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GENUS ONE CURVES AND BRAUER–SEVERI VARIETIES

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1. Introduction

Let K be a field. Let A be a central simple algebra over K and let X be the associated Brauer–Severi variety over K. An interesting question that has recently been asked [2, 7] is whether there exists a genus one curve C over K such that $K(C)$ splits A. In other words, is there a genus one curve C over K with a morphism $C \to X$?

In this short note, we explicitly construct such a genus one curve in case X has dimension ≤ 4 (equivalently, when A has degree $d \leq 5$). The methods we use do not seem to generalize to higher-dimensional X. One obstacle is that we rely upon certain explicit descriptions of the moduli spaces of genus one curves with degree d divisor classes, which are not known for higher d. In fact, we expect the question has a negative answer in general.

We would like to point the reader to related work [4, 1] regarding which Brauer classes are split by a given (genus 1) curve over K. Also, note that if K is a finite field, then the question is trivial because the Brauer group is, so in the rest of this note we assume K is infinite.

2. Index 2 and Index 3

These cases are covered by previous work (e.g., [8]). We briefly describe constructions for these two cases, since the higher cases below are similar in spirit.

Let A be a quaternion algebra over K, and let X be a genus zero curve representing the same Brauer class. Let L be a degree 2 line bundle on X, so a section of $L^{\otimes 2}$ cuts out a degree 4 subscheme D of X. Then there exists a double cover C of X ramified exactly at D by the cyclic covering trick [5, Proposition 4.1.3]. The genus of C is 1. For a general section, when the characteristic of K is different from 2, the curve C will be a smooth irreducible genus one curve.

Now let A be a central simple algebra over K of degree 3, with X the corresponding Brauer–Severi variety. Then the inverse of the canonical bundle of X is a line bundle whose general sections cut out genus one curves.

3. Index 4

Let A be a central simple algebra over K of degree 4 and let X be the corresponding Brauer–Severi variety. Let $\alpha \in \mathrm{Br}(K)$ be the class of A, so α is a nontrivial element of index 4 in $\mathrm{Br}(K)$ and has period 2 or 4. By [6, Corollary 15.2.a], the class of 2α has index 2 or 1. Let Y be a Brauer–Severi variety of dimension 1 whose Brauer class is 2α.

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It is well known that the intersection of two general sections of $\mathcal{O}_{\mathbb{P}^3}(2)$ is a smooth irreducible genus one curve. Another way to describe this curve is as the zero locus of a general section of the pushforward $\pi_*\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(2,1)$, where $\pi : \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^3$ is the first projection. To generalize this construction for our situation, we descend this vector bundle to $X \times Y$.

We claim that the line bundle $\mathcal{O}_{X_{\bar{K}}}(2) \boxtimes \mathcal{O}_{Y_{\bar{K}}}(1)$ on $X_{\bar{K}} \times Y_{\bar{K}}$ descends to a line bundle $\mathcal{L}$ on $X \times Y$. In other words, we want to show that $\mathcal{O}_{X_{\bar{K}}}(2) \boxtimes \mathcal{O}_{Y_{\bar{K}}}(1)$ is in the image of the map

$$\mathrm{Pic}(X \times Y) \to \mathrm{Pic}(X_{\bar{K}} \times Y_{\bar{K}}),$$

and more precisely, in the image of the map

$$\mathrm{Pic}(X \times Y) \to \mathrm{Pic}(X_{\bar{K}} \times Y_{\bar{K}})^{\mathrm{Gal}(\bar{K}/K)}.$$

The next term in the low degree exact sequence coming from the Leray spectral sequence for the map $X_{\bar{K}} \to X$ with coefficients in $\mathbb{G}_m$ is the Brauer group $\mathrm{Br}(K)$. Similarly, there is an exact sequence

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\bar{K}})^{\mathrm{Gal}(\bar{K}/K)} \to \mathrm{Br}(K).$$

The obstruction to the line bundle $\mathcal{O}_{X_{\bar{K}}}(1)$ coming from a line bundle on X is exactly the class α in $\mathrm{Br}(K)$, so because the differential is a homomorphism, the obstruction for $\mathcal{O}_{X_{\bar{K}}}(2)$ is 2α. Similarly, the obstruction for $\mathcal{O}_{Y_{\bar{K}}}(1)$ is 2α. By the Künneth formula, the obstruction for $\mathcal{O}_{X_{\bar{K}}}(2) \boxtimes \mathcal{O}_{Y_{\bar{K}}}(1)$ is $2\alpha + 2\alpha = 0$.

Therefore, there exists a line bundle $\mathcal{L}$ on $X \times Y$ as above, and the pushforward $\pi_*\mathcal{L}$ via the projection $\pi : X \times Y \to X$ is a rank 2 vector bundle on X. By base change, the bundle $\pi_*\mathcal{L}$ on X has many sections, and a general section cuts out a genus one curve on X.

4. Index 5

Let A be central simple algebra A over K of degree 5. Let $\alpha \in \mathrm{Br}(K)$ be the class of A, so α is a nontrivial element in $\mathrm{Br}(K)[5]$. Let X and Y be Brauer–Severi varieties representing the classes α and 2α, respectively. We construct a genus one curve in X by finding a general section of a vector bundle over $X \times Y$.

The following observation may be found in [3] and was explained to us by Laurent Gruson (private communication). The sheaf $\mathcal{O}_{X_{\bar{K}}}(1) \boxtimes \Omega^1_{Y_{\bar{K}}}(2)$ is a rank 4 vector bundle on $X_{\bar{K}} \times Y_{\bar{K}}$. The zero locus of a general section is a closed smooth subvariety of $X_{\bar{K}} \times Y_{\bar{K}}$ whose projection to $X_{\bar{K}}$ is a smooth irreducible genus one curve.

We claim that the vector bundle $\mathcal{O}_{X_{\bar{K}}}(1) \boxtimes \Omega^1_{Y_{\bar{K}}}(2)$ on $X_{\bar{K}} \times Y_{\bar{K}}$ descends to a vector bundle $\mathcal{E}$ over K. Because $\mathcal{O}_{X_{\bar{K}}} \boxtimes \Omega^1_{Y_{\bar{K}}}$ certainly descends, we want to show that the line bundle $\mathcal{O}_{X_{\bar{K}}\times Y_{\bar{K}}}(1,2)$ is in the image of the map

$$\mathrm{Pic}(X \times Y) \to \mathrm{Pic}(X_{\bar{K}} \times Y_{\bar{K}})^{\mathrm{Gal}(\bar{K}/K)}.$$

As in the index 4 case, the obstruction lies in $\mathrm{Br}(K)$, and an almost identical computation shows that it is $\alpha + 2(2\alpha) = 5\alpha = 0$ in $\mathrm{Br}(K)$.

By base change, the vector bundle $\mathcal{E}$ on $X \times Y$ has many sections, so we may take a general section as above. The projection to X of the zero locus of this section is a genus one curve, as desired.

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