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L^∞-VARIATIONAL PROBLEM ASSOCIATED TO DIRICHLET FORMS

Pekka Koskela, Nageswari Shanmugalingam and Yuan Zhou*

Abstract. We study the L^∞-variational problem associated to a general regular, strongly local Dirichlet form. We show that the intrinsic distance determines the absolute minimizer (infinite harmonic function) of the corresponding L^∞-functional. This leads to the existence and uniqueness of the absolute minimizer on a bounded domain, given a continuous boundary data. Applying this, we also obtain that an infinity harmonic function on $\mathbb{R}^n$ may be the minimizer for several different variational problems. Finally, we apply our results to Carnot–Carathéodory spaces.

1. Introduction

The classical L^∞-variational problem is to consider the local minimizers of the L^∞-functional

$$F(u, \Omega) = \operatorname*{esssup}_{x\in\Omega} |\nabla u(x)|^2$$

over the class of functions $u \in C(\Omega) \cap \mathrm{Lip}_{\,\mathrm{loc}}(\Omega)$ on an open subset $\Omega \subset \mathbb{R}^n$ with a given boundary data. This study was initiated by Aronsson [2–5], who introduced the idea of absolute minimizer. A function $u \in C(\Omega) \cap \mathrm{Lip}_{\,\mathrm{loc}}(\Omega)$ is said to be an *absolute minimizer* if for every open set $U \Subset \Omega$ and each function $v \in C(\overline{U}) \cap \mathrm{Lip}_{\,\mathrm{loc}}(U)$ with $u|_{\partial U} = v|_{\partial U}$, we have $F(u, U) \le F(v, U)$. An absolute minimizer is also called an infinity harmonic function since it is a viscosity solution of the ∞-Laplace equation

$$\sum_{i,\,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \tag{1.1}$$

which is a highly degenerate second-order partial differential equation; see for example [17]. The study of the absolute minimizers of the L^∞-variational problem has attracted considerable attention; see [6, 8–13, 17–19, 21, 22, 24, 29] and references therein.

Note that the above L^∞-variational problem corresponds to the standard Dirichlet energy form $\mathscr{E}_{\mathbb{R}^n}$ given by

$$\mathscr{E}_{\mathbb{R}^n}(u, v) = \int_{\mathbb{R}^n} \langle \nabla u(x), \nabla v(x)\rangle\, dx$$

for $u, v \in W^{1,2}(\mathbb{R}^n)$. Modeled on the above theory, for a given Dirichlet form, one can introduce the corresponding L^∞-variational problem as follows.

Let X be a locally compact, complete, connected and separable Hausdorff space and let m be a non-negative Radon measure with support X. Assume that $\mathscr{E}$ is

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*Corresponding author.

a regular, strongly local Dirichlet form on $L^2(X, m)$ with domain $\mathcal{D}$. Denote by Γ the corresponding energy measure. Then the Radon–Nikodym derivative $\frac{d}{dm}\Gamma(u, u)$ plays the role of the square of the length of the gradient of $u \in \mathcal{D}$, and for every pair $u, v \in \mathcal{D}$, we always have

$$\mathscr{E}(u, v) = \int_X d\Gamma(u, v).$$

For the details, see for example [15] and also Section 2 below. Given an open subset $\Omega \subset X$, we denote by $\mathcal{D}_{\mathrm{loc}}(\Omega)$ the class of functions u for which, whenever $U \Subset \Omega$ is open, there exists $w \in \mathcal{D}$ with $w = u$ on U.

Following the idea of Aronsson, we introduce the corresponding L^∞-functional and the absolute minimizers for $(X, \mathscr{E}, m)$. Such absolute minimizers play the role of infinity harmonic functions in view of the case $(\mathbb{R}^n, \mathscr{E}_{\mathbb{R}^n}, dx)$.

Definition 1.1. Let $\Omega \subset X$ be an open set. We define the L^∞-functional, $F_{\mathscr{E}}(\cdot, \Omega)$, on $\mathcal{D}(\Omega)$ by

$$F_{\mathscr{E}}(u, \Omega) = \operatorname*{esssup}_{x\in\Omega} \frac{d}{dm}\Gamma(u, u)(x). \tag{1.2}$$

(i) A function u is said to be an *absolute minimizer* (or *infinity harmonic function*) on Ω associated to $(\mathscr{E}, m)$ if $u \in C(\Omega) \cap \mathcal{D}_{\mathrm{loc}}(\Omega)$ and for all open sets $U \Subset \Omega$ and all $v \in C(\overline{U}) \cap \mathcal{D}_{\mathrm{loc}}(U)$ with $v|_{\partial U} = u|_{\partial U}$, we have $F_{\mathscr{E}}(u, U) \le F_{\mathscr{E}}(v, U)$. Denote by $\mathrm{AM}(\Omega; \mathscr{E}, m)$ the class of all absolute minimizers on Ω.

(ii) Given a boundary data $f \in C(\partial\Omega)$, a function u is said to be an *absolutely minimizing gradient extension* (or *infinity harmonic extension*) of f associated to $\mathscr{E}$ if $u \in C(\overline{\Omega}) \cap \mathrm{AM}(\Omega; \mathscr{E}, m)$ and $u|_{\partial\Omega} = f$. Denote by $\mathrm{AM}_f(\Omega; \mathscr{E}, m)$ the class of all absolutely minimizing gradient extensions of f.

Motivated by the classical theory of infinity harmonic functions on $\mathbb{R}^n$, some natural questions arise: given a bounded open subset $\Omega \Subset X$ and boundary data $f \in C(\partial\Omega)$, is $\mathrm{AM}_f(\Omega; \mathscr{E}, m)$ non-empty? Does it have only one function in it? Does it give the absolutely minimizing Lipschitz extension of f for some distance $\hat{d}$ on X, in the sense described below?

