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COMPLETELY INTEGRABLE TORUS ACTIONS ON COMPLEX MANIFOLDS WITH FIXED POINTS

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Abstract. We show that if a holomorphic n-dimensional compact torus action on a compact connected complex manifold of complex dimension n has a fixed point then the manifold is equivariantly biholomorphic to a smooth toric variety.

1. Introduction

We begin by recalling some notions from the theory of toric varieties.

We work in the vector space $\mathrm{Lie}(S^1)^n \cong \mathbb{R}^n$ with the lattice $\mathrm{Hom}(S^1, (S^1)^n) \cong \mathbb{Z}^n$. Here, we identify $\mathrm{Lie}(S^1)$ with $\mathbb{R}$ such that the exponential map $\exp\colon \mathbb{R} \to S^1$ is $t \mapsto e^{2\pi i t}$.

A *unimodular fan* is a finite set Δ of convex polyhedral cones with the following properties.

(1) A face of a cone in Δ is also a cone in Δ.
(2) The intersection of two cones in Δ is a common face.
(3) Every cone in Δ is unimodular, i.e., it has the form $\mathrm{pos}(\lambda_1, \dots, \lambda_k)$ where $\lambda_1, \dots, \lambda_k$ is part of a $\mathbb{Z}$-basis of the lattice. Here, pos denotes the positive span: the set of linear combinations with non-negative coefficients.[1]

A fan Δ is *complete* if the union of the cones in Δ is all of $\mathrm{Lie}(S^1)^n$.

The theory of toric varieties associates to a unimodular fan Δ a complex manifold M_Δ with a holomorphic $(\mathbb{C}^*)^n$-action with the following properties.

(1) The fixed points in M_Δ are in bijection with the n-dimensional cones in Δ.
(2) Let p be a fixed point in M_Δ. Then the isotropy weights at p are a $\mathbb{Z}$-basis to the lattice $\mathrm{Hom}((S^1)^n, S^1) \subset (\mathrm{Lie}(S^1)^n)^*$. Moreover, let $\lambda_1, \dots, \lambda_n$ be the dual basis; then the cone in Δ that corresponds to p is $\mathrm{pos}(\lambda_1, \dots, \lambda_n)$.
(3) The manifold M_Δ is compact if and only if the fan Δ is complete.

Explicitly, let $\lambda_1, \dots, \lambda_m \in \mathbb{Z}^n$ be the primitive generators of the one-dimensional cones in Δ. Each λ_i encodes a homomorphism $a \mapsto a^{\lambda_i}$ from $\mathbb{C}^*$ to $(\mathbb{C}^*)^n$; together they give a homomorphism $\pi\colon (a_1, \dots, a_m) \mapsto \prod_{j=1}^m a_j^{\lambda_j}$ from $(\mathbb{C}^*)^m$ to $(\mathbb{C}^*)^n$. Then $M_\Delta = U_\Delta / K_\Delta$, where $U_\Delta = \{z \in \mathbb{C}^m \mid \mathrm{pos}(\lambda_i \,|\, z_i = 0) \in \Delta\}$ and $K_\Delta = \ker \pi$. For the details of the construction and the proof of its properties, we refer the reader to the book [3] by Cox et al. and to the book [1] by Audin.

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[1] This property of a cone or a fan is also described in the literature by the adjectives *smooth*, *non-singular*, *regular*, and *Delzant*.

In fact, M_Δ is an *algebraic* variety. Moreover, every smooth complex algebraic variety that is equipped with an algebraic $(\mathbb{C}^*)^n$-action with an open dense free orbit is isomorphic to some M_Δ. (The proof of this fact appeared in the book [11] by Kempf et al. and in the paper [15] by Miyake and Oda and relies on a lemma of Sumihiro [16]; see Corollary 3.1.8 in [3].) Our main theorem is a complex analytic variant of this result:

Theorem 1. *Let M be a connected complex manifold of complex dimension n equipped with a faithful action of the torus $(S^1)^n$ by biholomorphisms. If M is compact and the action has fixed points, then there exists a unimodular fan Δ and an $(S^1)^n$-equivariant biholomorphism of M_Δ with M.*

Remark 2.

(1) Our theorem gives a negative answer to a question that was raised by Buchstaber and Panov in [2, Problem 5.23].

Let M be a closed $2n$-dimensional manifold with an $(S^1)^n$-action that is locally standard: every orbit has a neighbourhood that is equivariantly diffeomorphic, up to an automorphism of $(S^1)^n$, to an invariant open subset of $\mathbb{C}^n$ with the standard $(S^1)^n$-action. Also assume that the quotient $M/(S^1)^n$ is diffeomorphic, as a manifold with corners, to a simple convex polytope P in $\mathbb{R}^n$.[2] Such manifolds, introduced in [4] and studied in the toric topology community, are called *quasi-toric manifolds*[3].

The question of Buchstaber and Panov is whether there exists a non-toric quasitoric manifold that admits an $(S^1)^n$-invariant complex structure.

(2) Our theorem strengthens an earlier result of Ishida and Masuda, that if a closed complex manifold of complex dimension n admits an $(S^1)^n$-action, and if its odd-degree cohomology groups vanish, then the Todd genus of the manifold is equal to one. See [9, Theorem 1.1 and Remark 1.2].

