

## STRONG L-SPACES AND LEFT-ORDERABILITY

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ABSTRACT. We introduce the notion of a strong L-space, a closed, oriented three-manifold admitting a Heegaard diagram whose associated Heegaard Floer complex has rank equal to the order of the first homology of the manifold. Examples of strong L-spaces include the branched double covers of alternating links in the three-sphere. We prove that the fundamental group of a strong L-space is not left-orderable.

## 1. Introduction

Heegaard Floer homology, developed by Ozsváth and Szabó [12] in the early 2000s, has been an extremely effective tool for answering classical questions about 3-manifolds. However, surprisingly little is known about the relationship between Heegaard Floer homology and topological properties of Heegaard splittings, even though a Heegaard diagram is an essential ingredient in defining the Heegaard Floer homology of a closed 3-manifold  $Y$ . In particular, a Heegaard diagram provides a presentation of the fundamental group of  $Y$ , and it is natural to ask how this presentation is related to the Heegaard Floer chain complex. In this paper, we shall investigate one such connection.

A *left-ordering* on a non-trivial group  $G$  is a total order  $<$  on the elements of  $G$  such that  $g < h$  implies  $kg < kh$  for any  $g, h, k \in G$ . A group  $G$  is called *left-orderable* if it is non-trivial and admits at least one left-ordering. The question of which 3-manifolds have left-orderable fundamental group has been of considerable interest and is closely connected to the study of foliations. For instance, if  $Y$  admits a co-oriented,  $\mathbb{R}$ -covered foliation (i.e., a taut, co-oriented foliation such that the leaf-space of the induced foliation on the universal cover  $\tilde{Y}$  is homeomorphic to  $\mathbb{R}$ ), then  $\pi_1(Y)$  is left-orderable. Boyer, Rolfsen, and Wiest [3] showed that the fundamental group of any irreducible three-manifold  $Y$  with  $b_1(Y) > 0$  is left-orderable, reducing the question to that of rational homology spheres.

In its simplest form, Heegaard Floer homology associates to a closed, oriented three-manifold  $Y$  a  $\mathbb{Z}/2\mathbb{Z}$ -graded, finitely generated abelian group  $\widehat{\text{HF}}(Y)$ . This group is computed as the homology of a free chain complex  $\widehat{\text{CF}}(\mathcal{H})$  associated to a Heegaard diagram  $\mathcal{H}$  for  $Y$ ; different choices of diagrams for the same manifold yield chain-homotopy-equivalent complexes. The group  $\widehat{\text{CF}}(\mathcal{H})$  depends only on the combinatorics of  $\mathcal{H}$ , but the differential on  $\widehat{\text{CF}}(\mathcal{H})$  involves counts of holomorphic curves that rely on auxiliary choices of analytic data. If  $Y$  is a rational homology sphere, then the Euler characteristic of  $\widehat{\text{HF}}(Y)$  is equal to  $|H_1(Y; \mathbb{Z})|$ , which implies that the rank of  $\widehat{\text{HF}}(Y)$  is at least  $|H_1(Y; \mathbb{Z})|$ .  $Y$  is called an *L-space* if  $\widehat{\text{HF}}(Y) \cong \mathbb{Z}^{|H_1(Y; \mathbb{Z})|}$ ; thus, L-spaces have the simplest possible Heegaard Floer homology. Examples of L-spaces include  $S^3$ , lens

spaces (whence the name), all manifolds with finite fundamental group, and branched double covers of alternating (or, more broadly, *quasi-alternating*) links. Additionally, Ozsváth and Szabó [10] showed that if  $Y$  is an L-space, it does not admit any taut foliation; whether the converse is true is an open question.

The following related conjecture, stated formally by Boyer, Gordon, and Watson [2], has recently been the subject of considerable attention:

**Conjecture 1.** *Let  $Y$  be a closed, connected, irreducible 3-manifold. Then  $\pi_1(Y)$  is not left-orderable if and only if  $Y$  is an L-space.*

This conjecture is now known to hold for all geometric, non-hyperbolic three-manifolds [2].<sup>1</sup> Additionally, Boyer, Gordon, and Watson [2] and Greene [5] have shown that the branched double cover of any non-split alternating link in  $S^3$  — which is generically a hyperbolic three-manifold — has non-left-orderable fundamental group.