The main purpose of this paper is to give affirmative answers to these questions for an arbitrary regular, strongly local Dirichlet form under the following Standard assumption **(A)**.

Definition 1.2 (Standard assumption (A))**.** The topology induced by the intrinsic distance d of $\mathscr{E}$ and the original topology on X coincide. Here, the *intrinsic distance* d associated to $\mathscr{E}$ is defined by

$$d(x, y) = \sup\{u(x) - u(y) : \ u \in C(X) \cap \mathcal{D}_{\mathrm{loc}}(X), \ \Gamma(u, u) \le m\}$$

for all $x, y \in X$, where $\Gamma(u, u) \le m$ means that $\Gamma(u, u)$ is absolutely continuous with respect to m and its Radon–Nikodym derivative satisfies $\frac{d}{dm}\Gamma(u, u) \le 1$ almost everywhere.

We always make this assumption **(A)** throughout the whole paper. Recall that under it, (X, d) is a length space; see [26, 28].

For any $K \subset X$, $\mathrm{Lip}_d(K)$ denotes the class of functions u such that

$$\mathrm{Lip}_d(u, K) = \sup_{x,\, y \in K,\, x \neq y} \frac{|u(x) - u(y)|}{d(x, y)} < \infty.$$

Given an open set Ω, $\mathrm{Lip}_{d,\,\mathrm{loc}}(\Omega)$ denotes the class of all u, such that $u \in \mathrm{Lip}_d(K)$ for all $K \Subset \Omega$.

Definition 1.3. Let $\Omega \subset X$ be an open set.

(i) Denote by $\mathrm{AML}(\Omega; d)$ the class of functions $u \in C(\Omega) \cap \mathrm{Lip}_{d,\,\mathrm{loc}}(\Omega)$, such that for each open set $U \Subset \Omega$, $\mathrm{Lip}_d(u, U) = \mathrm{Lip}_d(u, \partial U)$.

(ii) For any given $f \in C(\partial\Omega)$, $\mathrm{AML}_f(\Omega; d)$ is the class of all functions $u \in \mathrm{AML}(\Omega;\, d) \cap C(\overline{\Omega})$ with $u|_{\partial\Omega} = f$. Usually, a function $u \in \mathrm{AML}_f(\Omega; d)$ is called an absolutely minimizing Lipschitz extension of f.

The main result of this paper reads as follows.

Theorem 1.4. *Let $\mathscr{E}$ be a regular, strongly local Dirichlet form on $L^2(X, m)$ that satisfies Standard assumption* **(A)**. *Let $\Omega \subset X$ be a bounded open set. Then $u \in \mathrm{AM}(\Omega; \mathscr{E}, m)$ if and only if $u \in \mathrm{AML}(\Omega; d)$. Moreover, for each $f \in C(\partial U)$, the set $\mathrm{AM}_f(\Omega, \mathscr{E}, m)$ has exactly one element.*

Theorem 1.4 follows from Theorem 3.3 below. The proof of Theorem 3.3 relies on the key lemma, Lemma 2.5. In Lemma 2.5, we establish a weak coincidence of the intrinsic distance and gradient structures induced by $\mathscr{E}$: $\sup_{x \in V}(\mathrm{Lip}_d u(x))^2 = F_{\mathscr{E}}(u, V)$ for each open set V. Note that usually one cannot expect that $(\mathrm{Lip}_d u)^2 = \frac{d}{dm}\Gamma(u, u)$ holds almost everywhere even for $u \in \mathrm{Lip}_d(X)$, as observed by Sturm [26]; also see [20, 21]. See Sections 2 and 4 for more details.

As a consequence of Theorem 1.4 and Corollary 3.4, we will see that the local behavior of the intrinsic distance totally determines the absolute minimizer (see Corollary 3.4). This together with Sturm's constructions shows that a classical infinity harmonic function on $\mathbb{R}^n$ may be an absolute minimizer for several different functionals.

There exists a variety of Dirichlet forms that satisfy our Standard assumption **(A)** (see for example [15]): in Section 5, we apply Theorem 1.4 to Carnot–Carathéodory spaces. Note that our results extend some of the results obtained in [8–10, 29] to more general settings where we may not have a corresponding Aronsson equation.

2. Dirichlet forms and a key lemma

In this section, we first recall some basic notions and properties of Dirichlet forms, and then establish the key Lemma 2.5, which gives a weak coincidence of the intrinsic distance and gradient structures induced by our Dirichlet form. This lemma plays a key role in the proof of Theorem 1.4.

Let X be a locally compact, connected and separable Hausdorff space and m be a non-negative Radon measure with support X. A *Dirichlet form* $\mathscr{E}$ on $L^2(X, m)$ is a closed, non-negative definite and symmetric bilinear form defined on a dense linear subspace $\mathcal{D}$ of $L^2(X)$, that satisfies the *Markov property*: for any $u \in \mathcal{D}$, the function $v = \min\{1, \max\{0, u\}\}$ satisfies $\mathscr{E}(v, v) \le \mathscr{E}(u, u)$. Moreover, $\mathscr{E}$ is said to be *strongly local* if $\mathscr{E}(u, v) = 0$ whenever $u, v \in \mathcal{D}$ with u constant on a neighborhood of the support of v; to be *regular* if there exists a subset of $\mathcal{D} \cap C_c(X)$ which is both dense in $C_c(X)$ with uniform norm and in $\mathcal{D}$ with the norm $\|\cdot\|_{\mathcal{D}}$ defined by

$$\|u\|_{\mathcal{D}} = [\|u\|^2_{L^2(X)} + \mathscr{E}(u, u)]^{1/2}$$

for each $u \in \mathcal{D}$.