(3) In Theorem 1, the assumption "complex" cannot be weakened to "almost complex". For example, for every two complex toric manifolds of complex dimension 2, their equivariant connected sum along a free orbit supports an invariant almost complex structure, has fixed points, but is not (equivariantly diffeomorphic to) a toric manifold; see [10, Section 11.2]. For higher-dimensional analogues, see [6, Section 13]; for more interesting four-dimensional examples, see [14, Theorem 5.1]. A necessary and sufficient condition for a quasitoric manifold to admit an invariant almost complex structure was given in [13, Theorem 1].

(4) The symplectic analogue of Theorem 1 is also true: a closed symplectic manifold of dimension $2n$ with a faithful $(S^1)^n$ action with at least one fixed point is a symplectic toric manifold. To see this, it is enough to show that such an action is Hamiltonian; being a toric manifold then follows from Delzant's

[2] A map from $M/(S^1)^n$ to P is a diffeomorphism of manifolds with corners if and only if it is a homeomorphism and, for every real valued function on P, the function extends to a smooth function on $\mathbb{R}^n$ if and only if its pullback to M is smooth. For every $k \in \{0, \dots, n\}$, a diffeomorphism carries the k-dimensional orbits in M to the relative interiors of the k-dimensional faces of P.

[3] Davis–Januszkiewicz [4] used the term *toric manifold*, but this term was already used in the literature to mean a smooth toric variety, so Buchstaber–Panov [2] introduced instead the term *quasitoric manifold*.

theorem [5, Théorème 2.1]. Let p a fixed point. There exist n subcircles of $(S^1)^n$ that span $(S^1)^n$ and whose isotropy weights are all positive. In order to show that the $(S^1)^n$ action is Hamiltonian, it is enough to show that each of these S^1 actions has a momentum map. Fix one of these S^1 actions. Since there is a fixed point, the S^1 orbits are null-homotopic, so the S^1 action lifts to an S^1 action on the universal bundle, $\tilde{M}$. As $H^1(\tilde{M}) = 0$, this lifted action is Hamiltonian. By Morse theory, at most one point of $\tilde{M}$ can be a strict local minimum for the momentum map (see, e.g., [7]). So the fibre of $\tilde{M}$ over the fixed point p can contain only one point. So $\tilde{M} = M$, and so there is a momentum map on M.

(5) It is necessary to assume that the action has fixed points: the complex torus $\mathbb{C}^*/(z \sim 2z)$ has a holomorphic S^1-action, induced from multiplication on $\mathbb{C}^*$, but it is not a toric variety: the $\mathbb{C}^*$-action is not faithful.

(6) It is necessary to assume that the manifold is compact: the open unit disc in $\mathbb{C}$ with the natural circle action has a fixed point, but it is not a toric variety: the circle action does not extend to a $\mathbb{C}^*$-action.

2. The complexified action

Let the torus $(S^1)^n$ act on a complex manifold M by biholomorphisms. If the manifold M is compact, then the $(S^1)^n$-action extends to a $(\mathbb{C}^*)^n$-action that is holomorphic not only in the sense that each element of $(\mathbb{C}^*)^n$ acts by a biholomorphism but also in the sense that the action map $(\mathbb{C}^*)^n \times M \to M$ is holomorphic. See, e.g., [8, Theorem 4.4]. For the convenience of the reader, we briefly recall here some of the details of this standard construction.

Let $\xi_1, \dots, \xi_n$ be the fundamental vector fields of the $(S^1)^n$-action with respect to the coordinate one-dimensional subtori. Let $J\colon TM \to TM$ be the multiplication by $\sqrt{-1}$. We claim that the vector fields $-J\xi_1, \dots, -J\xi_n$ are holomorphic (in the sense that their flows preserve the complex structure) and commute with each other and with the vector fields ξ_i.

As the $(S^1)^n$-action preserves J and ξ_j, it preserves $-J\xi_j$, for each j. So the vector fields $-J\xi_j$ commute with the vector fields ξ_i that generate this action. Since J is a complex structure, its Nijenhuis tensor,

$$N(Z,W) := 2\left([JZ,JW] - J[Z,JW] - J[JZ,W] - [Z,W]\right),$$

vanishes. Setting $Z = \xi_i$ and $W = \xi_j$, we get that $[J\xi_i, J\xi_j] = J[\xi_i, J\xi_j] + J[J\xi_i, \xi_j] + [\xi_i, \xi_j]$, and each of the three terms on the right hand side is zero. So the vector fields $-J\xi_j$ commute with each other. A vector field Y is holomorphic if and only if $[Y, JW] = J[Y,W]$ for each vector W; see [12, Proposition 2.10 in Chapter IX]. Set $Y := -J\xi_i$ and W arbitrary; because $JY (= \xi_i)$ is holomorphic, $[JY, JW] = J[JY, W]$; by the vanishing of the Nijenhuis tensor,

$$\begin{aligned} 0 = N(JY,W) &= 2\left([-Y,JW] - J[JY,JW] - J[-Y,W] - [JY,W]\right) \\ &= 2([-Y,JW] - J[-Y,W]), \end{aligned}$$

so Y is holomorphic.

