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WEAK TRACE MEASURES ON HARDY–SOBOLEV SPACES

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Abstract. In this paper, we obtain a characterization of the weak trace measures for the Hardy–Sobolev spaces H^p_s, that is, the positive Borel measures on $\mathbf{S}^n$ such that

$$\sup_{\lambda>0} \lambda^p \mu(\{\zeta \in \mathbf{S}^n;\, \mathcal{M}_{\mathrm{rad}}[f](\zeta) > \lambda\}) \leq C\|f\|^p_{H^p_s},$$

when $1 < p < +\infty$. Also, some partial results on weak q-trace measures for the non diagonal case are obtained.

1. Introduction

Let H^p_s be the Hardy–Sobolev space on $\mathbf{B}^n$, the unit ball of $\mathbf{C}^n$. If f is a function on $\mathbf{B}^n$ and $\mathbf{S}^n$ is the unit sphere, let $\zeta \in \mathbf{S}^n$ and let $\{\mathcal{M}_{\mathrm{rad}}[f](\zeta) = \sup_{0<\rho<1} |f(\rho\zeta)|$ be the radial maximal function. If μ is a non-negative Borel measure on $\mathbf{S}^n$, $0 < p, q < +\infty$ and $0 < s < n$, we say that μ is q- trace measure for H^p_s if there exists $C > 0$ such that for any $f \in H^p_s$

$$\|\mathcal{M}_{\mathrm{rad}}[f]\|_{L^q(d\mu)} \leq C\|f\|_{H^p_s}. \tag{1.1}$$

Analogously, μ is a weak q-trace measure for H^p_s, if there exists $C > 0$, such that for any $f \in H^p_s$

$$\|\mathcal{M}_{\mathrm{rad}}[f]\|_{L^{q,\infty}(d\mu)} := \sup_{\lambda>0} \lambda^q \mu(\{\zeta \in \mathbf{S}^n;\, \mathcal{M}_{\mathrm{rad}}[f](\zeta) > \lambda\}) \leq C\|f\|^q_{H^p_s}. \tag{1.2}$$

Clearly any q-trace measure for H^p_s is also a weak q-trace measure for H^p_s. When $p = q$, we will simply say trace and weak trace measures, respectively.

Observe that for $n - sp < 0$, the space H^p_s consists of continuous functions up to the boundary, and all finite positive measures on $\mathbf{S}^n$ are trace measures. From now on we will assume that $n - sp > 0$. With the appropriate changes, some of the results are also valid for the extreme case $n = sp$.

The problem of the characterization of the trace measures for the Hardy–Sobolev spaces H^p_s on the unit ball, when $0 < p < +\infty$, is not completely solved and only include some particular range of s and p. When $p \leq 1$, the trace measures for H^p_s have been characterized in [**Ah**]. If $p > 1$ and $n - sp < 1$, the trace measures for H^p_s coincide with the trace measures for the non isotropic potential space $K_s[L^p]$ defined below (see [**CohVe**]), which can be characterized in terms of non-isotropic Riesz capacities. Interesting results for a related problem on Carleson measures for H^2_s and $n - 2s \geq 1$, have been obtained in [**VoWi**] (see also [**Tch**] for the case $n - 2s = 1$). However, the problem is still open for the remaining cases.

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The object of this paper is to give a characterization of the weak trace measures for H^p_s for any $p < +\infty$ and $0 < sp < n$. Also, we will give some results on weak q-trace measures for the non diagonal case $q \neq p$.

Before we state our main results, we introduce some notations. We denote by $d\sigma$ the normalized Lebesgue measure on $\mathbf{S}^n$. The notation $\zeta\overline{\eta}$ will be used to indicate the complex inner product in $\mathbf{C}^n$ given by $\zeta\overline{\eta} = \sum_{i=1}^n \zeta_i\overline{\eta_i}$, if $\zeta = (\zeta_1, \ldots, \zeta_n)$, $\eta = (\eta_1, \ldots, \eta_n)$. If $\zeta \in \mathbf{S}^n$, we denote by $B(\zeta, R)$ the non-isotropic ball in $\mathbf{S}^n$, given by $B(\zeta, R) = \{\eta \in \mathbf{S}^n,\ |1 - \zeta\overline{\eta}| < R\}$.

The Hardy–Sobolev space H^p_s, $0 \leq s$, $0 < p < +\infty$, consists of those functions f holomorphic in $\mathbf{B}^n$ such that if $f(z) = \sum_k f_k(z)$ is its homogeneous polynomial expansion and the fractional radial derivative is defined by

$$\mathcal{R}^s f(z) := (I + R)^s f(z) = \sum_k (1 + k)^s f_k(z),$$

we have that $||f||^p_{H^p_s} := \int_{\mathbf{S}^n} \mathcal{M}_{\mathrm{rad}}[\mathcal{R}^s f](\zeta)^p d\sigma(\zeta) < +\infty$.

We can reformulate the problem stated in (1.2) observing that if $0 < s < n$, and $p > 1$, for any function f in H^p_s there exists $g \in L^p(\mathbf{S}^n)$ such that

$$f(z) = \mathcal{C}_s[g](z) := \int_{\mathbf{S}^n} \frac{g(\zeta)}{(1 - z\bar{\zeta})^{n-s}} d\sigma(\zeta) \tag{1.3}$$

and $||f||_{H^p_s} \approx \inf_g ||g||_{L^p}$ (see for instance [**CaOr**]).

Consequently, the estimate (1.2) can be rewritten as follows: there exists $C > 0$ such that for any $f \in L^p(\mathbf{S}^n)$,

$$\sup_{\lambda > 0} \lambda^q \mu(\{\zeta \in \mathbf{S}^n;\ \mathcal{M}_{\mathrm{rad}}[\mathcal{C}_s[f]](\zeta) > \lambda\} \leq C\|f\|^q_{L^p}. \tag{1.4}$$

The relationship between exceptional sets for H^p_s, i.e., sets $E \subset \mathbf{S}^n$ such that there exists $f \in H^p_s$ with $M_{\mathrm{rad}} f(\zeta) = \infty$ for $\zeta \in E$ and weak trace measures is one of the motivations to study the weak trace measures. We observe, for instance, that if μ is a weak trace measure for H^p_s, then its support cannot be an exceptional set for H^p_s.

The main result in this paper is the following:

Theorem 1.1. *Let $1 < p < +\infty$, $0 < s < n$ and μ a finite positive Borel measure on $\mathbf{S}^n$. We then have that the following assertions are equivalent:*

(i) *There exists $C > 0$ such that for any $f \in H^p_s$,*

$$\mu\left(\{\zeta \in \mathbf{S}^n;\ \mathcal{M}_{\mathrm{rad}}[f](\zeta) > \lambda\}\right) \leq C\frac{\|f\|^p_{H^p_s}}{\lambda^p}.$$

(ii) *There exists $C > 0$ such that for any $f \in L^p(\mathbf{S}^n)$,*

$$\mu\left(\{\zeta \in \mathbf{S}^n;\ \mathcal{M}_{\mathrm{rad}}[\mathcal{C}_s[f]](\zeta) > \lambda\}\right) \leq C\frac{\|f\|^p_{L^p}}{\lambda^p}.$$

(iii) *The measure μ satisfies the following two conditions:*

(a) *There exists $C > 0$ such that for any $\zeta \in \mathbf{S}^n$ and $r > 0$,*
$$\mu(B(\zeta, r)) \leq C r^{n-sp}.$$

(b) *There exists $C > 0$ such that for any $f \in L^p$ and any non-isotropic ball $B \subset \mathbf{S}^n$,*
$$\int_B \sup_{\rho<1} \left| \int_B \frac{f(\eta)}{(1-\rho\zeta\overline{\eta})^{n-s}} d\sigma(\eta) \right| d\mu(\zeta) \leq C \left(\int_B |f|^p d\sigma \right)^{\frac{1}{p}} \mu(B)^{\frac{1}{p'}}.$$

In order to give an interpretation of condition (iiib), we recall that $g \in L^{p,\infty}(\mu)$ if and only if,
$$\|g\|_{L^{p,\infty}(\mu)} = \sup_{\lambda>0} \lambda^p \mu(\{\zeta \in \mathbf{S}^n;\ |g(\zeta)| > \lambda\}) < +\infty.$$
The so-called Kolmogorov's condition (see Lemma 2.8, Chapter V in [**GaRu**]) gives an equivalent definition. Namely, $g \in L^{p,\infty}(\mu)$ if and only if, there exists $r < p$ and $C > 0$, such that for any subset $E \subset \mathbf{S}^n$, such that $\mu(E) > 0$,
$$\mu(E)^{\frac{r-p}{rp}} \left\{ \int_E |g(\zeta)|^r d\mu(\zeta) \right\}^{\frac{1}{r}} \leq C. \tag{1.5}$$

Consequently, condition (iiib) of the weak trace measure given in Theorem 1.1 can be interpreted as a Kolmogorov-type estimate where the family of sets E considered are restricted to non-isotropic balls, $g = \mathcal{M}_{\mathrm{rad}}[\mathcal{C}_s[f\chi_B]]$ and $r = 1 < p$.

Finally, observe that condition (iiib) is a localized condition, analogous to the one obtained by [**LaSaU**] for some maximal integral singular operators on $\mathbf{R}^n$.

Before we state our next result, we introduce some more definitions and notations.