In this paper, we prove the “if” direction of Conjecture 1 for manifolds that are “L-spaces on the chain level.” To be precise, we call a three-manifold  $Y$  a *strong L-space* if it admits a Heegaard diagram  $\mathcal{H}$  such that  $\widehat{\text{CF}}(\mathcal{H}) \cong \mathbb{Z}^{|\mathcal{H}_1(Y; \mathbb{Z})|}$ . This purely combinatorial condition implies that the differential on  $\widehat{\text{CF}}(\mathcal{H})$  vanishes, without any consideration of holomorphic discs. We call such a Heegaard diagram a *strong Heegaard diagram*. By considering the presentation for  $\pi_1(Y)$  associated to a strong Heegaard diagram, we prove:

**Theorem 1.** *If  $Y$  is a strong L-space, then  $\pi_1(Y)$  is not left-orderable.*

The standard Heegaard diagram for a lens space is easily seen to be a strong diagram. Moreover, Greene [6] constructed a strong Heegaard diagram for the branched double cover of any alternating link in  $S^3$ ; indeed, Boyer, Gordon, and Watson’s proof that the fundamental group of such a manifold is not left-orderable essentially makes use of the group presentation for  $\pi_1$  associated to that Heegaard diagram. In fact, it is quite tempting to conjecture that every strong L-space can be realized as the branched double cover of an alternating link in  $S^3$ . While this conjecture is perhaps overly ambitious, we do not know of any counterexamples at the present. Indeed, while it is not hard to construct families of low-genus strong Heegaard diagrams that *a priori* seem unrelated to Greene’s construction (in which the genus of the Heegaard diagram equals the number of crossings in an alternating diagram for the link), the manifolds presented by these diagrams nevertheless turn out to be branched double covers of alternating links. Although an affirmative answer to this conjecture would render Theorem 1 redundant, it would provide an alternate topological characterization of the manifolds arising as branched double covers of alternating links and thus partially answer R. H. Fox’s famous open question: “What is an alternating knot?” [7, p. 32]. In any case, Theorem 1 seems like a useful step in the direction of Conjecture 1 in that it relies only on data contained in the Heegaard Floer chain complex to prove a result about the fundamental group.

Additionally, the following theorem, which is well known but does not appear in the literature, indicates that being a strong L-space is a fairly restrictive condition:

<sup>1</sup>Specifically, the work of Boyer, Rolfsen, and Wiest [3] and Lisca and Stipsicz [8] gives the result for Seifert manifolds with base orbifold  $S^2$ , as was also observed by Peters [13]. The cases of Seifert manifolds with non-orientable base orbifold and of Sol manifolds follow from [3] and [2].

**Theorem 2.** *If  $Y$  is an integer homology sphere that is a strong L-space, then  $Y \cong S^3$ .*

In particular, there exist integer homology spheres that are L-spaces (e.g., the Poincaré homology sphere) but not strong L-spaces. The fact that the condition of being a strong L-space detects  $S^3$  suggests that it might be possible to obtain a more explicit characterization or even a complete classification of strong L-spaces. Below, we shall present a graph-theoretic proof of Theorem 2. In subsequent work with Josh Greene, we shall extend this argument to show that for any  $n$ , there are finitely many strong L-spaces with  $|H_1(Y; \mathbb{Z})| \leq n$ . Apropos of the previous discussion, this result is reminiscent of the classical theorem of Bankwitz [1] and Crowell [4] that there are only finitely many alternating knots with a given determinant.

**2. Proofs of Theorems 1 and 2**

To prove Theorem 1, we will use a simple obstruction to left-orderability that can be applied to group presentations.