If M is compact, the vector fields $-J\xi_1, \dots, -J\xi_n$ are complete, and we get an $\mathbb{R}^{2n}$-action, $\mathbb{R}^{2n} \times M \to M$, via

$$\left(\sum_{i=1}^{2n} a_i \mathbf{e}_i, x \right) \mapsto c_x(1),$$

where $c_x(r)$ is the integral curve of the vector field $\sum_{i=1}^{n} -a_i J\xi_i + a_{n+i}\xi_i$ with $c_x(0) = x$. This action descends to a $(\mathbb{C}^*)^n$-action by biholomorphisms that extends the given $(S^1)^n$-action. Finally, the action map $(\mathbb{C}^*)^n \times M \to M$ is holomorphic, because its differential, which at the point (z, m) is the map $\mathbb{C}^n \times T_m M \to T_{z \cdot m} M$ that takes $(2\pi(r_1 + i\theta_1, \dots, r_n + i\theta_n), v)$ to $\sum_j -r_j J\xi_j|_{z\cdot m} + \theta_j \xi_j|_{z \cdot m} + z_* v$, is complex linear.

Remark 3. In the next section we will see that if there exists a fixed point then the extended $(\mathbb{C}^*)^n$-action is faithful. In general, the extended $(\mathbb{C}^*)^n$-action might not be faithful.

Example 4. Let $(S^1)^n$ act on $\mathbb{C}^n$ with weights $\alpha_1, \dots, \alpha_n$:

$$g \cdot (z_1, \dots, z_n) = (g^{\alpha_1} z_1, \dots, g^{\alpha_n} z_n),$$

where $g^{\alpha_i} = g_1^{\alpha_{i1}} \cdots g_n^{\alpha_{in}}$ for $g = (g_1, \dots, g_n) \in (S^1)^n$ and for the isotropy weight $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in}) \in \mathbb{Z}^n$. Then the complexified action is given by the same formula applied to $g = (g_1, \dots, g_n) \in (\mathbb{C}^*)^n$.

3. Structures near fixed points

Let M be a connected complex manifold of complex dimension n. Let the torus $(S^1)^n$ act on M faithfully by biholomorphisms. Let p be a point in M that is fixed by the $(S^1)^n$-action. Let $\alpha_1, \dots, \alpha_n$ be the isotropy weights at p.

Let $\mathbb{C}_{\alpha_i}$ denote the one-dimensional complex vector space $\mathbb{C}$ with the $(S^1)^n$-action that is obtained by composing the homomorphism $(S^1)^n \to S^1$ that is encoded by the weight α_i with the standard action of S^1 on $\mathbb{C}$ by scalar multiplication.

We begin with a local result:

Lemma 5. *There exists an $(S^1)^n$-invariant neighbourhood U_p of p in M, an $(S^1)^n$-invariant neighbourhood $\widetilde{U}_p$ of the origin in T_pM, and an $(S^1)^n$-equivariant biholomorphism $\varphi_p \colon U_p \to \widetilde{U}_p$ whose differential at p is the identity map on T_pM.*

Proof. Let $\varphi \colon U \to \widetilde{U} \subseteq \mathbb{C}^n$ be a local holomorphic chart near p with $\varphi(p) = 0$. Identifying $\mathbb{C}^n$ with T_pM via the differential

$$(d\varphi)_p \colon T_pM \to T_0\mathbb{C}^n \cong \mathbb{C}^n,$$

we get a biholomorphism

$$\varphi' \colon U \to \widetilde{U}' \subseteq T_pM$$

whose differential at p is the identity map on T_pM. We want to obtain such a biholomorphism that is also equivariant.

Set

$$U' := \bigcap_{g \in (S^1)^n} gU.$$

Clearly, U' is invariant and contains p. We now show that U' is open. The complement of U' is the image of the closed subset $(S^1)^n \times (M \setminus U)$ of $(S^1)^n \times M$ under the action map $(S^1)^n \times M \to M$. Since $(S^1)^n$ is compact, the action map is proper. Being proper means that the preimage of every compact set is compact; when the target space M is a manifold[4] it implies that the map is closed. Thus, the complement $M \setminus U'$ is closed, and so U' is open.

To obtain an equivariant chart, we average φ': let

$$\widetilde{\varphi} := \int_{g \in (S^1)^n} (g \circ \varphi' \circ g^{-1})\, dg : \; U' \to T_pM,$$

where dg is Haar measure on $(S^1)^n$. The map $\widetilde{\varphi}$ is holomorphic and $(S^1)^n$-equivariant. Moreover, its differential at p is the identity map on T_pM. By the implicit function theorem, $\widetilde{\varphi}$ restricts to a biholomorphism from some smaller open neighbourhood U'' of p in M to an open neighbourhood of the origin in T_pM. The restriction of $\widetilde{\varphi}$ to the invariant neighbourhood $U_p := \bigcap_{g \in (S^1)^n} g \cdot U''$ of p in M satisfies the requirements of the lemma. □

Corollary 6. *There exists an $(S^1)^n$-equivariant local holomorphic chart*

$$\varphi_p \colon U_p \to \mathbb{D}^n$$

from an invariant open neighbourhood U_p of p to a polydisc $\mathbb{D}^n$ in $\mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$.

Proof. By the definition of the isotropy weights, there exists a complex linear $(S^1)^n$-equivariant isomorphism between the tangent space T_pM and the representation $\mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$. Corollary 6 then follows from Lemma 5 by restricting the chart to the preimage of a polydisc. □

We would like to extend the chart of Corollary 6 to a chart whose image is all of $\mathbb{C}^n$. We can do this when the $(S^1)^n$ extends to a $(\mathbb{C}^*)^n$-action; for example, if the manifold is compact; by "sweeping" by the $(\mathbb{C}^*)^n$-action.