Let $\mathscr{M}(X)$ denote the collection of all signed Radon measures on X with finite mass. Beurling and Deny [7] showed that a regular, strongly local Dirichlet form $\mathscr{E}$ can be written as

$$\mathscr{E}(u, v) = \int_X d\Gamma(u, v)$$

for all $u, v \in \mathcal{D}$, where Γ is an $\mathscr{M}(X)$-valued non-negative definite and symmetric bilinear form defined by the formula

$$\int_X \phi\, d\Gamma(u, v) \equiv \frac{1}{2}\left[\mathscr{E}(u, \phi v) + \mathscr{E}(v, \phi u) - \mathscr{E}(uv, \phi)\right] \tag{2.1}$$

for all $u, v \in \mathcal{D} \cap L^\infty(X)$ and $\phi \in \mathcal{D} \cap \mathscr{C}_0(X)$. We call $\Gamma(u, v)$ the *Dirichlet energy measure (squared gradient)*. The Radon–Nikodym derivative $\frac{d\Gamma(u,u)}{dm}(z)$ plays the role of the square of the length of the gradient of $u \in \mathcal{D}$ at $z \in X$. Observe that, since $\mathscr{E}$ is strongly local, Γ is local and satisfies the Leibniz rule and the chain rule, see for example [15]. Then for any open subset $\Omega \subset X$, $\Gamma(u, v)$ can be defined as a measure on Ω for every pair $u, v \in \mathcal{D}_{\rm loc}(\Omega)$, where $\mathcal{D}_{\rm loc}(\Omega)$ denotes the collection of all $u \in L^2_{\rm loc}(X)$ satisfying that for each open subset $U \Subset \Omega$, there exists a function $w \in \mathcal{D}$ such that $u = w$ almost everywhere on U.

Recall the definition of the intrinsic distance d from the introduction. In what follows, $B(x, r)$ refers to the open ball with respect to d, centered at x and with radius $r > 0$. We also refer to the characteristic function of a set U by 1_U.

Lemma 2.1 and Corollary 2.2 tell us that Lipschitz functions are in the domain $\mathcal{D}_{\rm loc}(X)$ of the Dirichlet form $\mathscr{E}$. Lemma 2.1 can also be found in [25, Appendix], and the proof given below is a simplification of the proof found in [25]. We provide the simplified proof here for the reader's convenience. Corollary 2.2 can also be found in [14].

Lemma 2.1. *Suppose that X is separable and satisfies the Standard assumption* **(A)**. *Then, given $x_0 \in X$, the function u given by $u(x) = d(x_0, x)$ is in $\mathcal{D}_{\rm loc}(X)$.*

Proof. By the definition of the intrinsic distance, for each $x \in X$ there is a sequence $\{u_k\}_k$ of functions in $\mathcal{D}_{\rm loc}(X) \cap C(X)$ such that $\Gamma(u_k, u_k) \le m$ for each k and $d(x, x_0) = \lim_{k\to\infty}[u_k(x) - u_k(x_0)]$. By the Markov property of $\mathscr{E}$, we can also assume that $u_k(x_0) = 0$, $0 \le u_k(x) \le d(x, x_0)$, and for each $z \in X$ we can further assume that $0 \le u_k(z) \le u_k(x)$. Observe that for each $z, w \in X$, by the definition of the intrinsic metric, $|u_k(z) - u_k(w)| \le d(z, w)$, that is, for each k the function u_k is Lipschitz continuous and bounded on X. By the local compactness and separability of X, by a use of the Arzela–Ascoli theorem we can find a subsequence, also denoted $\{u_k\}_k$, and a Lipschitz function $u_x : X \to \mathbb{R}$, such that $u_k \to u_x$ uniformly on X. Now the closedness property of $\mathscr{E}$ shows that $u_x \in \mathcal{D}_{\rm loc}(X) \cap C(X)$ with $\Gamma(u_x, u_x) \le m$. The latter property follows because the square-roots of the Radon–Nikodym derivatives of $\Gamma(u_k, u_k)$ form a bounded sequence in $L^2(X)$, and the Hilbert space property of

$L^2(X)$ implies that for a further subsequence of $\{u_k\}_k$ the square-roots of the Radon–Nikodym derivatives of the Dirichlet measures $\Gamma(u_k, u_k)$ converge to a function whose square gives $\Gamma(u_x, u_x)$.

Note that $u_x(x_0) = 0$ and $u_x(x) = d(x_0, x)$, and that whenever $v \in \mathcal{D}_{\mathrm{loc}}(X) \cap C(X)$ with $\Gamma(v, v) \le m$, it must be that $|v(x) - v(x_0)| \le d(x, x_0) = u_x(x)$.

Since by assumption X is separable, we can find a countable collection of points $\{x_j\}_j$ in X that forms a dense subset of X. For each k we consider $v_k = \max_{1\le j\le k} u_{x_j}$. The Markov property of $\mathscr{E}$ ensures that $v_k \in \mathcal{D}_{\mathrm{loc}}(X)$, and the locality property of $\mathscr{E}$ implies that

$$\Gamma(v_k, v_k) = \sum_{j=1}^{k} 1_{\{v_k = u_{x_j}\}}\, \Gamma(u_{x_j}, u_{x_j}) \le m.$$

Furthermore, for $j = 1, \dots, k$ we have $v_k(x_j) = u_{x_j}(x_j) = d(x_0, x_j)$ and v_k is Lipschitz continuous on X with $|v_k(w) - v_k(z)| \le d(w, z)$. Another application of the closedness property of $\mathscr{E}$ and the Arzela–Ascoli theorem, together with the density of the set $\{x_j\}_j$ in X yields the desired claim. □

Corollary 2.2. *If X is separable, satisfies the Standard assumption* **(A)***, and $v : X \to \mathbb{R}$ is Lipschitz continuous, that is, $v \in \mathrm{Lip}_d(X)$, then $v \in \mathcal{D}_{\mathrm{loc}}(X)$ with $\Gamma(v, v)$ absolutely continuous with respect to m.*

Proof. The result follows from a direct application of Lemma 2.1 together with the closedness property of $\mathscr{E}$ upon noting that if the Lipschitz constant of v is L, then v is also given by $v(x) = \sup\{v(x_j) - Ld(x, x_j)\}$. □

Under the Standard assumption **(A)** that the topology induced by the intrinsic distance d of $\mathscr{E}$ and the original topology on X coincide, it is known that d is a distance, $d(x, y) < \infty$ for all $x, y \in X$, and (X, d) is a length space; see [25,27,28]. The following relations between the intrinsic distance and the Dirichlet energy measure was proved in [14].

