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# CONGRUENCES BETWEEN HILBERT MODULAR FORMS: CONSTRUCTING ORDINARY LIFTS, II

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Abstract. In this paper, we improve on the results of our earlier paper [BLGG12], proving a near-optimal theorem on the existence of ordinary lifts of a mod $l$ Hilbert modular form for any odd prime $l$.

Contents



## 1. Introduction

Let $F$ be a totally real field with absolute Galois group $G_F$, and let $l$ be an odd prime number. In our earlier paper [BLGG12], we proved a general result on the existence of ordinary modular lifts of a given modular representation $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$; we refer the reader to the introduction of *op. cit.* for a detailed discussion of the problem of constructing such a lift, and of our techniques for doing so.

The purpose of this paper is to improve on the hypotheses imposed on $\overline{\rho}$, removing some awkward assumptions on its image; in particular, if $l = 3$ then the results of [BLGG12] were limited to some cases where $\overline{\rho}$ was induced from a quadratic character, whereas our main theorem is the following.

**Theorem A.** *Suppose that $l > 2$ is prime, that $F$ is a totally real field, and that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible and modular. Assume that $\overline{\rho}|_{G_{F_v}}$ is reducible at all places $v|l$ of $F$.*

*If $l = 5$ and the projective image of $\overline{\rho}|_{G_{F(\zeta_5)}}$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_5)$, assume further that there is a finite solvable totally real extension $F'/F$ such that $\overline{\rho}|_{G_{F'}}$ is conjugate to a representation valued in $\mathrm{GL}_2(\mathbb{F}_5)$.*

*Then $\overline{\rho}$ has a modular lift $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$, which is ordinary at all places $v|l$.*

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(Note that the assumption that $\overline{\rho}|_{G_{F_v}}$ is reducible at all places $v|l$ of $F$ is necessary.) Our methods are based on those of [BLGG12]. The reason that we are now able to prove a stronger result is that the automorphy lifting results that we employed in [BLGG12] have since been optimized in [BLGGT10] and [Tho12]; in particular, we make extensive use of the results of the appendix to [BLGG13], which improves on a lifting result of [BLGGT10], and classifies the subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$, which are adequate in the sense of [Tho12]. In Section 2, we use these results to prove Theorem A, except in the case that $l = 3$ or 5 and the projective image of $\overline{\rho}(G_{F(\zeta_l)})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_l)$, and certain cases where $\overline{\rho}$ is dihedral. In the dihedral cases, the result is proved in [All12]. In the remaining cases, the adequacy hypothesis we require fails, but in Section 3 we handle this case completely when $l = 3$ by making use of the Langlands–Tunnell theorem, and we prove a partial result when $l = 5$ using the results of [SBT97].

**1.1. Notation.** If $M$ is a field, we let $G_M$ denote its absolute Galois group. We write $\bar{\varepsilon}$ for the mod $l$ cyclotomic character. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and regard all algebraic extensions of $\mathbb{Q}$ as subfields of $\overline{\mathbb{Q}}$. For each prime $p$ we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. In this way, if $v$ is a finite place of a number field $F$, we have a homomorphism $G_{F_v} \hookrightarrow G_F$. We also fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

We normalize the definition of Hodge–Tate weights so that all the Hodge–Tate weights of the $l$-adic cyclotomic character $\varepsilon$ are $-1$. We refer to a two-dimensional potentially crystalline representation with all pairs of labelled Hodge–Tate weights equal to $\{0, 1\}$ as a weight 0 representation. (The reason for this terminology is that the Galois representations associated to an automorphic representation, which is cohomological of weight 0 have these Hodge–Tate weights.)

If $F$ is a totally real field, then a continuous representation $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is said to be *modular* if there exists a regular algebraic automorphic representation $\pi$ of $\mathrm{GL}_2(\mathbb{A}_F)$, such that $\bar{r}_l(\pi) \cong \bar{r}$, where $r_l(\pi)$ is the $l$-adic Galois representation associated with $\pi$.

We let $\zeta_l$ be a primitive $l$th root of unity.