We denote by K_s the non-isotropic potential operator defined on $L^p(\mathbf{S}^n)$, by
$$K_s[f](\zeta) = \int_{\mathbf{S}^n} \frac{f(\eta)}{|1-\zeta\overline{\eta}|^{n-s}} d\sigma(\eta), \quad \zeta \in \overline{\mathbf{B}}^n.$$

A measure μ on $\mathbf{S}^n$ is a weak trace measure for the non-isotropic potential space $K_s[L^p]$, if there exists $C > 0$ such that for any $f \geq 0$,
$$\sup_{\lambda>0} \lambda^p \mu((\{\zeta \in \mathbf{S}^n;\ \mathcal{M}_{\mathrm{rad}}[K_s[f]](\zeta) > \lambda\}) \leq C\|f\|_{L^p}^p.$$
A measure μ on $\mathbf{S}^n$ is a trace measure for the non-isotropic potential space $K_s[L^p]$, if there exists $C > 0$ such that for any $f \geq 0$,
$$\int_{\mathbf{S}^n} \mathcal{M}_{\mathrm{rad}}[K_s[f]]^p(\zeta)\, d\mu(\zeta) \leq C\|f\|_{L^p}^p.$$

Since for $\zeta \in \mathbf{S}^n$, $\mathcal{M}_{\mathrm{rad}}[\mathcal{C}_s[f]](\zeta) \leq \mathcal{M}_{\mathrm{rad}}[K_s[|f|]](\zeta) = K_s[|f|](\zeta)$, any trace measure (resp. weak trace measure) for $K_s[L^p]$ is also a trace measure (resp. weak trace measure) for the space H_s^p. The study of the traces on $K_s[L^p]$, which correspond to an operator with positive kernel is simpler than the same problem on H_s^p.

The trace measures and, more generally, the q-traces for the Riesz potential spaces in $\mathbf{R}^n$, $I_s[L^p]$, have been studied by several authors. When $1 < p < q$, it was shown by Adams (see [**AdHe**], Theorem 7.2.2) that μ is a q-trace for $I_s[L^p]$, if
$$\sup_{x\in\mathbf{R}^n,\, r>0} \frac{\mu(B(x,r))}{r^{(n-sp)q/p}} < +\infty,$$

where $B(x,r)$ is the Euclidean ball of radius $r>0$ centered at $x \in \mathbf{R}^n$. In the case $p=q$, a capacitary characterization was obtained by Adams et al. (see [**Ad1**] and [**AdHe**], Theorem 7.2.1): μ is a trace for $I_s[L^p]$ if and only if $\sup_G \frac{\mu(G)}{C^R_{s,p}(G)} < +\infty$, where the supremum is taken over all open sets and $C^R_{s,p}$ is the Riesz capacity. Other non capacitary characterizations have been given in [**KeSa**]. In the upper triangle case $1<q<p$, a capacitary characterization has been obtained by [**MaNe**]. All these results have a non-isotropic version for the non-isotropic potential space $K_s[L^p]$. In [**CaOrVe**] it has been obtained a characterization of the q trace measures, $1<q<p$, in terms of the non-isotropic Wolff potential.

The following theorem establishes, in particular, that when $n-sp<1$ the trace and weak trace for H^p_s and $K_s[L^p]$ coincide:

Theorem 1.2. *Let $1<p<+\infty$, $0<s<n$ and μ a finite positive Borel measure on $\mathbf{S}^n$. We then have:*

(i) *If $n-sp<1$, the following assertions are equivalent:*
 (a) *The measure μ is a weak trace measure for H^p_s.*
 (b) *The measure μ is a trace measure for H^p_s.*
 (c) *The measure μ is a weak trace measure for $K_s[L^p]$.*
 (d) *The measure μ is a trace measure for $K_s[L^p]$.*
 (e) *There exists $C>0$ such that for any open set $E \subset \mathbf{S}^n$*

$$\mu(E) \le C C_{s,p}(E), \tag{1.6}$$

where $C_{s,p}(E)$ is the (s,p)-non-isotropic Riesz capacity of E.

(ii) *If $n-sp \ge 1$, there exists a positive Borel measure μ on $\mathbf{S}^n$, such that μ is a weak trace measure for H^p_s, but μ is not a weak trace measure for $K_s[L^p]$.*

As we will see in the forthcoming sections, any weak trace measure μ for H^p_s satisfies that $\mu(B(\zeta,r)) \le C r^{n-sp}$. The following theorem shows that if $p>2$ this necessary condition can be strengthened.

Theorem 1.3. *If $p>2$, $n-sp\ge 1$ and μ is a weak trace measure for H^p_s, then there exists $C>0$ such that for any open set $G \subset \mathbf{S}^n$*

$$\mu(G) \le C C_{2s,\frac{p}{2}}(G). \tag{1.7}$$

For the non-diagonal case $p \ne q$, we obtain the following results:

Theorem 1.4. *Let $0<s<n$ and μ a finite positive Borel measure on $\mathbf{S}^n$ and suppose either $p \le 1$ and $p \le q$ or $1<p<q$. We then have that the following assertions are equivalent:*

(i) *The measure μ is a weak q-trace measure for H^p_s.*
(ii) *The measure μ is a q-trace measure for H^p_s.*
(iii) *There exists $C>0$ such that for any $\zeta \in \mathbf{S}^n$, $r>0$,*

$$\mu(B(\zeta,r) \le C r^{(n-sp)\frac{q}{p}}.$$

For the more difficult case when $q<p$, we give some partial results. If $q<p\le 1$, it was proved in [**CaOr1**] that μ is a q-trace measure for H^p_s if and only if $\sup_{r_B<\delta} \frac{\mu(B)}{r_B^{n-sp}} \chi_B(\zeta) \in L^{\frac{q}{p-q}}(d\mu)$. Here, for the weak q-traces we have the following theorem:

Theorem 1.5. *Let $0 < s < n$ and μ a finite positive Borel measure on $\mathbf{S}^n$ and suppose that $q < p \leq 1$. Assume that in addition $0 \leq n - sp < p$. We then have that the following assertions are equivalent:*

(i) *μ is a weak q-trace measure for H^p_s.*

(ii) *For any fixed $\delta > 0$, if r_B denotes the radius of the non-isotropic ball B, then,*

$$\sup_{r_B<\delta}\left(\frac{\mu(B)}{r_B^{n-sp}}\right)^{\frac{1}{p}}\chi_B(\zeta)\in L^{\frac{pq}{p-q},\infty}(d\mu). \tag{1.8}$$

When $p > 1$ we have

Theorem 1.6. *Let $0 < q < p$, $1 < p$, $0 \leq n - sp < 1$, μ a finite positive Borel measure on $\mathbf{S}^n$. If $q \leq 1$, in addition we assume that $p > 2 - \frac{s}{n}$. Then the following conditions are equivalent:*

(i) *There exists $C > 0$ such that*

$$\|M_{\mathrm{rad}}[f]\|_{L^{q,\infty}(d\mu)} \leq C\|f\|_{H^p_s}.$$

(ii) $\mathcal{W}_{s,p}[\mu] \in L^{\frac{q(p-1)}{p-q},\infty}(\mu)$.

The paper is organized as follows: In Section 2 we recall the non-isotropic dyadic decompositions and the properties that we will use in the proof of Theorem 1.1. In Section 3, we define and give the main properties of the non-isotropic capacities and give the lemmas that will be used to prove that (i) implies (iiia) of Theorem 1.1, and that (i) implies (ie) of Theorem 1.2. In Section 4, we obtain weak L^p-estimates for a non-isotropic fractional maximal operator that will also be used in the proof of Theorem 1.1. We finish the proof of Theorem 1.1 in Section 5, and the proofs of Theorems 1.2 and 1.3 in Section 6. In Section 7, we give the proof of results on q-trace measures for the non-diagonal case $p \neq q$.

Finally, the usual remark on notation: we will adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence, and we will write $A \lesssim B$ if there exists an absolute constant $C > 0$, such that $A \leq CB$. We will say that two quantities A and B are equivalent if both $A \lesssim B$ and $B \lesssim A$, and, in that case, we will write $A \approx B$.

2. Preliminares on non-isotropic dyadic decompositions

We recall a modification of the non-isotropic dyadic decomposition of a metric space of [**Chr**], due to [**HyKa**], which plays a similar role to the dyadic decomposition in $\mathbf{R}^n$. We will state this decomposition for $\mathbf{S}^n$. Given a fixed parameter $0 < \delta < 1$, small enough and a fixed point $x_0 \in \mathbf{S}^n$, there exists a finite collection of families of sets, $\mathcal{D}_j$, $j = 1, \dots, M$, called the adjacent dyadic systems. Each $\mathcal{D}_j$ is a family of Borel sets Q^k_α, $k \in \mathbf{Z}$, $\alpha \in I_j$, called the dyadic cubes, which are associated with points ζ^k_α, which we will call the center points of the cubes Q^k_α, having the following properties:

(i) $\mathbf{S}^n = \cup_{\alpha\in I_j} Q^k_\alpha$ (disjoint union), for each $k \in \mathbf{Z}$.

(ii) if $k < l$, then either $Q^l_\beta \cap Q^k_\alpha = \emptyset$ or $Q^l_\beta \subset Q^k_\alpha$.

(iii) There exist $c_1, C_1 > 0$ such that $B(\zeta^k_\alpha, c_1\delta^k) \subset Q^k_\alpha \subset B(\zeta^k_\alpha, C_1\delta^k) := B(Q^k_\alpha)$.

(iv) If $k \leq l$ and $Q^l_\beta \subset Q^k_\alpha$, then $B(Q^l_\beta) \subset B(Q^k_\alpha)$.

(v) For any $k \in \mathbf{Z}$, there exists α such that $x_0 = \zeta_\alpha^k$, the center point of $Q_\alpha^k \in \mathcal{D}_j$.

(vi) There exists $C > 0$ such that for any non-isotropic ball $B(\zeta, r) \subset \mathbf{S}^n$, with $\delta^{k+3} < r \leq \delta^{k+2}$, there exists j and $Q_\alpha^k \in \mathcal{D}_j$ such that $B(\zeta, r) \subset Q_\alpha^k$ and $\mathbf{diam}\, Q_\alpha^k \leq Cr$.