Let  $X$  denote the set of symbols  $\{0, +, -, *\}$ . These symbols are meant to represent the possible signs of real numbers:  $+$  and  $-$  represent positive and negative numbers, respectively, and  $*$  represents a number whose sign is not known. As such, we define a commutative, associative multiplication operation on  $X$  by the following rules: (1)  $0 \cdot \epsilon = \epsilon \cdot 0 = 0$  for any  $\epsilon \in X$ ; (2)  $+\cdot + = -\cdot - = +$ ; (3)  $+\cdot - = -\cdot + = -$ ; and (4)  $\epsilon \cdot * = * \cdot \epsilon = *$  for  $\epsilon \in \{+, -, *\}$ .

A group presentation  $\mathcal{G} = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$  gives rise to an  $m \times n$  matrix  $E(\mathcal{G}) = (\epsilon_{i,j})$  with entries in  $X$  by the following rule:

$$(2.1) \quad \epsilon_{i,j} = \begin{cases} 0 & \text{if neither } x_i \text{ nor } x_i^{-1} \text{ occurs in } r_j \\ + & \text{if } x_i \text{ appears in } r_j \text{ but } x_i^{-1} \text{ does not} \\ - & \text{if } x_i^{-1} \text{ appears in } r_j \text{ but } x_i \text{ does not} \\ * & \text{if both } x_i \text{ and } x_i^{-1} \text{ occur in } r_j. \end{cases}$$

**Lemma 1.** *Let  $\mathcal{G} = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$  be a group presentation such that for any  $d_1, \dots, d_m \in \{0, +, -\}$ , not all zero, the matrix  $M$  obtained from  $E(\mathcal{G})$  by multiplying the  $i^{\text{th}}$  row by  $d_i$  has a non-zero column whose non-zero entries are either all  $+$  or all  $-$ . Then the group  $G$  presented by  $\mathcal{G}$  is not left-orderable.*

*Proof.* Suppose that  $<$  is a left-ordering on  $G$ , and let  $d_i$  be  $0, +$ , or  $-$  according to whether  $x_i = 1, x_i > 1$ , or  $x_i < 1$  in  $G$ . Since  $G$  is non-trivial, at least one of the  $d_i$  is non-zero. If the  $j^{\text{th}}$  column of  $M$  is non-zero and has entries in  $\{0, +\}$ , the relator  $r_j$  is a product of generators  $x_i$  that are all non-negative in  $G$ , and at least one of which is strictly positive. Thus,  $r_j > 1$  in  $G$ , which contradicts the fact that  $r_j$  is a relator. An analogous argument applies for a non-zero column with entries in  $\{0, -\}$ .  $\square$

We shall focus on presentations with the same number of generators as relations. For a permutation  $\sigma \in S_n$ , let  $\text{sign}(\sigma) \in \{+, -\}$  denote the sign of  $\sigma$  ( $+$  if  $\sigma$  is even,  $-$  if  $\sigma$  is odd). The key technical lemma is the following:

**Lemma 2.** *Let  $\mathcal{G} = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$  be a group presentation such that  $E(\mathcal{G})$  has the following properties:*

- (1) *There exists at least one permutation  $\sigma_0 \in S_n$  such that the entries  $\epsilon_{1,\sigma_0(1)}, \dots, \epsilon_{n,\sigma_0(n)}$  are all non-zero.*

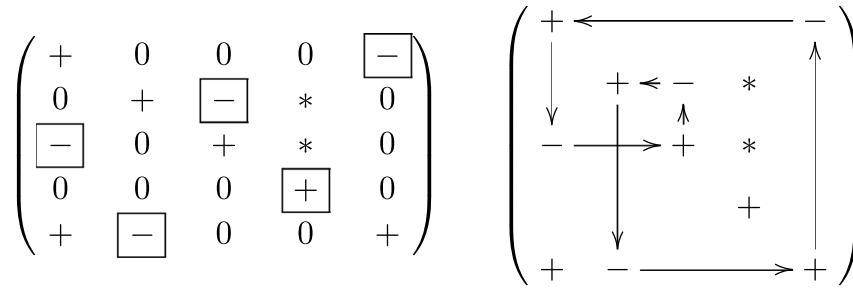


FIGURE 1. Illustration of the proof of Lemma 2. In the matrix  $M$  shown at left, the entries  $m_{i,\sigma(i)}$  are highlighted, where  $\sigma$  is the permutation constructed in the proof. To find  $\sigma$ , we start with the  $+$  in the upper left corner, travel to a  $-$  in the same column, and then travel to the diagonal entry in the same row as this  $-$ . Repeating this procedure, we eventually obtain a closed loop, as shown at right.