Lemma 7. *Suppose that the $(S^1)^n$-action extends to a $(\mathbb{C}^*)^n$-action. Then there exists an invariant open neighbourhood V_p of p in M and an $(S^1)^n$-equivariant biholomorphism of V_p with $\mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$.*

Proof. Let $\varphi_p \colon U_p \to \mathbb{D}^n$ be an $(S^1)^n$-equivariant holomorphic local chart, as in Corollary 6. Since φ_p is $(S^1)^n$-equivariant and holomorphic, it intertwines the restriction to U_p of the vector fields that generate the complexified $(\mathbb{C}^*)^n$-action on M with the restriction to $\mathbb{D}^n$ of the vector fields that generate the complexified $(\mathbb{C}^*)^n$-action on $\mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$. This, and the fact that φ_p is a diffeomorphism between U_p and $\mathbb{D}^n$, implies that φ_p also intertwines the partial flows on U_p and on $\mathbb{D}^n$ that are generated by these vector fields; in particular it intertwines the domains of definition of these partial flows.

For each $t \in \mathbb{R}$, let g_t be the element of $(\mathbb{C}^*)^n$ that acts on $\mathbb{C}^n$ as scalar multiplication by e^{-t}, and let $\eta \in \mathrm{Lie}(\mathbb{C}^*)^n$ be the generator of the one-parameter subgroup $t \mapsto g_t$. Since $e^{-t}\mathbb{D}^n \subset \mathbb{D}^n$ for all $t \geq 0$, and because φ_p intertwines the domains

[4]In fact, it is enough to assume that the target space is Hausdorff and compactly generated. Compactly generated means that a subset is closed if and only if its intersection with every compact set K is closed in K; this property holds if the space is locally compact or if the space is metrizable.

of definition of the partial flows on U_p and on $\mathbb{D}^n$ that correspond to η, we get that $g_t U_p \subset U_p$ for all $t \geq 0$. So, for every $t \geq 0$, the domain of definition of the $(S^1)^n$-equivariant biholomorphism

$$\varphi_p^{(t)} := (g_t)^{-1} \circ \varphi_p \circ g_t \; : \; g_{-t}U_p \to e^t \mathbb{D}^n$$

contains U_p. Here, $g_t \colon g_{-t}U_p \to U_p$ and $(g_t)^{-1} \colon \mathbb{D}^n \to e^t\mathbb{D}^n$ are given by the complexified actions on M and on $\mathbb{C}^n$. By the choice of g_t, the latter map is multiplication by e^t.

Moreover, because φ_p intertwines the partial flows that correspond to η and these partial flows are defined for all $t \geq 0$, the restriction to U_p of $\varphi_p^{(t)}$ coincides with φ_p for all $t \geq 0$. Substituting $t-s$ instead of t, we get that the maps $\varphi_p^{(t)}$ and $\varphi_p^{(s)}$ agree whenever they are both defined. Thus, all these maps fit together into a map

$$\bigcup_{t \geq 0} \varphi_p^{(t)} \colon V_p \to \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n},$$

where $V_p = \bigcup_{t \geq 0} g_{-t}U_p$. This map is onto, because its image is the union of the sets $e^t\mathbb{D}^n$ over all $t \geq 0$. The map is one to one, because it is one to one on each $g_{-t}U_p$, and for every two points in the domain there exists a $t \geq 0$ such that the points are both in $g_{-t}U_p$. Since V_p is covered by $(S^1)^n$-invariant open sets $g_{-t}U_p$ on which the map is an $(S^1)^n$-equivariant biholomorphism, we deduce that the map is itself an $(S^1)^n$-equivariant biholomorphism, as required. □

4. Obtaining a fan

Let M be a connected complex manifold of complex dimension n, let the torus $(S^1)^n$ act on M faithfully by biholomorphisms, and assume that this action extends to a holomorphic $(\mathbb{C}^*)^n$-action. The set of fixed points is discrete; assume that it is nonempty and finite.

In Lemma 7 we assigned to every fixed point p in M an open subset V_p that is biholomorphic to $\mathbb{C}^n$. By assumption, there exists at least one fixed point. So, the union X of the sets V_p over these fixed points,

$$X := \bigcup_{p \in M^{(S^1)^n}} V_p,$$

is nonempty.

Remark 8. In Section 6, we show that if M is compact and connected then the union X of the sets V_p is all of M. The proof relies on the results of Sections 4 and 5.

By its definition, X is a $(\mathbb{C}^*)^n$-invariant open submanifold of M. Moreover, we claim that there exists a unique open $(\mathbb{C}^*)^n$ orbit in M, this orbit and free and is dense in M, and it coincides with the free $(\mathbb{C}^*)^n$ orbit in V_p for each p. To see this, we consider the fundamental vector fields $\xi_1, \ldots, \xi_n$ of the $(S^1)^n$-action with respect to the coordinate one-dimensional subtori. We think of them as holomorphic sections $M \to T^{1,0}M \cong TM$ of the holomorphic tangent bundle $T^{1,0}M$ of M. The n-th exterior product $\bigwedge^n T^{1,0}M \to M$ is a holomorphic line bundle and $\xi_1 \wedge \cdots \wedge \xi_n$ is a holomorphic section of this line bundle. A point $x \in M$ belongs to an open $(\mathbb{C}^*)^n$ orbit if and only if $(\xi_1 \wedge \cdots \wedge \xi_n)(x)$ is not zero. This means that the union of the open

$(\mathbb{C}^*)^n$ orbits is the complement of the zero locus of a holomorphic section. Since the zero locus is a complex analytic subvariety of M and M is connected, the union of the open $(\mathbb{C}^*)^n$ orbits is either empty, or it is open, dense, and connected. The claim then follows from the facts that there exists at least one V_p, it contains a free and open $(\mathbb{C}^*)^n$ orbit, and every two distinct orbits are disjoint.