The family $\mathcal{D} = \bigcup_{j=1}^M \mathcal{D}_j$ is called a dyadic decomposition of $\mathbf{S}^n$ and we say that the set Q_α^k is a dyadic cube of generation k centered at ζ_α^k with radius $r_{Q_\alpha^k} := \delta^k$.

If $0 < s < n$, we define the fractional maximal function

$$\mathcal{M}_s[f](\zeta) = \sup_{\zeta \in B} \frac{1}{|B|^{1-\frac{s}{n}}} \int_B |f| d\sigma.$$

If $1 \leq j \leq M$, we consider the dyadic fractional maximal function

$$\mathcal{M}_s^j[f](\zeta) = \sup_{\zeta \in Q \in \mathcal{D}_j} \frac{1}{|Q|^{1-\frac{s}{n}}} \int_Q |f| d\sigma.$$

Here $|E| = \sigma(E)$. The following lemma is a version of a Whitney decomposition of an open set in $\mathbf{S}^n$ (see, for instance, p. 857 in [**SaWh**]). For a sake of completeness, we give a sketch of the proof.

Lemma 2.1. *Let $R > 1$ and let Ω be an open set in $\mathbf{S}^n$. Consider a dyadic adjacent system $\mathcal{D}_j$ in $\mathbf{S}^n$, $j \in \{1, \dots, M\}$. If j is fixed, Let Λ_j be the family of cubes $Q_\alpha^k \in \mathcal{D}_j$, which are maximal with respect to the property $RB(Q_\alpha^k) \subset \Omega$. We then have:*

(i) *$\Omega = \bigcup_{Q_\alpha^k \in \Lambda_j} Q_\alpha^k$ and for the cubes in Λ_j, either $Q_\alpha^k \cap Q_{\alpha_1}^{k_1} = \emptyset$ or $Q_\alpha^k = Q_{\alpha_1}^{k_1}$.*

(ii) *There exists $K > 0$ only depending on the constants C_1 and δ of the dyadic adjacent system, such that for every $Q_\alpha^k \in \Lambda_j$, we have that $KRB(Q_\alpha^k) \cap \Omega^c \neq \emptyset$.*

(iii) *There exists $C > 0$ only depending on the constants C_1 and δ of the dyadic adjacent system, such that $\sum_{Q_\alpha^k \in \Lambda_j} \chi_{RB(Q_\alpha^k)} \leq C \chi_\Omega$.*

Proof. Let $j \in \{1, \dots, M\}$ be fixed. Let Λ_j be the family of cubes $Q_\alpha^k \in \mathcal{D}_j$, maximal with respect to the property $RB(Q_\alpha^k) \subset \Omega$, where $B(Q_\alpha^k) = B(\zeta_\alpha^k, C_1 \delta^k)$ is as it was defined in (iii) of the properties of the adjacent dyadic systems.

Observe that if $\zeta \in \Omega$, for any k, there exists a cube $Q_\alpha^k \in \mathcal{D}_j$ such that $\zeta \in Q_\alpha^k$ and, if k is big enough, we also have that $RB(Q_\alpha^k) \subset \Omega$. We then have that $\Omega = \bigcup_{Q_\alpha^k \in \Lambda_j} Q_\alpha^k$ and the first assertion in (i) holds. The maximality of the choice of the cubes gives the second assertion of (i).

Next, we check that (ii) holds. The maximality of the cube Q_α^k gives that if $\hat{Q_\alpha}^{k-1}$ is the predecessor of the cube Q_α^k, then $RB(\hat{Q_\alpha}^{k-1}) \cap \Omega^c \neq \emptyset$. Let $\zeta \in RB(\hat{Q_\alpha}^{k-1}) \cap \Omega^c$. We then have that

$$|1 - \zeta \overline{\zeta_\alpha^k}| \leq 2 \left(|1 - \zeta \overline{\zeta_\alpha^{k-1}}| + |1 - \zeta_\alpha^{k-1} \overline{\zeta_\alpha^k}| \right) \leq 2RC_1 \delta^{k-1} + 2C_1 \delta^{k-1} \leq KR\delta^k,$$

for $K \geq \frac{2C_1(R+1)}{\delta R}$. Thus, $\emptyset \neq RB(\hat{Q_\alpha}^{k-1}) \cap \Omega^c \subset KRB(Q_\alpha^k) \cap \Omega^c$.

In order to prove (iii), let $\zeta \in \Omega$ be fixed. Since there exists $C > 0$ such that for any cube Q_α^k such that $\zeta \in RB(Q_\alpha^k)$, we have that $RB(Q_\alpha^k) \subset B(\zeta, C\delta^k)$, we deduce that

$$\sum_{Q^k_\alpha\in\Lambda_j}\chi_{RB(Q^k_\alpha)}(\zeta)\lesssim\sum_{Q^k_\alpha\in\Lambda_j,\,\zeta\in RB(Q^k_\alpha)}\frac{\sigma(Q^k_\alpha)}{\sigma(B(\zeta,C\delta^k))}=\frac{\sigma(\cup_{Q^k_\alpha;\,\zeta\in RB(Q^k_\alpha)}Q^k_\alpha)}{\sigma(B(\zeta,C\delta^k))}\leq 1.$$

□

3. Non-isotropic capacities and Wolff potentials

Let us recall (see [**CohVe**]), that if $1<p<+\infty$, $0<s<n$, the non-isotropic Riesz capacity of a set $E\subset\mathbf{S}^n$, is defined by

$$C_{s,p}(E)=\inf\{\|f\|^p_{L^p}\,;\,f\geq 0\ K_s[f]\geq 1\text{ on }E\}.$$

The non-isotropic Wolff potential of a positive measure ν on $\mathbf{S}^n$ is given by

$$\mathcal{W}_{s,p}[\nu](\zeta)=\int_0^1\left(\frac{\nu(B(\zeta,t))}{t^{n-sp}}\right)^{p'-1}\frac{dt}{t}. \tag{3.1}$$

We recall that if ν is a nonnegative Borel measure on $\mathbf{S}^n$, $V_{s,p}[\nu]$ is the non-isotropic Riesz potential defined by

$$V_{s,p}[\nu](\zeta):=K_s[K_s[\nu]^{p'-1}]. \tag{3.2}$$

We have that $\mathcal{W}_{s,p}[\nu]\lesssim V_{s,p}[\nu]$, and the non-isotropic version of the fundamental Wolff's theorem gives that in average the converse is also true, namely, if we denotes the energy of the measure ν by

$$\mathcal{E}_{s,p}[\nu]:=\int_{\mathbf{S}^n}V_{s,p}[\nu](\zeta)d\nu(\zeta),$$

them the following theorem holds:

$$\mathcal{E}_{s,p}[\nu]\approx\int_{\mathbf{S}^n}\mathcal{W}_{s,p}[\nu](\zeta)d\nu(\zeta). \tag{3.3}$$

The following proposition gives a necessary condition for a measure to satisfy a q-weak trace estimate.

Proposition 3.1. *Let $0<p,q<+\infty$, $0<s<n$. Let μ be a positive finite Borel measure on $\mathbf{S}^n$. Assume that there exists $C>0$ such that for every $\lambda>0$ and $f\in H^p_s$,*

$$\mu\left(\{\zeta\in\mathbf{S}^n;\,\mathcal{M}_{\rm rad}[f](\zeta)>\lambda\}\right)\leq C\frac{||f||^q_{H^p_s}}{\lambda^q}.$$

We then have that there exists $C>0$, such that for any $\zeta\in\mathbf{S}^n$, $0<r$,

$$\mu(B(\zeta,r))\leq Cr^{(n-sp)\frac{q}{p}}. \tag{3.4}$$

Proof. Let $\zeta\in\mathbf{S}^n$, $0<r<1$ be fixed. Let F be the holomorphic function on $\mathbf{B}^n$ defined by

$$F(z)=\frac{1}{(1-(1-r)z\overline{\zeta})^N},$$

with $N>0$ to be chosen later. We then have

$$B(\zeta,r)\subset\left\{\eta\in\mathbf{S}^n;\,\mathcal{M}_{\rm rad}F(\eta)>\frac{1}{r^N}\right\},$$

and, by hypothesis,

$$\mu(B(\zeta,r)) \le \mu\Big(\Big\{\eta\in \mathbf{S}^m;\ \mathcal{M}_{\rm rad}F(\eta)>\frac{1}{r^N}\Big\}\Big)\lesssim r^{Nq}\|F\|^q_{H^p_s}.$$

On the other hand, since

$$\left|\mathcal{R}^s\frac{1}{(1-(1-r)z\overline{\zeta})^N}\right|\lesssim \frac{1}{|1-(1-r)z\overline{\zeta}|^{N+s}},$$

we have that

$$\begin{aligned}||F||^p_{H^p_s}&\lesssim \int_{\mathbf{S}^n}\frac{1}{|1-(1-r)\eta\overline{\zeta}|^{(N+s)p}}d\sigma(\eta)\\ &=\int_{B(\zeta,r)}\frac{1}{|1-(1-r)\eta\overline{\zeta}|^{(N+s)p}}d\sigma(\eta)\\ &\quad+\sum_{k\ge 1}\int_{B(\zeta,2^{k+1}r)\setminus B(\zeta,2^kr)}\frac{1}{|1-(1-r)\eta\overline{\zeta}|^{(N+s)p}}d\sigma(\eta).\end{aligned}$$

If $k\ge 1$, and $\eta\in B(\zeta,2^{k+1}r)\setminus B(\zeta,2^kr)$, $|1-(1-r)\eta\overline{\zeta}|\approx 2^kr$. These estimates give, if N is chosen big enough, that the above is bounded by

$$\sum_{k\ge 0}\frac{(2^{k+1}r)^n}{(2^kr)^{(N+s)p}}\lesssim r^{n-(N+s)p}\sum_{k\ge 0}\left(\frac{C}{2^{(N+s)p}}\right)^k,$$

which gives the desired estimate. □

In particular, the above proposition gives that when $p=q$, and μ is a weak trace measure for H^p_s, then $\mu(B(\zeta,r))\lesssim r^{n-sp}$. When $p>1$ and $0<n-sp<1$, the necessary condition can be strengthened. This will be a consequence of the following Proposition, proved in [**CohVe**].