- (2) For any permutation  $\sigma \in S_n$  such that  $\epsilon_{1,\sigma(1)}, \dots, \epsilon_{n,\sigma(n)}$  are all non-zero, we have  $\epsilon_{1,\sigma(1)}, \dots, \epsilon_{n,\sigma(n)} \in \{+, -\}$ .
- (3) For any two permutations  $\sigma, \sigma'$  as in (2), we have

$$\text{sign}(\sigma) \cdot \epsilon_{1,\sigma(1)} \cdot \dots \cdot \epsilon_{n,\sigma(n)} = \text{sign}(\sigma') \cdot \epsilon_{1,\sigma'(1)} \cdot \dots \cdot \epsilon_{n,\sigma'(n)}.$$

Then the group  $G$  presented by  $\mathcal{G}$  is not left-orderable.

In other words, if we consider a formal determinant

$$\det(E(\mathcal{G})) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot \epsilon_{1,\sigma(1)} \cdot \dots \cdot \epsilon_{n,\sigma(n)},$$

viewed as a formal sum of elements of  $X$ , condition (1) says that at least one summand is non-zero, condition (2) says that no non-zero summand is  $*$ , and condition (3) says all the non-zero elements are the same.

*Proof.* By reordering the generators and relations, it suffices to assume that  $\sigma_0$  from condition (1) is the identity, so that  $\epsilon_{i,i} \neq 0$  for  $i = 1, \dots, n$ , and hence  $\epsilon_{i,i} \in \{+, -\}$  by condition (2). We shall show that  $E(\mathcal{G})$  satisfies the hypotheses of Lemma 1.

Suppose, then, toward a contradiction, that  $d_1, \dots, d_n$  are elements of  $\{0, +, -\}$ , not all zero, such that every non-zero column of the matrix  $M$  obtained as in Lemma 1 contains a non-zero off-diagonal entry (perhaps a  $*$ ) that is not equal to the diagonal entry in that column. Denote the  $(i, j)^{\text{th}}$  entry of  $M$  by  $m_{i,j}$ .

We may inductively construct a sequence of distinct indices  $i_1, \dots, i_k \in \{1, \dots, n\}$  such that

- (A)  $m_{i_j, i_j} \in \{+, -\}$  for each  $j = 1, \dots, m$ , and
- (B)  $m_{i_{j+1}, i_j} \neq 0$  and  $m_{i_{j+1}, i_j} \neq m_{i_j, i_j}$

for each  $j = 1, \dots, k$ , taken modulo  $k$ . This is done by “connecting the dots” as in Figure 1. Specifically, we begin by choosing any  $i_1$  such that  $m_{i_1, i_1} \neq 0$ . Given  $i_j$ , our assumption on  $M$  states that we can choose  $i_{j+1}$  satisfying assumption (B) above; we then have  $m_{i_{j+1}, i_{j+1}} \neq 0$  since otherwise the whole  $i_{j+1}^{\text{th}}$  row would have to be zero. Repeating this procedure, we eventually obtain an index  $i_k$  that is equal to some

previously occurring index  $i_{k'}$ , where  $k' + 1 < k$ . The sequence  $i_{k'+1}, \dots, i_k$ , relabeled accordingly, then satisfies the assumptions (A) and (B).