In particular, X is connected and dense in M.

The connected components of the fixed point sets of the circle subgroups of $(S^1)^n$ are closed complex submanifolds of X. If such a submanifold has complex codimension one, then, in analogy with the toric topology literature, we call it a *characteristic submanifold* of X (cf. [14, p. 240]).

Since X is a union of finitely many V_ps and each V_p has only finitely many characteristic submanifolds, there are only finitely many characteristic submanifolds in X. Denote them

$$X_1, \ldots, X_m.$$

Let T_i be the subgroup of T that fixes X_i. If a compact group acts faithfully on a connected manifold then at every fixed point the linear isotropy representation is faithful. Therefore, the linear isotropy representation of T_i at any point q of X_i is faithful. Since T_i acts holomorphically and fixes X_i, we get a faithful representation of T_i on the one-dimensional complex space T_qX/T_qX_i. This gives an injection $T_i \to S^1$, where S^1 acts on T_qX/T_qX_i by scalar multiplication. By continuity, this injection is independent of the choice of point q in X_i. Since, by assumption, T_i contains a circle subgroup of T, this injection is an isomorphism. Let

$$\lambda_i \colon S^1 \to T_i \subset (S^1)^n$$

be the inverse of this isomorphism, composed with the inclusion map into $(S^1)^n$.

We define an abstract simplicial complex:

$$\Sigma := \left\{ I \subseteq \{1, \ldots, m\} \;\middle|\; \bigcap_{i \in I} X_i \neq \emptyset \right\}.$$

To each simplex $I \in \Sigma$ we assign the cone

$$C_I := \operatorname{pos}(\lambda_i \mid i \in I) := \left\{ \sum_{i \in I} a_i \lambda_i \;\middle|\; a_i \geq 0 \right\}$$

in $\operatorname{Lie}(S^1)^n$.

Example 9. Take $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. Let $(S^1)^n$ act on it with weights $\alpha_1, \ldots, \alpha_n \in \operatorname{Hom}((S^1)^n, S^1) \subset (\operatorname{Lie}(S^1)^n)^*$. Suppose that the action is faithful; then $\alpha_1, \ldots, \alpha_n$ are a $\mathbb{Z}$-basis of $\operatorname{Hom}((S^1)^n, S^1)$. The characteristic submanifolds are the coordinate hyperplanes $\{z_i = 0\}$ for $i = 1, \ldots, n$. The homomorphisms $\lambda_1, \ldots, \lambda_n$ are the basis to $\operatorname{Hom}(S^1, (S^1)^n) \subset \operatorname{Lie}(S^1)^n$ that is dual to $\alpha_1, \ldots, \alpha_n$.

Recall that a cone in $\operatorname{Lie}(S^1)^n$ is *unimodular* if it is generated by part of a $\mathbb{Z}$-basis of $\operatorname{Hom}(S^1, (S^1)^n)$.

Returning to our general case –

Lemma 10. *The cones C_I, for $I \in \Sigma$, are unimodular.*

Proof. Let $I \in \Sigma$. By the definition of Σ, this means that the intersection $\bigcap_{i \in I} X_i$ is nonempty. Let q be a point in this intersection. Let p be a fixed point such that $q \in V_p$. Since V_p is isomorphic to some $\mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$ on which the action is faithful, the lemma follows from Example 9. □

Fix a point q in the free $(\mathbb{C}^*)^n$ orbit in X. For any $\xi \in \mathrm{Lie}(S^1)^n$, consider the curve

$$c_q^\xi \colon \mathbb{R} \to X$$

that is given by

$$c_q^\xi(r) := \exp(-rJ\xi) \cdot q \qquad \text{for } r \in \mathbb{R}$$

where $\exp\colon \mathrm{Lie}(\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is the exponential map and where J denotes multiplication by i in $\mathrm{Lie}(\mathbb{C}^*)^n$.

Denote by C_I^0 the relative interior of the cone C_I. Denote

$$X_I = \bigcap_{i \in I} X_i \quad \text{and} \quad X_I^0 = \bigcap_{i \in I} X_i \setminus \bigcup_{j \notin I} X_j.$$

Lemma 11. *Let $\xi \in \mathrm{Lie}(S^1)^n$ and $I \in \Sigma$. Then $\xi \in C_I^0$ if and only if the curve $c_q^\xi(r)$ converges as $r \to -\infty$ to a point q' in X_I^0. Moreover, in this case the limit point q' belongs to V_p for every p such that $V_p \cap X_I \neq \emptyset$.*