Proposition 3.2. *Let $1<p<+\infty$, $0<s<n$, $0<n-sp<1$ and ν a finite positive Borel measure on $\mathbf{S}^n$. Then there exists a holomorphic function $F_\nu\in H^p_s$, such that*

(i) *For any $\zeta\in\mathbf{S}^n$, $\lim_{r\to 1}{\rm Re}F_\nu(r\zeta)\ge C\mathcal{W}_{s,p}(\nu)(\zeta)$.*
(ii) $||F_\nu||^p_{H^p_s}\le C\mathcal{E}_{s,p}(\nu)$.

It is also well known the existence of a extremal measure for the Riesz capacity of an open set; see Proposition 7 in [**HeWo**]. Its non-isotropic version is the following lemma:

Lemma 3.3. *Let $0<s<n$, $1<p<+\infty$ and $G\subset\mathbf{S}^n$ be an open set. Then there exists a positive "extremal capacitary measure" ν_G on $\mathbf{S}^n$ such that*

(i) supp $\nu_G\subset\overline{G}$.
(ii) $\nu_G(G)=C_{s,p}(G)=\mathcal{E}_{s,p}[\nu_G]$.
(iii) *There exists a constant $C>0$, independent of G, such that $\mathcal{W}_{s,p}[\nu_G](\zeta)\ge C>0$, for any $\zeta\in G$.*

In fact, Proposition 7 in [**HeWo**] shows that the estimate (iii) holds for $C_{s,p}$-a.e. $\zeta\in G$, but in the same paper it is observed that in fact, it is also true for any $\zeta\in G$. The argument is as follows: If $\mathcal{W}_{s,p}[\nu_G](\zeta)\ge C$, for $C_{s,p}$-a.e. $\zeta\in G$, then it is also true that $\mathcal{W}_{s,p}[\nu_G](\zeta)\ge C$, for a.e. $\zeta\in G$ with respect to the Lebesgue measure on $\mathbf{S}^n$. Let $F\subset G$ such that $|F|=0$ and satisfying that $\mathcal{W}_{s,p}[\nu_G](\zeta)\ge C$, for any $\zeta\in G\setminus F$.

Assume that there exists $\zeta_0 \in F$ such that $\mathcal{W}_{s,p}[\nu_G](\zeta_0) < C$. If we denote $E = G \setminus F$, we then have that

$$\liminf_{\zeta\to\zeta_0;\,\zeta\in E} \mathcal{W}_{s,p}[\nu_G](\zeta) > \mathcal{W}_{s,p}[\nu_G](\zeta_0).$$

The non-isotropic version of Proposition 4 in [**HeWo**] gives that

$$\int_0^1 \left(\frac{C_{s,p}(E\cap B(\zeta_0,t))}{t^{n-sp}}\right)^{p'-1} \frac{dt}{t} < +\infty.$$

But $|E\cap B(\zeta_0,t)|^{\frac{n-sp}{n}} \lesssim C_{s,p}(E\cap B(\zeta_0,t))$ and since $|B(\zeta_0,t)\setminus E| = 0$, we then deduce that

$$\int_0^1 \left(\frac{t^{n-sp}}{t^{n-sp}}\right)^{p'-1} \frac{dt}{t} \lesssim \int_0^1 \left(\frac{C_{s,p}(E\cap B(\zeta_0,t))}{t^{n-sp}}\right)^{p'-1} \frac{dt}{t} < +\infty,$$

and we arrive to a contradiction.

In Section 6 we will use the above lemma to obtain a stronger necessary condition for μ to be a weak trace measure for H^p_s when $n - sp < 1$.

4. Weak L^p-estimates for the non-isotropic fractional maximal operator

In this section, we will prove the following theorem:

Theorem 4.1. *Let $0 < s < n$, $1 < p < +\infty$, μ a finite positive Borel measure on $\mathbf{S}^n$ and $\mathcal{D}_j$, $j = 1,\dots,M$ an adjacent dyadic system on $\mathbf{S}^n$. Then the following assertions are equivalent:*

(i) *There exists $C > 0$ such that for any $f \in L^p(\mathbf{S}^n)$ and $\lambda > 0$,*

$$\mu\left(\{\zeta\in\mathbf{S}^n;\, \mathcal{M}_s[f](\zeta) > \lambda\}\right) \le C\frac{\|f\|^p_{L^p}}{\lambda^p}.$$

(ii) *There exists $C > 0$ such that for any $f \in L^p(\mathbf{S}^n)$, $\lambda > 0$ and $j = 1,\dots,M$,*

$$\mu\left(\{\zeta\in\mathbf{S}^n;\, \mathcal{M}^j_s[f](\zeta) > \lambda\}\right) \le C\frac{\|f\|^p_{L^p}}{\lambda^p}.$$

(iii) *There exists $C > 0$ such that for any $\zeta \in \mathbf{S}^n$, $0 < r$,*

$$\mu\left(B(\zeta,r)\right) \le Cr^{n-sp}.$$

(iv) *There exists $C > 0$ such that for any $j = 1,\dots,M$ and $Q \in \mathcal{D}_j$,*

$$\mu\left(Q\right) \le Cr_Q^{n-sp}.$$

Proof. The proof follows closely the ideas in [**Sa**] and we will sketch briefly the main ingredients. The fact that (i) and (ii) are equivalents is a consequence of Lemma 2.9 in [**Ka**], where it is shown that

(i) $\mathcal{M}^j_s[f] \lesssim \mathcal{M}_s[f]$ and
(ii) $\mathcal{M}_s[f] \lesssim \sum_{j=1}^K \mathcal{M}^j_s[f]$.

Next, the fact that (iii) implies (iv) is an immediate consequence of property (iii) in the dyadic decomposition. Property (vi) in the dyadic decomposition gives that (iv) implies (iii).

So we are left to show that (ii) and (iv) are equivalents. Assume first that (ii) holds. Let $Q \in \mathcal{D}_j$, $j = 1, \ldots, M$ be fixed and let $f = \chi_Q$. If $\zeta \in Q$, we have that

$$\frac{1}{r_Q^{n-s}} \int_Q \chi_Q d\sigma \leq \mathcal{M}_s^j[\chi_Q f](\zeta),$$

and consequently, if $\lambda < r_Q^s$, $Q \subset \{\zeta;\, \mathcal{M}_s^j[\chi_Q](\zeta) > \lambda\}$, and

$$\mu(Q) \leq \mu(\{\zeta;\, \mathcal{M}_s^j[\chi_Q](\zeta) > \lambda\}) \lesssim \frac{r_Q^n}{\lambda^p}.$$

If $\lambda \to r_Q^s$, we deduce that $\mu(Q) \lesssim r^{n-sp}$. Finally, assume that (iv) holds, and let $f \in L^p$ non-negative and $j \in \{1, \ldots, M\}$. Let

$$\Omega_k = \{\zeta \in \mathbf{S}^n;\, \mathcal{M}_s^j[f](\zeta) > 2^k\} = \bigcup_{Q \in \mathcal{D}_j'} Q,$$

where $\mathcal{D}_j'$ is the family of cubs in $\mathcal{D}_j$, which are maximal with respect to the property that $\frac{1}{r_Q^{n-s}} \int_Q f d\sigma > 2^k$. In particular, these cubes are pairwise disjoint.

If $Q \in \mathcal{D}_j'$ the fact that Q is maximal gives that if $\hat{Q}$ is its first ascendant, then

$$\frac{1}{r_{\hat{Q}}^{n-s}} \int_{\hat{Q}} f d\sigma \leq 2^k,$$

and since by property (iii) of the dyadic decomposition, $r_{\hat{Q}} \approx r_Q$, we obtain that

$$2^k < \frac{1}{r_Q^{n-s}} \int_Q f d\sigma \lesssim \frac{1}{r_{\hat{Q}}^{n-s}} \int_{\hat{Q}} f d\sigma \leq 2^k.$$

Next, observe that the hypothesis on μ gives that

$$\left(\frac{1}{r_Q^{n-s}} \int_Q f d\sigma \right)^p \mu(Q) \leq r_Q^{sp} \frac{1}{r_Q^n} \int_Q f^p d\sigma \mu(Q) \lesssim \int_Q f^p d\sigma.$$

Altogether, we obtain that

$$\begin{aligned} \mu(\Omega_k) &= \sum_{Q \in \mathcal{D}_k'} \mu(Q) \\ &\lesssim \sum_{Q \in \mathcal{D}_k'} \left(\frac{1}{r_Q^{n-s}} \int_Q f d\sigma \right)^{-p} \int_Q f^p d\sigma \approx \sum_{Q \in \mathcal{D}_k'} 2^{-kp} \int_Q f^p d\sigma \\ &\leq \frac{1}{2^{kp}} \int_{\bigcup_{Q \in \mathcal{D}_k'}} f^p d\sigma \leq \frac{\|f\|_p^p}{2^{kp}}, \end{aligned}$$

and that gives (ii). □

5. Proof of Theorem 1.1

The proof of Theorem 1.1 is based in a version of a maximum principle that was first stated in [**Sa1**] for the study of two-weight norm inequalities for fractional integral operators.