Define a  $k$ -cycle  $\sigma \in S_n$  by  $\sigma(i_j) = i_{j+1}$  for  $j = 1, \dots, k \bmod k$ , and  $\sigma(i') = i'$  for  $i' \notin \{i_1, \dots, i_k\}$ . By construction,  $\epsilon_{i, \sigma(i)} \neq 0$  for each  $i = 1, \dots, n$ , so the sequence  $(\epsilon_{1, \sigma(1)}, \dots, \epsilon_{n, \sigma(n)})$  contains no \*s by condition (2). The sequences  $(\epsilon_{1, \sigma(1)}, \dots, \epsilon_{n, \sigma(n)})$  and  $(\epsilon_{1, 1}, \dots, \epsilon_{n, n})$  differ in exactly  $k$  entries, and the signature of  $\sigma$  is  $(-1)^{k-1}$ . This implies that

$$\text{sign}(\sigma) \cdot \epsilon_{1, \sigma(1)} \cdot \dots \cdot \epsilon_{n, \sigma(n)} = (-1)^{2k-1} \text{sign}(\text{id}) \cdot \epsilon_{1, 1} \cdot \dots \cdot \epsilon_{n, n},$$

which contradicts condition (3). This completes the proof. □

**Remark 1.** The classification of matrices satisfying these three conditions is actually a version of Pólya’s permanent problem [14], dating to 1913, which asks when the permanent of a matrix of real numbers can be computed as the determinant of a matrix obtained by changing the signs of some entries. The version described here was first studied by Samuelson [15] in 1947 in the context of linear differential equations arising in economics. For an excellent review of Pólya’s permanent problem, including a proof of a version of Lemma 2; see McCuaig [9]. (The authors discovered Lemma 2 independently.)

Now we will apply Lemma 2 to prove Theorem 1. We first recall some basic facts about the Heegaard Floer chain complex. A *Heegaard diagram* is a tuple  $\mathcal{H} = (\Sigma, \alpha, \beta)$ , where  $\Sigma$  is a closed, oriented surface of genus  $g$ ,  $\alpha = (\alpha_1, \dots, \alpha_g)$  and  $\beta = (\beta_1, \dots, \beta_g)$  are each  $g$ -tuples of pairwise disjoint simple closed curves on  $\Sigma$  that are linearly independent in  $H_1(\Sigma; \mathbb{Z})$ , and each pair of curves  $\alpha_i$  and  $\beta_j$  intersect transversely. A Heegaard diagram  $\mathcal{H}$  determines a closed, oriented three-manifold  $Y = Y_{\mathcal{H}}$  with a self-indexing Morse function  $f : Y \rightarrow [0, 3]$  such that  $\Sigma = f^{-1}(3/2)$ , the  $\alpha$  circles are the belt circles of the 1-handles of  $Y$ , and the  $\beta$  circles are the attaching circles of the 2-handles. If we orient the  $\alpha$  and  $\beta$  circles, the Heegaard diagram determines a group presentation

$$\pi_1(Y) = \langle a_1, \dots, a_g \mid b_1, \dots, b_g \rangle,$$

where the generators  $a_1, \dots, a_g$  correspond to the  $\alpha$  circles, and  $b_j$  is the word obtained as follows: If  $p_1, \dots, p_k$  are the intersection points of  $\beta_j$  with the  $\alpha$  curves, indexed according to the order in which they occur as one traverses  $\beta_i$ , and  $p_\ell \in \alpha_{i_\ell} \cap \beta_i$  for  $\ell = 1, \dots, k$ , then

$$(2.2) \quad b_j = \prod_{\ell=1}^k a_{i_\ell}^{\eta(p_i)},$$

where  $\eta(p_i) \in \{\pm 1\}$  is the local intersection number of  $\alpha_{i_\ell}$  and  $\beta_j$  at  $p_i$ .

Let  $\text{Sym}^g(\Sigma)$  denote the  $g^{\text{th}}$  symmetric product of  $\Sigma$ , and let  $\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \text{Sym}^g(\Sigma)$  be the  $g$ -dimensional tori  $\alpha_1 \times \dots \times \alpha_g$  and  $\beta_1 \times \dots \times \beta_g$ , which intersect transversally in a finite number of points. Assuming  $Y$  is a rational homology sphere,  $\widehat{\text{CF}}(\mathcal{H})$  is the free abelian group generated by points in  $\mathfrak{S}_{\mathcal{H}} = \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ .<sup>2</sup> More explicitly, these are tuples  $\mathbf{x} = (x_1, \dots, x_g)$ , where  $x_i \in \alpha_i \cap \beta_{\sigma(i)}$  for some permutation  $\sigma \in S_g$ . The differential on  $\widehat{\text{CF}}(\mathcal{H})$  counts holomorphic Whitney discs connecting points of  $\mathfrak{S}_{\mathcal{H}}$  (and depends