Proof. Suppose that $\xi \in C_I^0$. By the definition of Σ, X_I is nonempty. Let p be such that V_p meets X_I. Without loss of generality assume that $I = \{1, \ldots, k\}$ and that the characteristic submanifolds that meet V_p are $X_1, \ldots, X_n$. Let $\alpha_1, \ldots, \alpha_n$ denote the isotropy weights at p. The assumption that $\xi \in C_I^0$ exactly means that $\langle \xi, \alpha_i \rangle$ is positive for $i = 1, \ldots, k$ and zero for $i = k+1, \ldots, n$. Fix an isomorphism $(z_1, \ldots, z_n)\colon V_p \to \mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$ such that $z_i(q) = 1$ for all i. In these coordinates, the curve $c_q^\xi(r)$ is represented as

$$(z_1, \ldots, z_n)(c_q(r)) = (e^{2\pi r \langle \xi, \alpha_1 \rangle}, \ldots, e^{2\pi r \langle \xi, \alpha_n \rangle}).$$

As r approaches $-\infty$, the curve in $\mathbb{C}^n$ approaches the point $(\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{n-k})$. On the other hand, the coordinates take each intersection $V_p \cap X_i$ to the coordinate hyperplane $\{(z_1, \ldots, z_n) \mid z_i = 0\}$, and they take the intersection $V_p \cap X_I^0$ to the set $\{(z_1, \ldots, z_n) \mid z_i = 0 \text{ iff } 1 \le i \le k\}$. So the curve approaches a point in $V_p \cap X_I^0$, as required.

Now suppose that the curve $c_q^\xi(r)$ converges as $r \to -\infty$ to a point in X_I^0. Let p be such that this limit point is contained in V_p. As before, without loss of generality assume that $I = \{1, \ldots, k\}$ and that the characteristic submanifolds that meet V_p are exactly $X_1, \ldots, X_n$; fix an isomorphism $(z_1, \ldots, z_n)\colon V_p \to \mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n}$ such that $z_i(q) = 1$ for all i; the curve $c_q^\xi(r)$ is represented as $(z_1, \ldots, z_n)(c_q(r)) = (e^{2\pi r \langle \xi, \alpha_1 \rangle}, \ldots, e^{2\pi r \langle \xi, \alpha_n \rangle})$. Since the curve approaches a limit as $r \to -\infty$, the pairings $\langle \xi, \alpha_i \rangle$ are nonnegative for all $i = 1, \ldots, n$. Since this limit is in X_I^0, the pairings are positive for every $i \in I$ and they vanish for every $i \in \{1, \ldots, n\} \setminus I$. Thus, $\xi \in C_I^0$ as required. □

Corollary 12. (1) *For every $I, J \in \Sigma$, if $I \neq J$, then $C_I^0 \cap C_J^0 = \emptyset$.*

(2) *For every $I, J \in \Sigma$,*

$$C_I \cap C_J = C_{I \cap J}.$$

(3) *The collection of cones*

$$\Delta := \{ C_I \mid I \in \Sigma \}$$

is a fan, that is, every face of every cone in Δ is itself in Δ, and the intersection of every two cones in Δ is a common face.

Proof. Part (1) follows from Lemma 11 because the sets X_I^0 are disjoint. Part (3) follows from Part (2).

For Part (2), we only need to show the inclusion $C_I \cap C_J \subseteq C_{I\cap J}$, because the opposite inclusion is trivial. Let $\xi \in C_I \cap C_J$. Let $I' \subset I$ and $J' \subset J$ be the subsets such that $\xi \in C_{I'}^0$ and $\xi \in C_{J'}^0$. Then $C_{I'}^0 \cap C_{J'}^0 \neq \emptyset$. By Part (1), $I' = J'$. Let $L = I' = J'$. Then $L \subset I \cap J$, and $\xi \in C_L^0 \subset C_{I\cap J}$. □

Lemma 13. *For every $I \in \Sigma$, the set X_I is an $(S^1)^n$-invariant smooth closed complex submanifold of X of complex codimension $|I|$, it is connected, and it contains a fixed point.*

Proof. Fix $I \in \Sigma$.

Since each of the sets X_i, for $i \in I$, is closed in X, so is the intersection X_I of these sets.

Since X is the union of open subsets V_p, and because every intersection $V_p \cap X_I$ is an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in V_p, we deduce that X_I is itself an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in X. It remains to show that X_I is connected and contains a fixed point.

Choose any $\xi \in C_I^0$ (for example, we may take $\xi = \sum_{i\in I} \lambda_i$), and choose any q in the free $(\mathbb{C}^*)^n$ orbit in X. By Lemma 11, the curve $c_q^\xi(r)$ converges as $r \to -\infty$; let q' be its limit. Also by Lemma 11, for every p such that $V_p \cap X_I \neq \emptyset$, the limit point q' belongs to V_p. Since X_I is the union over such p of the subsets $V_p \cap X_I$, and because each of these subsets is connected and contains q', the union X_I is connected. Also, every p such that $V_p \cap X_I \neq \emptyset$ belongs to $V_p \cap X_I$; because the set of such ps is nonempty, X_I contains a fixed point. □

Corollary 14. *In the fan Δ, every cone is contained in an n-dimensional cone.*

Proof. Every cone in the fan has the form C_I for some $I \in \Sigma$. By Lemma 13, the set X_I contains a fixed point; let p be such a fixed point. Since V_p was chosen as in Lemma 7, by Example 9 there exist exactly n characteristic submanifolds, say, X_j for $j \in J \subset \{1, \ldots, m\}$ with $|J| = n$, that pass through p. Then $J \in \Sigma$, and C_J is an n-dimensional cone in Δ that contains C_I. □

5. Isomorphism of the subset X with a toric manifold

Let M be a connected complex manifold of complex dimension n, let the torus $(S^1)^n$ act on M faithfully by biholomorphisms, and assume that this action extends to a holomorphic $(\mathbb{C}^*)^n$-action. The set of fixed points is discrete; assume that it is nonempty and finite.