Let

$$\mathrm{Lip}_d u(x) = \lim \sup_{x \ne y \to x} \frac{|u(y) - u(x)|}{d(x, y)}.$$

Lemma 2.3. *We have $\mathrm{Lip}_d(X) \subset \mathcal{D}_{\mathrm{loc}}(X)$. Moreover, $\frac{d}{dm}\Gamma(u, u) \le (\mathrm{Lip}_d(u, X))^2$ almost everywhere for all $u \in \mathrm{Lip}_d(X)$.*

Next we prove a Cauchy–Schwarz type inequality for the energy measure form Γ.

Lemma 2.4. *We have the following Cauchy–Schwarz inequality for $u, \eta \in \mathcal{D}_{\mathrm{loc}}(U)$ whenever $U \subset X$ is open:*

$$\left| \frac{d}{dm}\Gamma(u, \eta)(z) \right| \le \frac{1}{2} \frac{d}{dm}\Gamma(u, u)(z) + \frac{1}{2} \frac{d}{dm}\Gamma(\eta, \eta)(z) \tag{2.2}$$

for almost all $z \in U$. Furthermore, if $\Gamma(u, u)$ and $\Gamma(\eta, \eta)$ are absolutely continuous with respect to the underlying measure $1_U m$, then so is $\Gamma(u, \eta)$.

Proof. The absolute continuity of $\Gamma(u, \eta)$ with respect to $1_U m$ follows from the definition of the intrinsic metric, the fact $u \pm \eta \in \mathcal{D}_{\mathrm{loc}}(U)$, Lemma 2.3, Corollary 2.2, and that

$$\Gamma(u, \eta) = \frac{1}{4}\{\Gamma(u + \eta, u + \eta) - \Gamma(u - \eta, u - \eta)\}.$$

Moreover, recall that for all $f,\, g \in L^\infty$ and $w,\, v \in \mathcal{D}$, the following Cauchy–Schwarz inequality holds:

$$\begin{aligned}\left|\int_X fg\, d\Gamma(w,\, v)\right| &\le \left(\int_X f^2\, d\Gamma(w,\, w)\right)^{1/2} \left(\int_X g^2\, d\Gamma(v,\, v)\right)^{1/2} \\ &\le \frac{1}{2}\int_X f^2\, d\Gamma(w,\, w) + \frac{1}{2}\int_X g^2\, d\Gamma(v,\, v).\end{aligned}$$

See [14, Theorem 3.7] for a proof of this fact. Now, for any open subset $V \Subset U$, taking $w \in \mathcal{D}$ such that $w = u$ on V and taking $f = g = 1_V \mathrm{sign}(\frac{d}{dm}\Gamma(u,\, \eta))$, we will see that

$$\int_V \left|\frac{d}{dm}\Gamma(u,\, \eta)\right| dm \le \frac{1}{2}\int_V \left(\frac{d}{dm}\Gamma(u,\, u) + \frac{d}{dm}\Gamma(\eta,\, \eta)\right) dm,$$

which gives (2.2) by a standard measure theory argument. $\square$

Recall that for the Dirichlet energy form $(\mathbb{R}^n,\, \mathscr{E}_{\mathbb{R}^n},\, m)$, we always have $\mathrm{Lip}\, u = |\nabla u|$ almost everywhere when $u \in \mathrm{Lip}\,(\mathbb{R}^n)$. Naturally, given a regular, strongly local Dirichlet form $\mathscr{E}$ satisfying the Standard assumption **(A)**, we would like to know whether for each $u \in \mathrm{Lip}_{\,d}(X)$,

$$\mathrm{Lip}_{\,d} u = \sqrt{\frac{d}{dm}\Gamma(u,\, u)} \tag{2.3}$$

almost everywhere. However, the answer to this question is not always in the positive as first observed by Sturm [26]; see Section 4 below, and also see [20,21] for a different example constructed via a large Cantor set. Instead of the above point-wise equality (2.3), we obtain the following weak coincidence of the intrinsic distance and differential structures, which is crucial for obtaining Theorem 1.4.

Lemma 2.5. *For each open set $U \subset X$ and every $u \in \mathcal{D}_{\mathrm{loc}}(U)$, we have*

$$\operatorname*{esssup}_{x\in U} \sqrt{\frac{d}{dm}\Gamma(u,\, u)(x)} = \sup_{x\in U} \mathrm{Lip}_{\,d} u(x). \tag{2.4}$$

To prove this, we need some auxiliary lemmas. For every $U \subset X$, define a local intrinsic distance d_U on U by

$$d_U(x,\, y) = \sup\{u(x) - u(y)\ :\ u \in C(U) \cap \mathcal{D}_{\mathrm{loc}}(U),\ 1_U\Gamma(u,\, u) \le 1_U m\}.$$

Lemma 2.6. *d_U is locally finite on U. Furthermore, if U is connected, then d_U is finite on U and hence is a distance function on U.*

Proof. It is easy to see from the definitions of d and d_U that on $U \times U$ we have $d \le d_U$. It therefore suffices to prove the reverse inequality. To this end, fix $x \in U$ and $0 < r < \min\{1, d(x, \partial U)\}/10$ such that $\overline{B}(x, r)$ is compact (recall that we assume X to be locally compact, so this is possible). We now fix $y \in B(x, r)$.