## 2. The adequate case

**2.1.** The notion of an *adequate* subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is defined in [Tho12]. We will not need to make use of the actual definition; instead, we will use the following classification result. Note that by definition an adequate subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ necessarily acts irreducibly on $\overline{\mathbb{F}}_l^n$.

**Proposition 2.1.1.** *Suppose that $l > 2$ is a prime, and that $G$ is a finite subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$, which acts irreducibly on $\overline{\mathbb{F}}_l^2$. Then precisely one of the following is true:*

- *We have $l = 3$, and the image of $G$ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_3)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$.*
- *We have $l = 5$, and the image of $G$ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_5)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$.*
- *$G$ is adequate.*

*Proof.* This is Proposition A.2.1 of [BLGG13]. ∎

In the case that $\overline{\rho}(G_{F(\zeta_l)})$ is adequate, our main result follows exactly as in section 6 of [BLGG12], using the results of Appendix A of [BLGG13] (which in turn build on the results of [BLGGT10]). We obtain the following theorem.

**Theorem 2.1.2.** *Suppose that $l > 2$ is prime, that $F$ is a totally real field, and that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible and modular. Suppose also that $\overline{\rho}(G_{F(\zeta_l)})$ is adequate. Then:*

(1) *There is a finite solvable extension of totally real fields $L/F$ which is linearly disjoint from $\overline{F}^{\ker \overline{\rho}}$ over $F$, such that $\overline{\rho}|_{G_L}$ has a modular lift $\rho_L : G_L \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ of weight $0$, which is ordinary at all places $v|l$.*
(2) *If furthermore $\overline{\rho}|_{G_{F_v}}$ is reducible at all places $v|l$, then $\overline{\rho}$ itself has a modular lift $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ of weight $0$, which is ordinary at all places $v|l$.*

*Proof.* First, note that (2) is easily deduced from (1) using the results of Section 3 of [Gee11] (which build on Kisin's reinterpretation of the Khare–Wintenberger method). Indeed, the proofs of Theorems 6.1.5 and 6.1.7 of [BLGG12] go through unchanged in this case.

Similarly, (1) is easily proved in the same way as Proposition 6.1.3 of [BLGG12] (and in fact the proof is much shorter). First, note that the proof of Lemma 6.1.1 of [BLGG12] goes through unchanged to show that there is a finite solvable extension of totally real fields $L/F$ which is linearly disjoint from $\overline{F}^{\ker \overline{\rho}}$ over $F$, such that $\overline{\rho}|_{G_L}$ has a modular lift $\rho' : G_L \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ of weight 0 which is potentially crystalline at all places dividing $l$, and in addition both $\overline{\rho}|_{G_{L_w}}$ and $\overline{\varepsilon}|_{G_{L_w}}$ are trivial for each place $w|l$ (and in particular, $\overline{\rho}|_{G_{L_w}}$ admits an ordinary lift of weight 0), and $\overline{\rho}$ is unramified at all finite places. By Lemma 4.4.1 of [GK12], $\rho'|_{G_{L_w}}$ is potentially diagonalizable in the sense of [BLGGT10] for all places $w|l$ of $L$.

Choose a CM quadratic extension $M/L$ that is linearly disjoint from $L(\zeta_l)$ over $L$, in which all places of $L$ dividing $l$ split. We can now apply Theorem A.4.1 of [BLGG13] (with $F' = F = M$, $S$ the set of places of $L$ dividing $l$, and $\rho_v$ an ordinary lift of $\overline{\rho}|_{G_{L_w}}$ for each $w|l$) to see that $\overline{\rho}|_{G_M}$ has an ordinary automorphic lift $\rho_M : G_M \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ of weight 0.