Proposition 5.1. *Let $0 < s < n$ and $c > 2$. Let f be an integrable function on $\mathbf{S}^n$. Let $l \in \mathbf{Z}$, $\Omega_l = \{\zeta \in \mathbf{S}^n; \mathcal{M}_{\mathrm{rad}}[\mathcal{C}_s[f]](\zeta) > 2^l\}$ and $\{Q_\alpha^k\}_{Q_\alpha^k \in \Lambda_j}$ be the non-isotropic Whitney decomposition of the open set Ω_l given by Lemma 2.1, for $R > 0$ big enough. Then there exists $C > 0$ only depending on s, n and c such that*

$$\sup_{\zeta \in Q_\alpha^k} \mathcal{M}_{\mathrm{rad}}\left[\mathcal{C}_s\left[\chi_{(cB(Q_\alpha^k))^c} f\right]\right](\zeta) \le C\left(2^l + \sup_{Q_\alpha^k \subset B'} \frac{\int_{B'} |f| d\sigma}{|B'|^{1-\frac{s}{n}}}\right). \tag{5.1}$$

Proof. Let $\zeta \in Q_\alpha^k$. By (ii) in Lemma 2.1, there exists $\xi \in \Omega_k^c \cap KRB(Q_\alpha^k)$. Hence, $\mathcal{M}_{\mathrm{rad}}[f](\xi) \le 2^l$.

Next, fixed this point $\zeta \in Q_\alpha^k$, there exists $\rho > 0$ such that

$$\mathcal{M}_{\mathrm{rad}}\left[\mathcal{C}_s\left[\chi_{(cB(Q_\alpha^k))^c} f\right]\right](\zeta) \le 2\left|\int_{\mathbf{S}^n} \frac{\chi_{(cB(Q_\alpha^k))^c} f(\eta)}{(1-\rho\zeta\overline{\eta})^{n-s}} d\sigma(\eta)\right|.$$

Let $\rho_1 < 1$ such that $\delta^k \lesssim 1-\rho_1$ and $|\rho - \rho_1| \approx \delta^k$. We fix $A > c$ to be chosen later. We then have,

$$\begin{aligned}
&|\mathcal{C}_s[\chi_{(cB(Q_\alpha^k))^c} f](\rho\zeta) - \mathcal{C}_s[f](\rho_1\xi)| \\
&\le \left|\int_{AB(Q_\alpha^k)} \frac{\chi_{(cB(Q_\alpha^k))^c} f(\eta)}{(1-\rho\zeta\overline{\eta})^{n-s}} d\sigma(\eta)\right| + \left|\int_{AB(Q_\alpha^k)} \frac{f(\eta)}{(1-\rho_1\xi\overline{\eta})^{n-s}} d\sigma(\eta)\right| \\
&\quad + \left|\int_{(AB(Q_\alpha^k))^c} \left(\frac{1}{(1-\rho\zeta\overline{\eta})^{n-s}} - \frac{1}{(1-\rho_1\xi\overline{\eta})^{n-s}}\right) f(\eta) d\sigma(\eta)\right| = \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

We begin with the estimate of the integral in I. Since $\zeta \in Q_\alpha^k$, for any $\eta \in (cB(Q_\alpha^k))^c$, we have that

$$|1-\zeta\overline{\eta}| \ge \frac{1}{2}|1-\eta\overline{\zeta_\alpha^k}| - |1-\zeta_\alpha^k\overline{\zeta}| \ge \frac{1}{2}cC_1\delta^k - C_1\delta^k = \left(\frac{c}{2}-1\right)C_1\delta^k.$$

Since we are assuming that $c > 2$, we then have,

$$\mathrm{I} \lesssim \frac{1}{|AB(Q_\alpha^k)|^{\frac{n-s}{n}}} \int_{AB(Q_\alpha^k)} |f(\eta)| d\sigma(\eta) \le \sup_{Q_\alpha^k \subset B'} \frac{\int_{B'} |f| d\sigma}{|B'|^{1-\frac{s}{n}}}.$$

For the estimate in II, we observe that since $\delta^k \lesssim 1-\rho_1$,

$$\mathrm{II} \lesssim \sup_{Q_\alpha^k \subset B'} \frac{\int_{B'} |f| d\sigma}{|B'|^{1-\frac{s}{n}}}.$$

Finally, we estimate the integral in III.

We will use an estimate that can be found, for instance, in the proof of Proposition 2.13 in [**Tch**], (see also [**HyMa**]). Let us consider a metric d on $\overline{\mathbf{B}}^n$ defined for $z, w \in \overline{\mathbf{B}}^n$ as $d(z,w) = ||z|-|w|| + |1 - \frac{z\overline{w}}{|z||w|}|$. Observe that for any $z, w \in \overline{\mathbf{B}}^n$, $d(z,w) \le |1-z\overline{w}|$, and that $d(z,w) \approx |1-z\overline{w}|$ if either z or w are on $\mathbf{S}^n$. We then have that there exists $M > 0$ such that if $z, z', w \in \mathbf{B}^n$, $d(z,w) \ge Md(z,z')$, then

$$\left|\frac{1}{(1-z\overline{w})^{n-s}} - \frac{1}{(1-z'\overline{w})^{n-s}}\right| \lesssim \left(\frac{d(z,z')}{|1-z\overline{w}|}\right)^{\frac{1}{2}} \frac{1}{|1-z\overline{w}|^{n-s}}. \tag{5.2}$$

Consequently, applying (5.2) to $z=\rho\zeta$, $z'=\rho_1\xi$, $w=\eta$ and $\zeta,\xi,\eta\in\mathbf{S}^n$, there exists $M_1>0$, such that if $|1-\zeta\overline{\eta}|\geq M_1 d(\rho\zeta,\rho_1\xi)$, we have that,

$$\left|\frac{1}{(1-\rho\zeta\overline{\eta})^{n-s}}-\frac{1}{(1-\rho_1\xi\overline{\eta})^{n-s}}\right|\lesssim\left(\frac{d(\rho\zeta,\rho_1\xi)}{|1-\zeta\overline{\eta}|}\right)^{\frac{1}{2}}\frac{1}{|1-\zeta\overline{\eta}|^{n-s}}. \tag{5.3}$$

Observe that there exists N such that $d(\rho\zeta,\rho_1\xi)=|\rho-\rho_1|+|1-\zeta\overline{\xi}|\leq N\delta^k$. We also observe that for any $\eta\in(AB(Q_\alpha^k))^c$ we have that $|1-\zeta\overline{\eta}|\geq(\frac{A}{2}-1)C_1\delta^k$. Choosing $A>0$, so that $(\frac{A}{2}-1)C_1>NM_1$, we have that $|1-\zeta\overline{\eta}|>M_1d(\rho\zeta,\rho_1\xi)$, and consequently, applying (5.3),

$$\mathrm{III}\lesssim\int_{(AB(Q_\alpha^k))^c}\left(\frac{d(\rho\zeta,\rho_1\xi)}{|1-\zeta\overline{\eta}|}\right)^{\frac{1}{2}}\frac{1}{|1-\zeta\overline{\eta}|^{n-s}}|f(\eta)|d\sigma(\eta).$$

Since $d(\rho\zeta,\rho_1\xi)\lesssim\delta^k$, and if $m\geq 0$ and $\eta\in 2^{m+1}AB(Q_\alpha^k)\setminus 2^mAB(Q_\alpha^k)$, $|1-\zeta\overline{\eta}|\approx 2^m\delta^k$, we deduce that

$$\begin{aligned}\mathrm{III}&\lesssim\sum_{m\geq 0}\left(\frac{1}{2^m}\right)^{\frac{1}{2}}\frac{1}{(2^m\delta^k)^{n-s}}\int_{2^{m+1}AB(Q_\alpha^k)\setminus 2^mAB(Q_\alpha^k)}|f(\eta)|d\sigma(\eta)\\&\lesssim\sup_{Q_\alpha^k\subset B'}\frac{\int_{B'}|f|d\sigma}{|B'|^{1-\frac{s}{n}}}.\end{aligned}$$

□

5.1. Proof of Theorem 1.1. As we have already mentioned in the introduction, conditions (i) and (ii) of Theorem 1.1 are equivalent, since any $f\in H_s^p$ can be written as $f=\mathcal{C}_s[g]$, where $g\in L^p$ and $\|f\|_{H_s^p}\approx\inf\|g\|_{L^p}$ for $g\in L^p$ such that $f=\mathcal{C}_s[g]$.

Next, we show that condition (ii) implies (iii) in Theorem 1.1. Assume that condition (ii) in Theorem 1.1 holds. By Proposition 3.1 we have that condition (iiia) is satisfied.

We next check condition (iiib). Let $B\subset\mathbf{S}^n$ be a non-isotropic ball such that $\mu(B)\neq 0$. We then have that for any $f\in L^p$,

$$\begin{aligned}\int_B\mathcal{M}_{\mathrm{rad}}\left[\mathcal{C}_s[\chi_B]f\right]d\mu&=\int_0^\infty\mu\left(\{\zeta\in B;\,\mathcal{M}_{\mathrm{rad}}\left[\mathcal{C}_s[\chi_Bf]\right](\eta)>\lambda\}\right)d\lambda\\&\leq\int_0^\infty\min\left(\mu(B),\mu\left(\{\zeta\in\mathbf{S}^n;\,\mathcal{M}_{\mathrm{rad}}\left[\mathcal{C}_s[\chi_Bf]\right](\eta)>\lambda\}\right)\right)d\lambda\\&\lesssim\int_0^\alpha\mu(B)d\lambda+\int_\alpha^\infty\frac{\int_B|f|^pd\sigma}{\lambda^p}d\lambda\leq\alpha\mu(B)+C\alpha^{1-p}\int_B|f|^pd\sigma\\&=\int_B|f|^pd\sigma\mu(B)^{\frac{1}{p'}},\end{aligned}$$

if we choose $\alpha=\left(\frac{\int_B|f|^pd\sigma}{\mu(B)}\right)^{\frac{1}{p}}$.