Let $u \in \mathcal{D}_{\mathrm{loc}}(U) \cap C(U)$ such that $1_U\Gamma(u,\, u) \le 1_U m$. If we knew that u could be extended to a function $\tilde{u}$ in $\mathcal{D}_{\mathrm{loc}}(X) \cap C(X)$, such that $\Gamma(\tilde{u}, \tilde{u}) \le m$, then we would be done. Since we are not able to directly extend u in this manner, we use a truncation argument as follows.

First, we may assume by the Markov property of $\mathscr{E}$ that $u(x) = 0$ and that $0 \le u(z) \le u(y)$. For $z \in X$ we set

$$\eta(z) = (r - d(z, \overline{B}(x, r)))_+.$$

Then Lemma 2.1 shows that $\eta \in \mathcal{D}_{\mathrm{Lip}}(X)$, and Lemma 2.3 that $\Gamma(\eta,\eta) \le m$ on X. By the Leibniz rule,

$$\Gamma(\eta u, \eta u) = u^2\Gamma(\eta,\eta) + \eta^2\Gamma(u,u) + 2\eta u\,\Gamma(\eta,u).$$

By (2.2), we have

$$\left|\frac{d}{dm}\Gamma(\eta,u)\right| \le \frac{1}{2}\left[\frac{d}{dm}\Gamma(u,u) + \frac{d}{dm}\Gamma(\eta,\eta)\right] \le 1,$$

and from the argument subsequent to (2.2) we also know that $\Gamma(\eta,u)$ is absolutely continuous with respect to the underlying measure m. It follows that on $\overline{B}(x,2r)$ we have

$$\begin{aligned}\frac{d}{dm}\Gamma(\eta u,\eta u) &\le \left[|u(y)-u(x)|^2 + r^2 + 2r|u(y)-u(x)|\right]\\ &= (r+|u(y)-u(x)|)^2,\end{aligned}$$

and on $X \setminus B(x,2r)$ we know that $\Gamma(\eta u,\eta u) = 0$. So, the function $\tilde{u}$ given by

$$\tilde{u} = \frac{1}{r+|u(y)-u(x)|}\,\eta u$$

belongs to $\mathcal{D}_{\mathrm{loc}}(X) \cap C(X)$ with $\Gamma(\tilde{u},\tilde{u}) \le m$. Therefore, by the definition of $d(x,y)$, it follows that

$$\frac{1}{r+|u(y)-u(x)|}\,[\eta(y)u(y) - \eta(x)u(x)] = \frac{r|u(y)-u(x)|}{r+|u(y)-u(x)|} \le d(x,y) < r. \tag{2.5}$$

This immediately tells us that the quantity $d_U(y,x)$ is finite when $y \in B(x,r)$, for we can take the supremum over all such u and note that if $|u(y)-u(x)| \to \infty$, then the ratio $|u(y)-u(x)|/(r+|u(y)-u(x)|)$ tends to 1. □

Let us continue the argument from the above proof. By (2.5) we know that

$$|u(y)-u(x)| \le \frac{r\,d(x,y)}{r-d(x,y)},$$

and so, taking the supremum over all such u we obtain

$$1 \le \frac{d_U(x,y)}{d(x,y)} \le \frac{r}{r-d(x,y)}.$$

Therefore, when $y \in B(x,r/2)$ we know that $d(x,y) \le d_U(x,y) \le 2\,d(x,y)$, that is, the topology on U induced by d_U coincides with the subspace topology induced by d on U. It follows that for $x \in U$, we have that

$$\lim_{y\to x}\frac{d_U(x,y)}{d(x,y)} = 1.$$

Therefore, if $u \in \mathcal{D}_{\mathrm{loc}}(U) \cap C(U)$ such that $1_U\Gamma(u,u) \le 1_U m$, then for $x \in U$ we see that

$$1 \ge \mathrm{Lip}_{d_U} u(x) = \mathrm{Lip}_{d} u(x).$$

As X is a length space with respect to the metric d (see [28]), by the definition of d_U, when u is a candidate in computing d_U, we know that u is Lipschitz continuous on U with respect to the metric d_U and hence is locally Lipschitz continuous on U with respect to the metric d.

Now, for $z \in U$ and $y \in B(z,r)$ with $0 < r < \min\{1, d(z, \partial U)\}/10$, and $u \in \mathcal{D}_{\rm loc}(U) \cap C(U)$ with $1_U \Gamma(u,u) \le 1_U m$, we can choose for every $\epsilon > 0$ a curve γ_ϵ in U with end points z, y and $\ell_d(\gamma_\epsilon) \le \epsilon + d(z,y)$. By the discussion above, we know that u is locally Lipschitz continuous on U with respect to the metric d, and so $\mathrm{Lip}_d u$ is an upper gradient of u with respect to the metric d. It follows that

$$|u(z) - u(y)| \le \int_{\gamma_\epsilon} \mathrm{Lip}_d u \, ds \le \ell_d(\gamma_\epsilon) \le d(x,y) + \epsilon.$$

Taking the supremum over all such possible u, and then letting $\epsilon \to 0$, we obtain that $d_U(z,y) \le d(z,y)$, thus proving the following lemma, Lemma 2.7.

Lemma 2.7. *Let U be an open subset of X. Then for every $x \in U$, there exists $r_x \in (0, d(x, \partial U))$ such that $d_U(x, y) = d(x, y)$, whenever $y \in B(x, r_x)$.*

Proof of Lemma 2.5. We first show that

$$\Lambda := \sup_{x \in U} \mathrm{Lip}_d u(x) \le L := \operatorname*{esssup}_{x \in U} \sqrt{\frac{d}{dm}\Gamma(u, u)(x)}.$$

Indeed, for any $\epsilon > 0$, applying $(L+\epsilon)^{-1}u$ and $-(L+\epsilon)^{-1}u$ into the definition of d_U, we have $|u(x) - u(y)| \le (L+\epsilon) d_U(x, y)$ for all $x, y \in U$. Hence, by the arbitrariness of ϵ and Lemma 2.7, we have $|u(x) - u(y)| \le L\, d_U(x, y) = L\, d(x, y)$, which gives $\sup_{x\in U} \mathrm{Lip}_d u(x) \le L$ as desired.