<sup>2</sup>For general three-manifolds, we must restrict to a particular class of so-called admissible diagrams.

on an additional choice of a basepoint  $z \in \Sigma$ ), but we do not need to describe this in any detail here.

Orienting the  $\alpha$  and  $\beta$  circles determines orientations of  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta$ . For  $\mathbf{x} \in \mathfrak{S}_\mathcal{H}$ , let  $\eta(\mathbf{x})$  denote the local intersection number of  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta$  at  $\mathbf{x}$ . If  $\mathbf{x} = (x_1, \dots, x_g)$  with  $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ , we have

$$(2.3) \quad \eta(\mathbf{x}) = \text{sign}(\sigma) \prod_{i=1}^g \eta(x_i).$$

These orientations determine a  $\mathbb{Z}/2$ -valued grading  $\text{gr}$  on  $\widehat{\text{CF}}(Y)$  by the rule that  $(-1)^{\text{gr}(\mathbf{x})} = \eta(\mathbf{x})$ ; the differential shifts this grading by 1. If  $Y$  is a rational homology sphere, then with respect to this grading, we have  $\chi(\widehat{\text{CF}}(\mathcal{H})) = \pm |H_1(Y; \mathbb{Z})|$ , and we may choose the orientations such that the sign is positive. (See [11, Section 5] for further details.)

The proof of Theorem 1 is completed with the following:

**Lemma 3.** *If  $\mathcal{H}$  is a strong Heegaard diagram for a strong  $L$ -space  $Y$ , then the corresponding presentation for  $\pi_1(Y)$  satisfies the hypotheses of Lemma 2.*

*Proof.* If  $\text{rank}(\widehat{\text{CF}}(\mathcal{H})) = \chi(\widehat{\text{CF}}(\mathcal{H})) = |H_1(Y; \mathbb{Z})|$ , then  $\widehat{\text{CF}}(\mathcal{H})$  is supported in a single grading, so  $\eta(\mathbf{x}) = 1$  for all  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ . The result then follows quickly from equations (2.1), (2.2), and (2.3). Specifically, since  $\mathfrak{S}_\mathcal{H} \neq \emptyset$ , there exists  $\sigma_0 \in S_g$  such that  $\alpha_i \cap \beta_{\sigma_0(i)} \neq \emptyset$  for each  $i$ , and hence  $\epsilon_{i, \sigma_0(i)} \neq 0$ . If  $\alpha_i$  and  $\beta_j$  contain a point  $x$  that is part of some  $\mathbf{x} \in \mathfrak{S}_\mathcal{H}$ , then every other point  $x' \in \alpha_i \cap \beta_j$  has  $\eta(x') = \eta(x)$ , and hence  $\epsilon_{i,j} = \eta(\mathbf{x}) \in \{+, -\}$ . Finally, if  $\mathbf{x} = (x_1, \dots, x_g)$  and  $\mathbf{x}' = (x'_1, \dots, x'_g)$ , with  $x_i \in \alpha_i \cap \beta_{\sigma(i)}$  and  $x'_i \in \alpha_i \cap \beta_{\sigma'(i)}$ , then equation (2.3) and the fact that  $\eta(\mathbf{x}) = \eta(\mathbf{x}')$  imply the final hypothesis.  $\square$

Finally, to prove Theorem 2, we use a simple graph-theoretic argument due to Josh Greene. Given a Heegaard diagram  $\mathcal{H}$ , let  $\Gamma_\mathcal{H}$  denote the bipartite graph with vertex sets  $\mathcal{A} = \{A_1, \dots, A_g\}$  and  $\mathcal{B} = \{B_1, \dots, B_g\}$ , with an edge connecting  $A_i$  and  $B_j$  for each intersection point in  $\alpha_i \cap \beta_j$ . The set  $\mathfrak{S}_\mathcal{H}$  thus corresponds to the set of perfect matchings on  $\Gamma_\mathcal{H}$ .