So, we are left to show that condition (iii) implies condition (ii) in Theorem 1.1. Assume, then, that condition (iii) holds. We first claim that if C is the constant in the maximum principle (Proposition 5.1) it is enough to check that if $\beta<1$ is small enough and fixed,

$$\begin{aligned}(5.4)\qquad &\mu\left(\zeta\in \mathbf{S}^n;\, \mathcal{M}_{\text{rad}}[\mathcal{C}_s[f]](\zeta)>2C\lambda,\, \mathcal{M}_s[f](\zeta)\le \beta\lambda\}\right)\\ &\quad\lesssim \beta\mu(\{\zeta\in \mathbf{S}^n;\, \mathcal{M}_{\text{rad}}[\mathcal{C}_s[f]](\zeta)>\lambda\})+\frac{1}{\beta^p\lambda^p}\int_{\mathbf{S}^n}|f|^p d\sigma.\end{aligned}$$

The fact that this claim implies the desired weak estimate is standard (see for instance, [**Sa**]) using the fact that since by hypothesis $\mu(B(\zeta,r))\lesssim r^{n-sp}$, Theorem 4.1 gives that the fractional maximal function satisfies a weak type inequality.

So, we are left to show (5.4). The proof follows closely the ideas in [**Sa**], and we will just sketch it. Let $\lambda=2^l$, and denote $\Omega_l=\{\zeta\in\mathbf{S}^n;\, \mathcal{M}_{\text{rad}}[\mathcal{C}_s[f]](\zeta)>2^l\}$. If $j\in\{1,\dots,M\}$, we consider the Whitney-type cubes $\{Q_\alpha^k\}_{Q_\alpha^k\in\Lambda_j}$ associated to the open set Ω_l, given in Lemma 2.1. Consider the sets

$$E_{\alpha,k}=\{\zeta\in Q_\alpha^k;\, \mathcal{M}_{\text{rad}}[\mathcal{C}_s[f]](\zeta)>2C\lambda,\, \mathcal{M}_s[f](\zeta)\le\beta\lambda\}.$$

For each $\zeta\in E_{\alpha,k}$, Proposition 5.1 gives that for $c>2$, and then we have that

$$\mathcal{M}_{\text{rad}}[\mathcal{C}_s[\chi_{(cB(Q_\alpha^k))^c}f]](\zeta)\le C(1+\beta)\lambda.$$

We fix an R as the constant given in Lemma 2.1 with $2<c<R$ and fix $\beta<\frac12$. We then have that for $\zeta\in E_{\alpha,k}$,

$$\begin{aligned}2C\lambda&<\mathcal{M}_{\text{rad}}[\mathcal{C}_s[f]](\zeta)\\ &\le \mathcal{M}_{\text{rad}}[\mathcal{C}_s[\chi_{cB(Q_\alpha^k)}f]](\zeta)+\mathcal{M}_{\text{rad}}[\mathcal{C}_s[\chi_{(cB(Q_\alpha^k))^c}f]](\zeta)\\ &\le \mathcal{M}_{\text{rad}}[\mathcal{C}_s[\chi_{cB(Q_\alpha^k)}f]](\zeta)+\frac32 C\lambda.\end{aligned}$$

Thus $\frac12 C\lambda<\mathcal{M}_{\text{rad}}[\mathcal{C}_s[\chi_{cB(Q_\alpha^k)}f]](\zeta)$, and, consequently

$$\lambda\mu(E_{\alpha,k})\lesssim\int_{E_{\alpha,k}}\mathcal{M}_{\text{rad}}[\mathcal{C}_s[\chi_{cB(Q_\alpha^k)}f]](\zeta)\,d\mu(\zeta).$$

Since

$$\mu\left(\{\zeta\in Q_\alpha^k;\, \mathcal{M}_{\text{rad}}[\mathcal{C}_s[f]](\zeta)>2C\lambda,\, \mathcal{M}_s[f](\zeta)\le\beta\lambda\}\right)=\sum_{\alpha,\,k,\,Q_\alpha^k\in\Lambda_j}\mu(E_{\alpha,k}),$$

we have that

$$\begin{aligned}\sum_\alpha\mu(E_{\alpha,k})\le\beta&\sum_{\mu(E_{\alpha,k})\le\beta\mu(cB(Q_\alpha^k))}\mu(cB(Q_\alpha^k))+\sum_{\mu(E_{\alpha,k})>\beta\mu(cB(Q_\alpha^k))}\mu(E_{\alpha,k})\\ &\times\left(\frac{\int_{E_{\alpha,k}}\mathcal{M}_{\text{rad}}[\mathcal{C}_s[\chi_{cB(Q_\alpha^k)}f]](\zeta)\,d\mu(\zeta)}{\lambda\mu(E_{\alpha,k})}\right)^p:=\mathrm{I}+\mathrm{II}.\end{aligned}$$

Next, since we are assuming the finite overlapping of the family of balls $RB(Q_\alpha^k)$ given in (iii) of Lemma 2.1, and $c<R$, we have that

$$\mathrm{I}\le\beta\sum_{\mu(E_{\alpha,k})\le\beta\mu(cB(Q_\alpha^k))}\mu(cB(Q_\alpha^k))=\beta\sum_{\mu(E_{\alpha,k})\le\beta\mu(cB(Q_\alpha^k))}\int\chi_{cB(Q_\alpha^k)}d\mu\lesssim\beta\mu(\Omega_l).$$

The hypothesis (iiib) on the measure μ and, once more, the finite overlapping property, gives that

$$\begin{aligned}\mathrm{II} &\le \frac{1}{\lambda^p}\sum_{\mu(E_{\alpha,k})>\beta\mu(cB(Q_\alpha^k))}\mu(E_{\alpha,k})\left(\frac{\int_{cB(Q_\alpha^k)}\mathcal{M}_{\mathrm{rad}}[\mathcal{C}_s[\chi_{cB(Q_\alpha^k)}f]](\zeta)\,d\mu(\zeta)}{\mu(E_{\alpha,k})}\right)^p\\
&\lesssim \frac{1}{\lambda^p}\sum_{\mu(E_{\alpha,k})>\beta\mu(cB(Q_\alpha^k))}\frac{\int_{cB(Q_\alpha^k)}|f|^p d\sigma\mu(cB(Q_\alpha^k))}{(\mu(E_{\alpha,k}))^{p-1}}\\
&\le \frac{1}{\lambda^p\beta^{q-1}}\sum_{\mu(E_{\alpha,k})>\beta\mu(cB(Q_\alpha^k))}\int_{cB(Q_\alpha^k)}|f|^p d\sigma \lesssim \frac{1}{\lambda^p}\int_{\Omega_k}|f|^p d\sigma \le \frac{1}{\lambda^p}\int_{\mathbf{S}^n}|f|^p d\sigma.\end{aligned}$$

6. Proof of Theorems 1.2 and 1.3

6.1. Proof of Theorem 1.2. We begin with the proof of assertion (i) in Theorem 1.2. Observe that trivially, (ib) implies (ia), (id) implies (ic), (ic) implies (ia) and (id) implies (ib). In [**CohVe**], it is shown that (ib) and (ie) are equivalent, and the methods in [**AdHe**] can be used to show that (id) and (ie) are also equivalent. So in order to finish the proof of (i) it is enough to show that (ia) implies (ie). This will be a consequence of the following proposition.

Proposition 6.1. *Let $0 < p < +\infty$, $0 < s < n$ and assume that $n - sp < 1$. Let μ be a positive finite Borel measure on $\mathbf{S}^n$. Assume that there exists $C > 0$ such that*

$$\mu\left(\{\zeta \in \mathbf{S}^n;\, \mathcal{M}_{\mathrm{rad}}[f](\zeta) > \lambda\}\right) \le C\frac{||f||^p_{H^p_s}}{\lambda^p},$$

for any $f \in H^p_s$ and $\lambda > 0$. We then have that there exists $C > 0$, such that for any open set $G \subset \mathbf{S}^n$,

$$\mu(G) \le CC_{s,p}(G). \tag{6.1}$$

Proof. Let G be an open set on $\mathbf{S}^n$ and let ν_G be the extremal capacitary measure of G.

Let F_{ν_G} is the holomorphic function given in Proposition 3.2. The fact that ν_G is extremal gives (see Lemma 3.3) that for any $\zeta \in G$,

$$\lim_{r\to 1}\mathrm{Re}\, F_{\nu_G}(\zeta) \ge C\mathcal{W}_{s,p}(\nu_G)(\zeta) \ge C > 0.$$

In particular, we deduce that for any $\zeta \in G$, $\mathcal{M}_{\mathrm{rad}}F_{\nu_G}(\zeta) \ge \frac{C}{2}$.

Hence, since we are assuming that μ is a weak trace measure for H^p_s,

$$\mu(G) \le \mu\left(\left\{\zeta \in \mathbf{S}^n;\, \mathcal{M}_{\mathrm{rad}}F_{\nu_G}(\zeta) > \frac{C}{2}\right\}\right) \lesssim ||F||^p_{H^p_s} \lesssim \mathcal{E}_{s,p}(\nu_G) = C_{s,p}(G).$$

□

Next, we prove (ii) in Theorem 1.2.