Now we show $L \le \Lambda$ by using Lemma 2.3. Fix $x \in U$ and $r < \frac{1}{4} d(x, \partial U)$, such that $\overline{B}(z,r)$ is compact. Then by the result of [28], for any $y, z \in B(x, r)$ we can find a rectifiable curve $\gamma \subset U$ joining z, y with length $\ell_d(\gamma) = d(z, y)$. So

$$|u(z) - u(y)| \le \int_0^1 \mathrm{Lip}_d u(\gamma(t))\, dt \le \Lambda d(z, y),$$

which means that $\mathrm{Lip}_d(u, B(x, r)) \le \Lambda$. Define the McShane extension

$$\widetilde{u}(y) = \inf_{z \in B(x,r)} \{u(z) + \Lambda d(z, y)\}.$$

We know that $\widetilde{u} \in \mathrm{Lip}_d(X)$, and hence, by Lemma 2.3, $\widetilde{u} \in \mathcal{D}_{\rm loc}(X)$ with

$$\frac{d}{dm}\Gamma(\widetilde{u}, \widetilde{u})(z) \le \Lambda.$$

Therefore, $\frac{d}{dm}\Gamma(u, u)(z) \le \Lambda$ for almost all $z \in B_d(x, r)$. This implies that $L \le \Lambda$ and hence gives (2.4). $\square$

3. Equivalent characterizations of AM(Ω; $\mathscr{E}$, m)

In this section, we give characterizations for the class AM(Ω; $\mathscr{E}$, m) and prove Theorem 1.4. The following class AM(Ω; d) is equivalent to the class AM(Ω; $\mathscr{E}$, m) by Corollary 2.2 and Lemma 2.5.

Definition 3.1. Let $\Omega \subset X$ be an open set. Denote by AM(Ω; d) the class of functions $u \in \mathrm{Lip}_{d,\,\rm loc}(\Omega)$ satisfying that for each open set $U \Subset \Omega$ and function $v \in C(\overline{U}) \cap \mathrm{Lip}_{d,\,\rm loc}(U)$ with $v|_{\partial U} = u|_{\partial U}$,

$$\sup_{x\in U} \mathrm{Lip}_d u(x) \le \sup_{x \in U} \mathrm{Lip}_d v(x).$$

The comparison property with cones was first introduced by Crandall–Evans–Gariepy [11] on $\mathbb{R}^n$ to study infinity harmonic functions. The following metric variant have been studied in [6,19].

Definition 3.2. Let $\Omega \subset X$ be an open set. Denote by $\mathrm{CC}(\Omega;\, d)$ the *class of all functions* $u \in \mathrm{Lip}_{d,\,\mathrm{loc}}(\Omega)$ *satisfying the comparison property with cones*, that is, for each open subset $U \Subset \Omega$, and for all $a \geq 0$, $b \in \mathbb{R}$ and $x_0 \in X \setminus U$, we have that

$$\max_{x\in\partial U}[\pm u(x) - C_{b,a,x_0}(x)] \leq 0 \text{ implies } \max_{x\in U}[\pm u(x) - C_{b,a,x_0}(x)] \leq 0,$$

where $C_{b,a,x_0}(x) = b + a\,d(x,\, x_0)$.

The following result gives Theorem 1.4.

Theorem 3.3. *Let $\Omega \subset X$ be a bounded open set.*

(I) *The following are equivalent.*
- (i) $u \in \mathrm{AML}(\Omega;\, d)$*;*
- (ii) $u \in \mathrm{AM}(\Omega;\, d)$*;*
- (iii) $u \in \mathrm{CC}(\Omega;\, d)$*;*
- (iv) $u \in \mathrm{AM}(\Omega;\, \mathscr{E},\, m)$.

(II) *For every $f \in C(\partial\Omega)$ there is one, and only one, function in* $\mathrm{AM}_f(\Omega;\, \mathscr{E},\, m)$.

Proof. For the proof of the equivalence of (i), (ii) and (iii), see for example [6,19]. The existence of $u \in \mathrm{AML}_f(\Omega; d)$ follows from Perron's method; for details see [6,18,22]. The uniqueness $u \in \mathrm{AML}_f(\Omega; d)$ is obtained in [24]; for a simple proof see [1] (and also [21]) with the observation that the argument in [1] also works for length spaces. With these, it suffices to show the equivalence of (ii) and (iv). However, this follows from Lemma 2.5 together with Corollary 2.2. □

We have the following corollary.

Corollary 3.4. *Let $\widetilde{\mathscr{E}}$ be another regular, strongly local Dirichlet form such that the topology induced by the intrinsic distance $\widetilde{d}$ of $\widetilde{\mathscr{E}}$ and the original topology on X coincide. If*

$$\lim_{x\neq y\to x} \frac{d(x,\, y)}{\widetilde{d}(x,\, y)} = 1 \tag{3.1}$$

for all $x \in \Omega$, then $u \in \mathrm{AM}(\Omega; \mathscr{E},\, m)$ if and only if $u \in \mathrm{AM}(\Omega; \widetilde{\mathscr{E}},\, m)$.

A special case of (3.1) is that for every $x \in \Omega$, there exists $r_x > 0$ such that $d(x,\, y) = \widetilde{d}(x,\, y)$ for all $y \in B_d(x,\, r_x)$.