The argument of the last paragraph of the proof of Proposition 6.1.3 of [BLGG12] (which uses the Khare–Wintenberger method to compare deformation rings for $\overline{\rho}|_{G_L}$ and $\overline{\rho}|_{G_M}$) now goes over unchanged to complete the proof. ☐

## 3. Inadequate cases

**3.1. The first inadequate case.** We now consider the case that $l = 3$ and $\overline{\rho}|_{G_{F(\zeta_3)}}$ is irreducible, but $\overline{\rho}(G_{F(\zeta_3)})$ is not adequate. By Proposition 2.1.1, this means that the projective image of $\overline{\rho}(G_{F(\zeta_3)})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_3)$, and is in particular solvable. We now use the Langlands–Tunnell theorem to prove our main theorem in this case.

**Theorem 3.1.1.** *Suppose that $F$ is a totally real field, and that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_3)$ is irreducible and modular. Assume that $\overline{\rho}|_{G_{F_v}}$ is reducible at all places $v|3$ of $F$, and that the projective image of $\overline{\rho}(G_{F(\zeta_3)})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_3)$.*

*Then $\overline{\rho}$ has a modular lift $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_3)$ which is ordinary at all places $v|3$.*

*Proof.* First, note that since the projective image of $\overline{\rho}(G_{F(\zeta_3)})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_3)$, the projective image of $\overline{\rho}$ itself is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_3)$ or $\mathrm{PGL}_2(\mathbb{F}_3)$ (see, for example, Theorem 2.47(b) of [DDT97]).

Choose a finite solvable extension of totally real fields $L/F$ which is linearly disjoint from $\overline{F}^{\ker\overline{\rho}}$ over $F$, with the further property that $\overline{\rho}|_{G_{L_w}}$ is unramified for each place $w|l$ of $L$. Exactly as in the proof of Theorem 2.1.2, by the results of Section 3 of [Gee11] it suffices to show that $\overline{\rho}|_{G_L}$ has a modular lift of weight 0, which is potentially crystalline at each place $w|l$. By Hida theory, it in fact suffices to find some ordinary modular lift of $\overline{\rho}|_{G_L}$ (not necessarily of weight 0).

Since the projective image of $\overline{\rho}$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_3)$ or $\mathrm{PGL}_2(\mathbb{F}_3)$, the image of $\overline{\rho}$ is contained in $\overline{\mathbb{F}}_3^{\times}\,\mathrm{GL}_2(\mathbb{F}_3)$. Then the Langlands–Tunnell theorem implies that $\overline{\rho}|_{G_L}$ has a modular lift $\rho$ corresponding to a Hilbert modular form of parallel weight one. This follows from the discussion after Theorem 5.1 of [Wil95] which also shows that the natural map $\rho(G_L) \to \overline{\rho}(G_L)$ may be assumed to be an isomorphism. Since $\overline{\rho}|_{G_{L_w}}$ is unramified at each place $w|l$ of $L$, this implies that $\rho$ is ordinary, as required. □

**3.2. The second inadequate case.** We now suppose that $l = 5$, that $\overline{\rho}|_{G_{F(\zeta_5)}}$ is irreducible but its image is not adequate. Then $\overline{\rho}(G_{F(\zeta_5)})$ has projective image conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$, and we see that $\overline{\rho}(G_F)$ has projective image conjugate to either $\mathrm{PGL}_2(\mathbb{F}_5)$ or $\mathrm{PSL}_2(\mathbb{F}_5)$. (This follows from [DDT97, Prop. 2.47].) Thus, after conjugating, we may assume that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_5)$ takes values in $\overline{\mathbb{F}}_5^{\times}\,\mathrm{GL}_2(\mathbb{F}_5)$.

In order to apply the results of [SBT97], we need to assume further that there is a finite solvable totally real extension $F'/F$ such that $\overline{\rho}|_{G_{F'}}$ is valued in $\mathrm{GL}_2(\mathbb{F}_5)$. (This condition is not automatic, but it holds if the projective image of $\overline{\rho}(G_F)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_5)$.)