Now if $1 < p \le 2$ and $n - sp \ge 1$, it is proved in [**AhCo**], p. 33 and Theorem 3.2 that there exists a measure μ on $\mathbf{S}^n$, which is a trace measure for H^p_s and it is not a trace measure for $K_s[L^p]$. In particular, it is a weak trace measure for H^p_s. Next, observe that if a measure ν is a weak trace measure for $K_s[L^p]$, then ν satisfies the capacitary condition on open sets and, consequently, ν is also a trace measure for

$K_s[L^p]$. Indeed, if $G \subset \mathbf{S}^n$, let Λ be the set of functions f on $\mathbf{S}^n$, non-negative satisfying that $K_s[f] \geq \chi_G$. Then if $f \in \Lambda$, $G \subset \{\zeta \in \mathbf{S}^n;\ K_s[f] \geq 1\}$, and consequently, since ν is a weak trace measure for $K_s[L^p]$, $\nu(G) \lesssim \inf_{f\in\Lambda} \|f\|_{L^p}^p = C_{s,p}(G)$.

This observation gives that since the measure μ is not a trace measure for $K_s[L^p]$, then μ is not a weak measure for $K_s[L^p]$. An that gives the example. If $2 < p < +\infty$ and $n - sp \geq 1$, it is proved in Theorems 4 and 5 in [**CohVe**] that there exists a measure μ on $\mathbf{S}^n$, which is a trace measure for H_s^p, and it is not a trace measure for $K_s[L^p]$. The conclusion is obtained as in the previous case.

6.2. Proof of Theorem 1.3. We recall that $f \in L^{p,\infty}(\mu)$ if and only if,

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{\lambda>0} \lambda^p \mu(\{\zeta \in \mathbf{S}^n;\ |f(\zeta)| > \lambda\}) < +\infty.$$

The Kolmogorov's condition (1.5) gives an equivalent reformulation of the weak trace measures for H_s^p, which will be used to prove Theorem 1.3. Namely, a measure μ is a weak trace for H_s^p, if and only if there exists $r < p$, such that

$$\sup_{\mu(E)>0} \mu(E)^{\frac{r-p}{rp}} \left\{ \int_E \sup_{\rho<1} \left| \int_{\mathbf{S}^n} \frac{f(\eta)}{(1-\rho\zeta\overline{\eta})^{n-s}} d\sigma(\eta) \right|^r d\mu(\zeta) \right\}^{\frac{1}{r}} \lesssim \|f\|_{L^p}. \tag{6.2}$$

We now prove Theorem 1.3. The proof follows the arguments in Theorem 2 in [**CohVe**], where it is obtained the same necessary condition for the trace measures. We will give an sketch of the proof. For $j \geq 0$, let $B_k = B(\zeta_k, \delta_k)$, $k = 0, 1, \dots$ be a sequence of non-isotropic balls in $\mathbf{S}^n$ satisfying that

$$1 \leq \sum_{2^{-j} \leq \delta_k < 2^{-j+1}} \chi_{B_k}(\zeta) \leq M,$$

for any $\zeta \in \mathbf{S}^n$, where the constant M depends only on the dimension n. If ν is a positive Borel measure on $\mathbf{S}^n$, it is proved in [**CohVe**] that there exists $C > 0$ such that if $B_k^* := B(\zeta_k, C\delta_k)$, then there exist C_1, C_2 such that

$$C_1 \sum_k (\nu(B_k)\delta_k^{sp-n})^{p'-1} \chi_{B_k}(\zeta) \leq \mathcal{W}_{s,p}[\nu](\zeta) \leq C_2 \sum_k (\nu(B_k)\delta_k^{sp-n})^{p'-1} \chi_{B_k^*}(\zeta) \tag{6.3}$$

Let G be an open subset of $\mathbf{S}^n$ and let $\nu := \nu_G$ be the extremal capacitary measure on G (with respect to the $C_{2s\,\frac{p}{2}}$ capacity). Let $z_k = (1-\delta_k)\zeta_k$, $k \geq 0$ and $(r_k(t))_k$, $0 < t < 1$, the Rademacher functions. For any $\lambda > n$, we define the holomorphic functions F_t on $\mathbf{B}^n$ by

$$F_t(z) = \sum_{k\geq 0} \frac{c_k r_k(t)}{(1 - z\overline{z_k})^\lambda},$$

$t \in (0,1)$. We choose c_k so that

$$c_k^2 \delta_k^{-2\lambda} = (\nu(B_k)\delta_k^{sp-n})^{\frac{2}{p-2}}.$$

In [**CohVe**], p. 1087, it is proved that $F_t \in H_s^p$ for almost any $t \in (0,1)$. Let us show, that in our situation that if we fix $2 < r < p$, $\int_0^1 \|F_t\|_{H_s^p}^r dt \lesssim C_{2s,\frac{p}{2}}(G)^{\frac{r}{p}}$. We follow closely their arguments, based on Khintchine estimate. Let us give a sketch of the proof.

Let $\varphi_k(z) = R^s((1-z\overline{z_k})^{-\lambda})$. We then have that $R^sF_t(z) = \sum_k c_k r_k(t)\varphi_k(z)$, where $|\varphi_k(z)| \lesssim \frac{1}{|1-z\overline{z_k}|^{\lambda+s}}$. Hence, since $r/p \le 1$,

$$\begin{aligned}\int_0^1 \|F_t\|^r_{H^p_s}dt &= \int_0^1 \Big\|\sum_k c_k r_k(t)\varphi_k(z)\Big\|^r_{H^p}dt \lesssim \left(\int_0^1 \Big\|\sum_k c_k r_k(t)\varphi(t)\Big\|^p_{H^p}dt\right)^{\frac{r}{p}}\\ &\lesssim \left(\int_{\mathbf{S}^n}\left(\sum_k c_k^2\frac{1}{|1-\zeta\overline{z_k}|^{2(\lambda+s)}}\right)^{\frac{p}{2}}d\sigma(\zeta)\right)^{\frac{r}{p}}\\ &\lesssim \left(\int_{\mathbf{S}^n}\left(\sum_k c_k^2\delta_k^{-2(\lambda+s)}\chi_{B_k(\zeta)}\right)^{\frac{p}{2}}d\sigma(\zeta)\right)^{\frac{r}{p}}\\ &\lesssim \left(\int_{\mathbf{S}^n}\left(\sum_k \left(\nu(B_k)\delta_k^{2s-n}\right)^{\frac{2}{p-2}}\chi_{B_k(\zeta)}\right)^{\frac{p}{2}}d\sigma(\zeta)\right)^{\frac{r}{p}}\end{aligned}$$

Applying that the discrete Wolff potential is bounded, up to a constant, by the non-isotropic Wolff potential, we deduce that the above integral is bounded by

$$\begin{aligned}&\left(\int_{\mathbf{S}^n}\left(\int_0^1\left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-2s}}\right)^{\frac{2}{p-2}}\frac{dr}{1-r}\right)^{\frac{p}{2}}d\sigma(\zeta)\right)^{\frac{r}{p}}\\ &\qquad\lesssim \int_{\mathbf{S}^n}\mathcal{W}_{2s,\frac{p}{2}}[\nu](\zeta)d\nu(\zeta) \lesssim \left(C_{2s,\frac{p}{2}}(G)\right)^{\frac{r}{p}},\end{aligned}$$

where the previous to the last estimate is an inequality which generalize the Wolff's inequality (see (10) in [**CohVe**]) and the last estimate is property (iii) of Lemma 3.3.

Next, Khintchine's inequality gives that

$$\left(\sum_k\frac{c_k^2}{|1-z\overline{z_k}|^{2\lambda}}\right)^{\frac{r}{2}} \lesssim \int_0^1 |F_t(z)|^r dt.$$

Then,

$$\int_0^1(\mathcal{M}_{\mathrm{rad}}[F_t](\zeta))^r dt \ge \mathcal{M}_{\mathrm{rad}}\left[\int_0^1 |F_t(z)|^r dt\right](\zeta) \ge C\sup_\rho\left(\sum_k\frac{c_k^2}{|1-\rho\zeta\overline{z_k}|^{2\lambda}}\right)^{\frac{r}{2}}. \tag{6.4}$$

If $\zeta\in B_k^*$,

$$\sup_\rho\sum_k\frac{c_k^2}{|1-\rho\zeta\overline{z_k}|^{2\lambda}} \approx \sup_\rho\sum_k\frac{c_k^2}{(1-\rho+\delta_k)^{2\lambda}} = \sum_k\frac{c_k^2}{\delta_k^{2\lambda}}.$$

Hence, we obtain from (6.3) and (6.4) that

$$\sup_\rho\left(\sum_k\frac{c_k^2}{|1-\rho\zeta\overline{z_k}|^{2\lambda}}\right)^{\frac{r}{2}} \ge C\left(\sum_k c_k^2\delta_k^{-2\lambda}\chi_{B_k^*}(\zeta)\right)^{\frac{r}{2}} \ge C\left(\mathcal{W}_{2s,\frac{p}{2}}[\nu](\zeta)\right)^{\frac{r}{2}}.$$

Integrating with respect to μ on G the above estimate and using that by property (iii) of Lemma 3.3, we have that $\mathcal{W}_{2s,\frac{p}{2}}[\nu](\zeta) \geq C$, for any $\zeta \in G$, we deduce that

$$\mu(G) \lesssim \int_G \int_0^1 (\mathcal{M}_{\mathrm{rad}}[F_t](\zeta))^r dt d\mu(\zeta).$$

Since by hypothesis, μ is a weak trace measure for H_s^p, the above estimate and (6.2), gives that

$$\mu(G) \lesssim \int_0^1 \int_G \mathcal{M}_{\mathrm{rad}}[F_t](\zeta)^r d\mu(\zeta) dt \lesssim \mu(G)^{\frac{p-r}{p}} \int_0^1 \|F_t\|_{H_s^p}^r dt \lesssim \mu(G)^{\frac{p-r}{p}} C_{2s,\frac{p}{2}}(G)^{\frac{r}{p}}.$$

Hence, $\mu(G)^{\frac{r}{p}} \lesssim C_{2s,\frac{p}{2}}(G)^{\frac{r}{p}}$, and consequently, $\mu(G) \lesssim C_{2s,\frac{p}{2}}(G)$.