Remark 3.5. In view of Theorem 3.3 and Corollary 3.4, the local behavior of the intrinsic distance totally determines the absolute minimizer.

4. Infinity harmonic functions on $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a proper open subset. Let u be an infinity harmonic function on Ω, that is, a viscosity solution of the ∞-Laplace equation (1.1). Jensen [17] proved that $u \in \mathrm{AM}(\Omega;\, \mathscr{E}_{\mathbb{R}^n},\, dx)$, that is, u is an absolute minimizer on Ω associated to the Dirichlet energy form $(\mathbb{R}^n,\, \mathscr{E}_{\mathbb{R}^n},\, dx)$. Recall that the intrinsic distance of $\mathscr{E}_{\mathbb{R}^n}$ is just the Euclidean distance $|\cdot|$.

On the other hand, Sturm [26, Theorem 2] proved that there exist infinitely many matrix valued maps $\widetilde{A}$ such that the intrinsic distances $d_{\widetilde{A}}$ of the associated Dirichlet forms $\mathscr{E}_{\widetilde{A}}$ are equivalent to the Euclidean distance in the sense of (3.1), and that for all $x \in \mathbb{R}^n$, we have

$$\delta\langle\xi,\, \xi\rangle \leq \langle\widetilde{A}(x)\xi,\, \xi\rangle < \langle\xi,\, \xi\rangle$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $\delta \in (0,\, 1)$ is a constant. Note that the associated Dirichlet form $\mathscr{E}_{\widetilde{A}}$ is given by: for every pair $f,\, h \in W^{1,\,2}(\mathbb{R}^n)$

$$\mathscr{E}_{\widetilde{A}}(f,\, h) = \int_{\mathbb{R}^n} \langle\widetilde{A}(x)\nabla f(x), \nabla h(x)\rangle\, dx;$$

and it is a regular, strongly local Dirichlet form. Indeed, $\mathscr{E}_{\widetilde{A}}$ and $\mathscr{E}_{\mathbb{R}^n}$ are comparable. So by Corollary 3.4, we have the following conclusion for these $\widetilde{A}$.

Proposition 4.1. *The following are equivalent:*

(i) *u is an infinity harmonic function on Ω.*
(ii) *$u \in \mathrm{AM}(\Omega;\, \mathscr{E}_{\mathbb{R}^n},\, dx)$.*
(iii) *$u \in \mathrm{AM}(\Omega;\, \mathscr{E}_{\widetilde{A}},\, dx)$.*

Remark 4.2. The phenomenon revealed by the above proposition also holds for viscosity solutions to a large number of Aronsson equations. Indeed, given any elliptic matrix-valued map A, Sturm actually constructed infinitely many perturbations $\widetilde{A}$ that generate the same intrinsic metric. For the ellipticity condition see Section 5 below. For these we still have $u \in \mathrm{AM}(\Omega;\, \mathscr{E}_A,\, dx)$ if and only if $u \in \mathrm{AM}(\Omega;\, \mathscr{E}_{\widetilde{A}},\, dx)$. If one further assumes that $A \in C^1$, then $u \in \mathrm{AM}(\Omega;\, \mathscr{E}_A,\, dx)$ is a solution to the Aronsson equation:

$$\langle\nabla_x H(x,\, \nabla u(x)), \nabla_\xi H(x,\, \nabla u(x))\rangle = 0,$$

where $H(x,\, \xi) = \langle A(x)\xi,\, \xi\rangle$; see [12]. Moreover, if $A \in C^2$, then by [30], a solution to this equation also belongs to $\mathrm{AM}(\Omega;\, \mathscr{E}_A,\, dx)$. So under the assumption $A \in C^2$, we have that u is a solution to the above Aronsson equation if and only if $u \in \mathrm{AM}(\Omega;\, \mathscr{E}_A,\, dx)$, and if and only if $u \in \mathrm{AM}(\Omega;\, \mathscr{E}_{\widetilde{A}},\, dx)$.

5. Absolute minimizers on Carnot–Carathéodory spaces

The infinity harmonic functions on Heisenberg groups, Carnot groups and also Carnot–Caratheodory spaces have been studied by many authors; see for example [8–10, 29]. Applying our Theorem 3.3, we extend some of their results to settings in which we may not have the corresponding Aronsson equation.

Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq m \leq n$. Assume that $\mathbf{X} = \{\mathbf{X}_i\}_{i=1}^m \subset C(\Omega; \mathbb{R}^n)$ is a family of smooth vector fields satisfying Hörmander's finite rank condition, that is, there is an integer $r \geq 1$ such that $\{\mathbf{X}_i(x)\}_{i=1}^m$ and their commutators up to order r span $\mathbb{R}^n$ at every point $x \in \Omega$. For every $x \in \Omega$, the horizontal tangent space $\mathscr{H}(x)$ is given by

$$\mathscr{H}(x) = \mathrm{span}\{\mathbf{X}_1(x), \ldots,\, \mathbf{X}_m(x)\}.$$

The horizontal Sobolev space $\dot{W}^{1,\,p}_{\mathbf{X}}(\Omega)$ is the completion of the collection of all functions $u \in C_c^\infty(\Omega)$ with $\mathbf{X}u \in L^p(\Omega; \mathbb{R}^m)$, and $\|u\|_{\dot{W}^{1,\,p}_{\mathbf{X}}(\Omega)} = \||\mathbf{X}u|\|_{L^p(\Omega)}$; see [16, Section 11.2]. Let $W^{1,p}_{\mathbf{X}}(\Omega) = L^p(\Omega;\mathbb{R}^m) \cap \dot{W}^{1,p}_{\mathbf{X}}(\Omega)$ with the norm $\|u\|_{W^{1,p}_{\mathbf{X}}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + \|u\|^p_{\dot{W}^{1,p}_{\mathbf{X}}(\Omega)})^{1/p}$.

