 \times X_2$ with symplectic form $\omega = \omega_1 + \omega_2$ is symplectic toric with Delzant polytope $P = P_1 \times P_2$. In the symplectic side, we have symplectic potentials u_i satisfying Guillemin boundary conditions of P_i. We let x be the variable of P_1 and y be variable of P_2. Our assumption shows that there exists a symplectic potential u on P and
$$u(x,y) = u_1(x) + u_2(y) + f(x,y), \quad f(x,y) \in C^\infty(\bar{P})$$
such that the scalar curvature of $u(x,y)$ is an affine function. Our goal is to show that $f(x,y)$ is separable. Let
$$f_1(x) = \frac{1}{vol(P_2)}\int_{P_2} f(x,y)\,dy, \quad f_2(y) = \frac{1}{vol(P_1)}\int_{P_1} f(x,y)\,dx.$$
Then we have

Proposition 3.1. $v(x,y) = u_1(x)+u_2(y)+f_1(x)+f_2(y)$ *is a symplectic potential of* P *satisfying the Guillemin boundary conditions.*

Proof. It is easy to see that $f_1(x) + f_2(y)$ is a smooth function on $\bar{P}$. Thus, we only need to show that (D^2v) is a positive matrix in order to prove that v is a symplectic potential. To show $(D^2v) > 0$ is equivalent to show that $(D^2(u_1(x) + f_1(x))) > 0$ and $(D^2(u_2(y)+f_2(y))) > 0$. However, $(D^2(u_1(x)+f_1(x))) > 0$ and $(D^2(u_2(y)+f_2(y))) > 0$ just follow from the fact that $(D^2u) > 0$. □

Let $\mathcal{S}$ be the set of all symplectic potentials. We define a subset of $\mathcal{S}$.

Definition 3.2.

$$\mathcal{M} = \Big\{ \underline{u}(x,y) \in \mathcal{S} \mid \underline{u}(x,y) = u_1(x) + u_2(y) + g_1(x) + g_2(y) \, s.t.$$
$$g_1(x) \in C^\infty(\bar{P}_1), \int_{P_1} f_1(x)\, dx = \int_{P_1} g_1(x)\, dx,$$
$$g_2(y) \in C^\infty(\bar{P}_2), \int_{P_2} f_2(y)\, dy = \int_{P_2} g_2(y)\, dy \Big\}.$$

Then we have

Proposition 3.3. *For any $\underline{u} \in \mathcal{M}$, we have*

$$\int_P (u(x,y) - v(x,y))^2 \, dxdy \leq \int_P (u(x,y) - \underline{u}(x,y))^2 \, dxdy. \tag{3.1}$$

And the equality holds if and only if $v = \underline{u}$.

Proof. (3.1) is equivalent to show that

$$\int_P (f(x,y) - f_1(x) - f_2(y))^2 \, dxdy \leq \int_P (f(x,y) - g_1(x) - g_2(y))^2 \, dxdy.$$

Expressing it out, we have

$$\int_P -2f(x,y)(f_1(x) + f_2(y)) + f_1^2(x) + f_2^2(y) \, dxdy$$
$$\leq \int_P -2f(x,y)(g_1(x) + g_2(y)) + g_1^2(x) + g_2^2(y) \, dxdy$$

which is equivalent to

$$0 \leq \int_P (f_1(x) - g_1(x))^2 + (f_2(y) - g_2(y))^2 \, dxdy.$$

The equality holds if and only if $f_1(x) = g_1(x)$ and $f_2(y) = g_2(y)$. □

Proof of Theorem (1.2). We use the Calabi flow to show that v is an extremal symplectic potential. Let $u(t)$ be a sequence of symplectic potentials satisfying the modified Calabi flow equation on P and $u(0) = v$. By Lemma 2.5, we have

$$\frac{d}{dt} \int_P (u(t) - u)^2 \, dxdy \leq 0.$$

Since $u(t) \in \mathcal{M}$, we obtain $u(t) = v$. This shows that v is a separable extremal symplectic potential on P. By the uniqueness of the extremal symplectic potential modulo affine functions [18], it follows that u is also separable. So, f is a separable function. □

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