7. Weak q-trace measures for H_s^p, $q \neq p$

In this section, we will give some results for a measure μ to be a weak q-trace measure for H_s^p, $0 < p, q < +\infty$, when $q \neq p$.

7.1. The case $p < q$. We begin recalling some results when $p < q$. In Theorem A in [**CaOr1**], it is shown that the condition (3.4), $\mu(B(\zeta, r)) \lesssim r^{(n-sp)\frac{q}{p}}$ is necessary and sufficient for a measure μ to be a q-trace measure for H_s^p, when $p \leq 1$ and $p \leq q$. On the other hand, when $1 < p < q$, the non-isotropic version of a theorem in [**Ad2**] shows that condition (3.4) is also necessary and sufficient for a measure μ to be a q-trace measure for $K_s[L^p]$. In particular, it can be deduced that in both cases the q-trace and the weak q trace for $K_s[L^p]$ coincide. Since in Proposition 3.1 it has been proved that (3.4) is necessary for a measure μ to be a weak q-trace measure for H_s^p, we have that in both when $p \leq 1$ and $p \leq q$ or $1 < p < q$, the q-trace measures and the weak q-trace measures for H_s^p coincide.

Summarizing we have the following theorem that include the statements of Theorem 1.4 of the introduction.

Theorem 7.1. *Let $0 < s < n$ and μ a finite positive Borel measure on $\mathbf{S}^n$ and suppose either $p \leq 1$ and $p \leq q$ or $1 < p < q$. We then have that the following assertions are equivalent*

(i) *The measure μ is a weak q-trace measure for H_s^p.*
(ii) *The measure μ is a q-trace measure for H_s^p.*
(iii) *The measure μ is a weak q-trace measure for $K_s[L^p]$.*
(iv) *The measure μ is a q-trace measure for $K_s[L^p]$.*
(v) *There exists $C > 0$ such that for any $\zeta \in \mathbf{S}^n$, $r > 0$,*

$$\mu(B(\zeta, r)) \leq C r^{(n-sp)\frac{q}{p}}.$$

7.2. The case $q < p$. The case $q < p$ is more difficult and the problem of the q-trace measures and even the weak q-trace measures is still far from being solved. We will give some results for some particular cases.

We begin dealing with the case $q < p$ and $p \leq 1$. In Theorem C in [**CaOr1**] was obtained a characterization of the q-trace measures for H_s^p when $q < p$ and $p \leq 1$. Here for the weak q-trace measures we have the Theorem 1.5.

7.3. Proof of Theorem 1.5. Assume that $\sup_{r_B<\delta}\left(\frac{\mu(B)}{r_B^{n-sp}}\right)^{\frac{1}{p}}\chi_B(\zeta)\in L^{\frac{pq}{p-q},\infty}(d\mu)$. If we fix $r<q$, Kolmogorov's condition and Hölder's inequality with exponent $\frac{p}{r}>1$, gives that

$$\begin{aligned}
&\|\mathcal{M}_{\text{rad}}[f]\|_{L^{q,\infty}(d\mu)}\\
&\approx \sup_E \mu(E)^{\frac{r-q}{rq}}\left(\int_E \mathcal{M}_{\text{rad}}[f](\zeta)^r d\mu(\zeta)\right)^{\frac{1}{r}}\\
&\le \sup_E \mu(E)^{\frac{r-q}{rq}}\left(\int_{\mathbf{S}^n} \mathcal{M}_{\text{rad}}[f](\zeta)|^p \frac{d\mu_E(\zeta)}{\sup_{r_B<\delta}\frac{\mu_E(B)}{r_B^{n-sp}}\chi_B(\zeta)}\right)^{\frac{1}{p}}\\
&\times\left(\int_E\left(\sup_{r_B<\delta}\frac{\mu_E(B)}{r_B^{n-sp}}\chi_B\right)^{\frac{r}{p-r}}d\mu(\zeta)\right)^{\frac{p-r}{rp}}.
\end{aligned}$$

Since $r<p$, we have that $\frac{pr}{p-r}<\frac{pq}{p-q}$ and $\frac{r-q}{rq}=\frac{\frac{pr}{p-r}-\frac{pq}{p-q}}{\frac{prpq}{(p-r)(p-q)}}$. Consequently, Kolmogorov's condition and the hypothesis give that the above supremmum can be bounded, up to a constant, by

$$\sup_E\left(\int_{\mathbf{S}^n}|\mathcal{M}_{\text{rad}}[f](\zeta)|^p\frac{d\mu_E(\zeta)}{\sup_{r_B<\delta}\frac{\mu_E(B)}{r_B^{n-sp}}\chi_B(\zeta)}\right)^{\frac{1}{p}}.$$

If we denote by $d\mu_1(\zeta)=\frac{d\mu_E(\zeta)}{sup_B\frac{\mu_E(B)}{r_B^{n-sp}}\chi_B(\zeta)}$, then the measure μ_1 satisfies that for any $r_B<\delta$,

$$\mu_1(B)=\int_B d\mu_1(\zeta)\le\int_B\frac{d\mu_E(\zeta)}{\frac{\mu_E(B)}{r_B^{n-sp}}}=r_B^{n-sp}.$$

Hence, since $p\le 1$, Theorem 7.1 gives that there exists $C>0$ such that for any $f\in H_s^p$,

$$\left(\int_{\mathbf{S}^n}|\mathcal{M}_{\text{rad}}[f](\zeta)|^p d\mu_1(\zeta)\right)^{\frac{1}{p}}\le C\|f\|_{H_s^p}.$$

and that finishes the proof of the sufficiency.

For the proof of the necessity, from the properties of the adjacent system we easily deduce that it is enough to prove that if μ is a weak q- trace measure for H_s^p, then for $j\in\{1,\dots,M\}$,

$$\sup_{Q\in\mathcal{D}_j,\,r_Q<1}\left(\frac{\mu(Q)}{r_Q^{n-sp}}\right)^{\frac{1}{p}}\chi_B(\zeta)\in L^{\frac{pq}{p-q},\infty}(d\mu).$$

We fix $0<\beta<1$ such that $n-sp<\beta p$. We then have that if $j\in\{1,\dots,M\}$, $Q\in\mathcal{D}_j$, $\zeta_Q\in Q$, and $(\lambda_Q)_{Q\in\mathcal{D}_j}$, $\lambda_Q\ge 0$, the holomorphic function on $\mathbf{B}^n$, F, given by

$$F(z)=\sum_{Q\in\mathcal{D}_j,\,r_Q<1}\lambda_Q\frac{1}{(1-z(1-r_Q)\overline{\zeta_Q})^\beta},$$

satisfies the following assertions:

(i) For any $\zeta \in \mathbf{S}^n$,

$$\begin{aligned}\mathrm{Re} M_{\mathrm{rad}}[F](\zeta) &\geq M_{\mathrm{rad}}\left[\sum_{Q\in\mathcal{D}_m,\ r_Q<1} \lambda_Q \mathrm{Re}\frac{1}{(1-z(1-r_Q)\overline{\zeta_Q})^\beta}\right](\zeta)\\ &\succeq \sum_{Q\in\mathcal{D}_m,\ r_Q<1} \lambda_Q \frac{1}{r_Q^\beta}\chi_Q(\zeta).\end{aligned}$$

(ii) $\|F\|_{H^p_s} \lesssim \left(\sum_{Q\in\mathcal{D}_m,\ r_Q<1} \lambda_Q^p r_Q^{n-(\beta+s)p}\right)^{\frac{1}{p}}$.

Using again the Kolmogorov expression of the $L^{q,\infty}(d\mu)$-norm, we deduce from the above estimates that

$$\left\|\sum_{Q\in\mathcal{D}_j,\ r_Q<1} \lambda_Q \frac{\chi_Q}{r_Q^\beta}\right\|_{L^{q,\infty}(d\mu)} \lesssim \left(\sum_{Q\in\mathcal{D}_j,\ r_Q<1} \lambda_Q^p r_Q^{n-(\beta+s)p}\right)^{\frac{1}{p}}.$$

This condition can be rewritten, denoting $\gamma_Q = \lambda_Q r_Q^{n/p-(\beta+s)}$, as

$$\left\|\sum_{Q\in\mathcal{D}_j,\ r_Q<1} \gamma_Q r_Q^{s-\frac{n}{p}} \chi_Q\right\|_{L^{q,\infty}(d\mu)} \lesssim \left(\sum_{Q\in\mathcal{D}_j,\ r_Q<1} \gamma_Q^p\right)^{\frac{1}{p}},$$

which by Theorem 1.1 in [**Ve**] is equivalent to the condition

$$\sup_{Q\in\mathcal{D}_j,\ r_Q<1}\left(\frac{\mu(Q)}{r_Q^{n-sp}}\right)^{\frac{1}{p}} \chi_Q(\zeta) \in L^{\frac{pq}{p-q},\infty}(d\mu).$$

7.4. Proof of Theorem 1.6. If $n - sp < 1$ and q and p are both strictly bigger than one, it can be deduced from [**CaOrVe**] a characterization of the weak q-trace measures for H^p_s in terms of the non-isotropic Wolff potential, similar to the one obtained for the strong q-trace measures. If $n-sp < 1$ but now $q \leq 1 < p$, an argument using the Kolmogorov expression of $L^{p,\infty}(d\mu)$ together with Theorem D in [**CaOr1**], where it is proved the strong q-trace characterization can be adapted to obtain the characterization of Theorem 1.6 in a similar way to the proof of Theorem 1.5.

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