In Section 4 we described an open subset X of M and a unimodular fan Δ. Let M_Δ be the toric variety that is associated to the fan Δ.

Lemma 15. *There exists an $(S^1)^n$-equivariant biholomorphism between M_Δ and X.*

We recall some properties of the set X and the fan Δ. Let $F = M^{(S^1)^n}$ denote the fixed point set. For every fixed point $p \in F$, let $\alpha_{p,1}, \dots, \alpha_{p,n}$ denote the isotropy weights of the torus action at p.

(1) The set X is the union over $p \in F$ of subsets V_p, such that each V_p is an invariant open neighbourhood of p that is equivariantly biholomorphic to the linear representation $\mathbb{C}_{\alpha_{p,1}} \oplus \cdots \oplus \mathbb{C}_{\alpha_{p,n}}$.

(2) The n-dimensional cones in Δ are in bijection with the fixed point sets $p \in F$, and the cone corresponding to the fixed point p is $\text{pos}(\lambda_{i_1}, \dots, \lambda_{i_n})$, where $\lambda_{i_1}, \dots, \lambda_{i_n}$ is a basis of $\text{Lie}(S^1)^n$ that is dual to the basis $\alpha_{p,1}, \dots, \alpha_{p,n}$ of $(\text{Lie}(S^1)^n)^*$.

The toric variety M_Δ that is associated to the fan Δ has similar properties: it is the union over $p \in F$ of invariant subsets V'_p, and every V'_p is equivariantly biholomorphic to $\mathbb{C}_{\alpha_{p,1}} \oplus \dots \oplus \mathbb{C}_{\alpha_{p,n}}$.

Lemma 15 follows immediately from these properties of X and M_Δ, by the following lemma.

Lemma 16. *Let X and X' be complex manifolds of complex dimension n, equipped with holomorphic $(\mathbb{C}^*)^n$-actions. Suppose that there exist open dense $(\mathbb{C}^*)^n$ orbits $\mathcal{O}$ in X and $\mathcal{O}'$ in X'. Suppose that there exist invariant open subsets V_p in X and V'_p in X', both indexed by $p \in F$, such that X is the union of the sets V_p and X' is the union of the sets V'_p, and that for each $p \in F$ there exists an equivariant biholomorphism $\varphi_p \colon V_p \to V'_p$. Then X is equivariantly biholomorphic to X'.*

Proof. Necessarily, $\mathcal{O}$ is contained in each V_p and $\mathcal{O}'$ is contained in each V'_p. Fix a point q in $\mathcal{O}$ and a point q' in $\mathcal{O}'$. After possibly composing each φ_p by the action of an element of $(\mathbb{C}^*)^n$, we may assume that $\varphi_p(q) = q'$ for each $p \in F$. So, for each p and $\tilde{p} \in F$, the maps φ_p and $\varphi_{\tilde{p}}$ coincide at the point q. By equivariance, φ_p and $\varphi_{\tilde{p}}$ coincide on all of $\mathcal{O}$; by continuity, they coincide on the entire overlap $V_p \cap V_{\tilde{p}}$. Thus, the φ_p fit together into a map

$$\varphi = \bigcup_p \varphi_p \colon X \to X'.$$

This map is holomorphic, equivariant, and onto. Similarly, the inverses $\psi_p := {\varphi_p}^{-1}$ fit together into a map

$$\psi = \bigcup_p \psi_p \colon X' \to X.$$

We have that $\psi \circ \varphi = \text{id}_X$ and $\varphi \circ \psi = \text{id}_{X'}$; thus, $\varphi \colon X \to X'$ is an equivariant biholomorphism, as required. □

6. The compact case

Let M be a connected complex manifold of complex dimension n, with a faithful $(S^1)^n$-action, with fixed points.

Suppose that M is compact. In Section 2 we extended the $(S^1)^n$-action to a holomorphic $(\mathbb{C}^*)^n$-action. In Section 4 we described an open subset X of M and we associated to it a fan Δ.

Lemma 17. *The fan Δ is complete.*

We begin by proving a special case:

Lemma 18. *Let M' be a complex manifold of complex dimension one, equipped with a faithful holomorphic action of S^1 with at least one fixed point. Suppose that M' is compact and connected. Then M' is equivariantly biholomorphic to $\mathbb{CP}^1$ with a standard $\mathbb{C}^*$-action.*

Proof. Consider the S^1-action on M'. Near a fixed point, it is isomorphic to the restriction of either the standard S^1-action on $\mathbb{C}$ or the opposite S^1-action on $\mathbb{C}$ to an invariant neighbourhood of the origin in $\mathbb{C}$.

Consider the flow that is generated by $-J\xi$, where ξ generates the S^1-action. If the S^1-action near a fixed point is standard, then the trajectories of this flow converge to the fixed point as their parameter approaches $-\infty$. If the S^1-action near a fixed point is opposite from standard, then the trajectories of this flow converge to the fixed point as their parameter approaches ∞.

Outside the fixed point set, the action is free. The quotient M'/S^1 is[5] a real one-manifold with boundary; its boundary is exactly the image of the fixed point set. Since M' is compact and connected and contains a fixed point, and by the classification of one-manifolds, the quotient M'/S^1 must be a closed segment.