**Theorem 3.2.1.** *Suppose that $F$ is a totally real field, and that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_5)$ is irreducible and modular. Assume that $\overline{\rho}|_{G_{F_v}}$ is reducible at all places $v|5$ of $F$, and that the projective image of $\overline{\rho}(G_{F(\zeta_5)})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_5)$. Assume further that there is a finite solvable totally real extension $F'/F$ so that $\overline{\rho}|_{G_{F'}}$ is conjugate to a representation valued in $\mathrm{GL}_2(\mathbb{F}_5)$.*

*Then $\overline{\rho}$ has a modular lift $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_5)$ which is ordinary at all places $v|5$.*

*Proof.* Since $\overline{\rho}$ is totally odd, we can replace $F'/F$ by a further finite solvable totally real extension and assume that $\overline{\rho}|_{G_{F'}}$ takes values in $\mathrm{GL}_2(\mathbb{F}_5)$ and has determinant equal to the cyclotomic character. Now, as in the proof of Theorem 2.1.2, to prove the current theorem, it suffices to show that $\overline{\rho}|_{G_{F'}}$ has a modular lift of weight 0, which is ordinary at each $v|5$. (The only thing that needs to be checked is that Proposition 3.1.5 of [Gee11] applies to $\overline{\rho}|_{G_{F'}}$. The only hypothesis which is not immediate is that if the projective image of $\overline{\rho}|_{G_{F'}}$ is $\mathrm{PGL}_2(\mathbb{F}_5)$, then $[F'(\zeta_5) : F'] = 4$. To see this, note that if $[F'(\zeta_5) : F'] = 2$, then since the determinant of $\overline{\rho}|_{G_{F'}}$ is the mod 5 cyclotomic character, it has image $\{\pm 1\}$. This implies that the projective image is $\mathrm{PSL}_2(\mathbb{F}_5)$, as required.)

By [SBT97, Theorem 1.2], there exists an elliptic curve $E/F'$ such that $E[5] \cong \overline{\rho}|_{G_{F'}}$ and the image of $G_{F'}$ in $\mathrm{Aut}(E[3])$ contains $\mathrm{SL}_2(\mathbb{F}_3)$ (and hence its image is equal to $\mathrm{Aut}(E[3])$ since the determinant is totally odd). We may further suppose that $E$ has good ordinary reduction at each prime of $F'$ dividing 5. (To see this, note that we may incorporate Ekedahl's effective version of the Hilbert Irreducibility Theorem [Eke90] into the proof of [SBT97, Theorem 1.2] exactly as is done in [Tay03, Lemma 2.3].) By the Langlands–Tunnell theorem, $E[3]$ has a modular lift corresponding to a Hilbert modular form $f_0$ of parallel weight 1. Replacing $F'$ by a finite totally real solvable extension linearly disjoint from $\overline{F'}^{\ker E[3]}$, we may assume that $f_0$ is ordinary at each prime dividing 3. By Hida theory, $E[3]$ then has a modular lift corresponding to a Hilbert modular form of parallel weight 2, which is ordinary at each prime dividing 3. Note that the conditions of the modularity lifting theorem [Gee09, Theorem 1.1], applied to $\rho := T_3E$, are satisfied. (For the third condition, note that $E[3]|_{G_{F'(\zeta_3)}}$ is irreducible as $E[3]|_{G_{F'}}$ has non-dihedral image.) It follows that $T_3E$ is modular and hence that $T_5E$ is modular. Thus we have exhibited a modular lift of $\overline{\rho}|_{G_{F'}} \cong E[5]$ which has weight 0 and is ordinary at each prime above 5. □

Finally, we deduce our main result from Theorems 2.1.2, 3.1.1 and 3.2.1.

*Proof of Theorem A.* If $\overline{\rho}|_{G_{F(\zeta_l)}}$ is reducible, then $\overline{\rho}$ is dihedral, and the result follows from Lemma 5.1.2 of [All12]. If $l = 3$ (respectively $l = 5$) and the projective image of $\overline{\rho}(G_{F(\zeta_l)})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_l)$, then the result follows from Theorem 3.1.1 (respectively, from Theorem 3.2.1). In all other cases, we see from Proposition 2.1.1 that $\overline{\rho}(G_{F(\zeta_l)})$ is adequate and the result follows from Theorem 2.1.2(2). □

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