**Lemma 4.** *If  $\mathcal{H}$  is a Heegaard diagram of genus  $g > 1$ , and  $\Gamma_\mathcal{H}$  contains a leaf (a 1-valent vertex), then  $Y_\mathcal{H}$  admits a Heegaard diagram  $\mathcal{H}'$  of genus  $g - 1$  with a bijection between  $\mathfrak{S}_\mathcal{H}$  and  $\mathfrak{S}_{\mathcal{H}'}$ .*

*Proof.* If  $A_i$  is 1-valent, then the curve  $\alpha_i$  intersects one  $\beta$  curve, say  $\beta_j$ , in a single point and is disjoint from the remaining  $\beta$  curves. By a sequence of handleslides of the  $\alpha$  curves, we may remove any intersections of  $\beta_j$  with any  $\alpha$  curve other than  $\alpha_i$ , without introducing or removing any other intersection points. We may then destabilize to obtain  $\mathcal{H}'$ . Since every element of  $\mathfrak{S}_\mathcal{H}$  includes the unique point of  $\alpha_i \cap \beta_j$ , we have a bijection between  $\mathfrak{S}_\mathcal{H}$  and  $\mathfrak{S}_{\mathcal{H}'}$ . (Indeed,  $\Gamma'_\mathcal{H}$  is obtained from  $\Gamma_\mathcal{H}$  by deleting  $A_i$  and  $B_j$ , which does not change the number of perfect matchings.) The case where  $B_i$  is 1-valent is analogous.  $\square$

*Proof of Theorem 2.* Let  $\mathcal{H}$  be a strong Heegaard diagram for  $Y$  whose genus  $g$  is minimal among all strong Heegaard diagrams for  $Y$ . Suppose, toward a contradiction, that  $g > 1$ . By Lemma 4,  $\Gamma_\mathcal{H}$  has no leaves. By assumption,  $\Gamma_\mathcal{H}$  has a single perfect

matching  $\mu$ . We direct the edges of  $\Gamma_{\mathcal{H}}$  by the following rule: an edge points from  $\mathcal{A}$  to  $\mathcal{B}$  if it is included in  $\mu$  and from  $\mathcal{B}$  to  $\mathcal{A}$  otherwise. Thus, every vertex in  $\mathcal{A}$  has exactly one outgoing edge, and every vertex in  $\mathcal{B}$  has exactly one incoming edge. We claim that  $\Gamma_{\mathcal{H}}$  contains a directed cycle  $\sigma$ . To see this, let  $\gamma$  be a maximal directed path in  $\Gamma_{\mathcal{H}}$  that visits each vertex at most once, and let  $v$  be the initial vertex of  $\gamma$ . If  $v \in \mathcal{B}$ , then there is a unique directed edge  $e$  in  $\Gamma_{\mathcal{H}}$  from some point  $w \in \mathcal{A}$  to  $v$ , and  $e$  is not included in  $\gamma$ . Likewise, if  $v \in \mathcal{A}$ , then there is an edge  $e$  not in  $\gamma$  connecting  $v$  and some point  $w \in \mathcal{B}$  since  $v$  is not a leaf, and  $e$  is directed from  $w$  to  $v$  since the only outgoing edge from  $v$  is in  $\gamma$ . In either case, the maximality of  $\gamma$  implies that  $w \in \gamma$ , which means that  $\gamma \cup e$  contains a directed cycle. However,  $(\mu - \sigma) \cup (\sigma - \mu)$  is then another perfect matching for  $\Gamma_{\mathcal{H}}$ .

Thus, the Heegaard diagram  $\mathcal{H}$  is a torus with a single  $\alpha$  curve and a single  $\beta$  curve intersecting in a single point, which describes the standard genus-1 Heegaard splitting of  $S^3$ .  $\square$

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