The flow on M' that is generated by $-J\xi$ descends to a flow on the interior of M'/S^1 that does not have fixed points. For each boundary component, the flow approaches that component either as its parameter approaches ∞ or as the parameter approaches $-\infty$. Necessarily, it approaches one boundary component when the parameter approaches ∞ and it approaches the other boundary component when the parameter approaches $-\infty$.

The corresponding fan must then be equal to the fan of $\mathbb{CP}^1$, and the manifold is equivariantly biholomorphic to $\mathbb{CP}^1$ by Lemma 16. □

We now return to the setup of Lemma 17: We have a connected complex manifold M of complex dimension n, with a faithful $(S^1)^n$-action, with fixed points. We assume that M is compact. We consider the open subset X of M and the associated fan Δ as described in Section 4.

Lemma 19. *Every $n-1$ dimensional cone in Δ is a common face of two n-dimensional cones in Δ.*

Proof. Let C_I be an $n-1$ dimensional cone in Δ, corresponding to the subset $I = \{i_1, \ldots, i_{n-1}\}$ of $\{1, \ldots, m\}$.

Let T_I be the codimension one subtorus of $(S^1)^n$ that is generated by the circles T_i for $i \in I$. By Lemma 13, X_I is a connected complex manifold of dimension one, equipped with a faithful holomorphic action of the circle $(S^1)^n/T_I$ with at least one fixed point. We will now show that X_I is compact, and will deduce Lemma 19 from Lemma 18.

First note that X_I is a connected component of the fixed point set of T_I in X. This follows from the facts that X_I is connected (by Lemma 13) and that, for each of the subsets V_p, if the intersection $V_p \cap X_I$ is nonempty then it is a connected component of the fixed point set of T_I in V_p. Let N denote the connected component of the

[5]Here, "is" means that there exists a unique manifold-with-boundary structure on M'/S^1 such that a function is smooth if and only if its pullback to M' is smooth.

fixed point set of T_I in M that contains X_I. As in any holomorphic torus action on a complex manifold, N is an $(S^1)^n$-invariant closed complex submanifold of M. By examining N near a point of X_I, we deduce that N has complex dimension one. Since N is closed in M and M is compact, N is compact. By Lemma 18, N is equivariantly biholomorphic to $\mathbb{CP}^1$ with a standard action of the circle $(S^1)^n/T_I$. In particular, N contains two fixed points; denote them p' and p''. The intersection $V_{p'} \cap N$, being a $(\mathbb{C}^*)^n$-invariant neighbourhood of p' in N, must be all of $N \setminus \{p''\}$. Similarly, the intersection $V_{p''} \cap N$ is all of $N \setminus \{p'\}$. So N is contained in the union X of the sets V_p, and so N must be equal to X_I. Thus, X_I is equivariantly biholomorphic to $\mathbb{CP}^1$ with a standard action of the circle $(S^1)^n/T_I$. This implies the result of Lemma 19. □

We are now ready to prove Lemma 17.

Proof of Lemma 17. Let $|\Delta|$ denote the union of the cones in Δ, and let $|\Delta^{n-2}|$ denote the union of the cones in Δ that have codimension ≥ 2. The complement $\mathrm{Lie}(S^1)^n \setminus |\Delta^{n-2}|$ is connected, open, and dense in $\mathrm{Lie}(S^1)^n$.

By Lemma 19, the union of the relative interiors of the faces of Δ of dimension $(n-1)$ and of dimension n is open in $\mathrm{Lie}(S^1)^n$. This union is $|\Delta| \setminus |\Delta^{n-2}|$. Thus, $|\Delta| \setminus |\Delta^{n-2}|$ is also open in $\mathrm{Lie}(S^1)^n \setminus |\Delta^{n-2}|$.

However, because $|\Delta|$ is closed in $\mathrm{Lie}(S^1)^n$, we also have that $|\Delta| \setminus |\Delta^{n-2}|$ is closed in $\mathrm{Lie}(S^1)^n \setminus |\Delta^{n-2}|$.

Since $|\Delta| \setminus |\Delta^{n-2}|$ is open and closed in $\mathrm{Lie}(S^1)^n \setminus |\Delta^{n-2}|$ and $\mathrm{Lie}(S^1)^n \setminus |\Delta^{n-2}|$ is connected, we deduce that $|\Delta| \setminus |\Delta^{n-2}|$ is either empty or is equal to all of $\mathrm{Lie}(S^1)^n \setminus |\Delta^{n-2}|$.

As, by assumption, M has a fixed point, Δ has at least one n-dimensional cone, so $|\Delta| \setminus |\Delta^{n-2}|$ is not empty. So $|\Delta| \setminus |\Delta^{n-2}|$ is equal to all of $\mathrm{Lie}(S^1)^n \setminus |\Delta^{n-2}|$. Taking the closures, we deduce that $|\Delta| = \mathrm{Lie}(S^1)^n$, as required. □

We are now ready to prove our main theorem.

Proof of Theorem 1. Lemma 16 gives an equivariant biholomorphism

$$\varphi \colon M_\Delta \to X.$$

By Lemma 17, the fan Δ is complete. This implies that the toric variety M_Δ is compact. So X must be compact. Since M is Hausdorff and connected, and X is a subset that is both compact and open, X is all of M. So φ defines an equivariant biholomorphism from M_Δ to M, as required. □

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