Denote by $\mathscr{A}$ the collection of all matrix-valued measurable maps

$$A = (a_{ij})_{1\le i,\, j\le m} :\ \Omega \to \mathbb{R}^{m\times m},$$

which are *elliptic*, that is, for each $A \in \mathscr{A}$, there exists a continuous function $\lambda :\ \Omega \to [1, \infty)$ such that

$$\frac{1}{\lambda(x)}|\xi|^2 \le \langle A(x)\xi,\, \xi\rangle_{\mathbb{R}^m} \le \lambda(x)|\xi|^2 \tag{5.1}$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^m$, where

$$\langle A(x)\xi,\, \xi\rangle_{\mathbb{R}^m} = \sum_{i,\, j=1}^{m} \xi_i a_{ij}\xi_j.$$

An *Hamiltonian* associated to A is given by $H(x,\, \xi) = \langle A(x)\xi,\, \xi\rangle_{\mathbb{R}^m}$.

Then an absolute minimizer of A and $\mathbf{X}$ is defined as follows. Here $\mathscr{L}$ is the n-dimensional Lebesgue measure on Ω.

Definition 5.1. Let $U \subset \Omega$ be an open set. A function $u : U \to \mathbb{R}$ is said to be an *absolute minimizer of* A *and* $\mathbf{X}$ (for short, $u \in \mathrm{AM}(U;\, A,\, \mathbf{X},\, \mathscr{L})$) if for any $V \Subset U$ and any $v \in W^{1,\infty}_{\mathbf{X}}(V) \cap C(\overline{V})$ with $u|_{\partial V} = v|_{\partial V}$, we have $F_{A,\mathbf{X}}(u,\, V) \le F_{A,\mathbf{X}}(v,\, V)$, where

$$F_{A,\mathbf{X}}(u,\, V) = \operatorname*{esssup}_{x\in V} H(x,\, \mathbf{X}u(x)).$$

Proposition 5.2. *Let $A \in \mathscr{A}(\Omega)$ and $U \Subset \Omega$ be a bounded open set.*

(I) *The following are equivalent*
  (i) $u \in \mathrm{AM}(U;\, A,\, \mathbf{X},\, \mathscr{L})$;
  (ii) $u \in \mathrm{AM}(U; d_{A,\mathbf{X}})$;
  (iii) $u \in \mathrm{AML}(U; d_{A,\mathbf{X}})$.

(II) *For each $f \in C(\partial U)$, there exists a unique $u \in \mathrm{AM}_f(U;\, A,\, \mathbf{X},\, dx)$.*

Proof. Without loss of generality, we assume that $\lambda(x)$ is bounded on Ω. Otherwise, we consider a domain $\widetilde{\Omega} \Subset \Omega$ with $U \Subset \widetilde{\Omega}$. Now define a bilinear form on Ω associated to A and $\mathbf{X}$ by: for all $f,\, h \in W^{1,2}_{\mathbf{X}}(\Omega)$, we have

$$\mathscr{E}_{A,\mathbf{X}}(f,\, h) = \int_{\mathbb{R}^n} \langle A(x)\mathbf{X}f(x),\, \mathbf{X}h(x)\rangle_{\mathbb{R}^m}\, d\mathscr{L}(x).$$

We claim that $\mathscr{E}_{A,\mathbf{X}}$ is a regular, strongly local Dirichlet form with domain $\mathcal{D} = W^{1,2}_{\mathbf{X}}(\Omega)$. Indeed, obviously, $\mathscr{E}_{A,\mathbf{X}}$ is a non-negative definite and symmetric bilinear form. The closedness follows from the completeness of $W^{1,2}_{\mathbf{X}}(\Omega)$. The Markov property also follows via an argument similar to that for the classical Dirichlet form $\mathscr{E}_{\mathbb{R}^n}$. The strong locality is easy to see. The regularity condition follows from the fact that $C^\infty_c(\Omega)$ is dense in $\dot{W}^{1,2}_{\mathbf{X}}(\Omega)$ by a standard approximating argument.

Moreover, we also see that the topology induced by the intrinsic distance $d_{A,\mathbf{X}}$ of $\mathscr{E}_{A,\mathbf{X}}$ coincides with original topology on $\mathbb{R}^n$. Here, observing

$$\frac{d}{dx}\Gamma(f,\, h)(x) = \langle A(x)\mathbf{X}f(x),\, \mathbf{X}h(x)\rangle_{\mathbb{R}^m},$$

we have

$$d_{A,\mathbf{X}}(x, y) = \sup\{u(x) - u(y) : u \in W^{1,\infty}_{X,\,\mathrm{loc}}(\Omega) : \ H(x, \mathbf{X}u(x)) \le 1 \ a.e.\}.$$

Indeed, this follows from the fact that, on each bounded open set $\widetilde{\Omega} \Subset \Omega$,

$$d_{I_m,\mathbf{X}}(x, y) \sim d_{A,\mathbf{X}}(x, y)$$

and

$$\frac{1}{C(\widetilde{\Omega})}|x - y| \le d_{I_m,\mathbf{X}}(x, y) \le C(\widetilde{\Omega})|x - y|^{1/m}$$

for all $x, y \in \widetilde{\Omega}$; see [23].

Note that $u \in \mathrm{AM}(U; A, \mathbf{X}, \mathscr{L})$ if and only if $u \in \mathrm{AM}(U; \mathscr{E}_{A,\mathbf{X}}, \mathscr{L})$. Then applying Theorem 3.3, we have Proposition 5.2. □

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Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35 (MaD), FI-40014, University of Jyväskylä, Finland
E-mail address: `pkoskela@maths.jyu.fi`

Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, OH 45221-0025, USA
E-mail address: `shanmun@uc.edu`

Department of Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100191, People's Republic of China
E-mail address: `yuanzhou@buaa.edu.cn`