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WEAK GEODESICS IN THE SPACE OF KÄHLER METRICS

Tamás Darvas and László Lempert

Abstract. Given a compact Kähler manifold (X, ω_0), according to Mabuchi, the set $\mathcal{H}_0$ of Kähler forms cohomologous to ω_0 has the natural structure of an infinite-dimensional Riemannian manifold. We address the question whether points in $\mathcal{H}_0$ can be joined by a geodesic, and strengthening the finding of [LV], we show that this cannot always be done even with a certain type of generalized geodesics. As in [LV], the result is obtained through the analysis of a Monge–Ampère equation.

1. Introduction

Let X be a connected compact complex manifold of dimension $m > 0$ and ω_0 a smooth Kähler form on it. In the 1980s Mabuchi discovered that there is a natural infinite-dimensional Riemannian manifold structure on the set $\mathcal{H}_0$ of smooth Kähler forms cohomologous to ω_0, and on the set

$$\mathcal{H} = \{v \in C^\infty(X) : \omega_0 + i\partial\bar\partial v > 0\}$$

of smooth strongly ω_0-plurisubharmonic functions. He also showed that $\mathcal{H}$ is isometric to the Riemannian product $\mathcal{H}_0 \times \mathbb{R}$, [M]. In [LV], answering a question posed by Donaldson, Vivas and the second author proved that in general there is no geodesic of class C^2 between two points in $\mathcal{H}$, resp. in $\mathcal{H}_0$; in fact, there is not even one of Sobolev regularity $W^{1,2}$.

Since geodesics and their generalizations, weak geodesics, potentially play an important role in the study of special Kähler metrics (for geodesics, see [D,M]), it is of interest to know whether two points in $\mathcal{H}$ can be connected at least by a weak geodesic. What the notion of weak geodesic should be is suggested by Semmes' reformulation of the geodesic equation in $\mathcal{H}$, see [S]. Let $S = \{s \in \mathbb{C} : 0 < \text{Im } s < 1\}$ and ω the pullback of ω_0 by the projection $\overline{S} \times X \to X$. With any C^2 curve $[0,1] \ni t \mapsto v_t \in \mathcal{H}$ associate a function $u : \overline{S} \times X \to \mathbb{R}$,

$$u(s, x) = v_{\text{Im}s}(x),$$

itself a C^2 function. Then $t \mapsto v_t$ is a geodesic if and only if u satisfies the Monge–Ampère equation $(\omega + i\partial\bar\partial u)^{m+1} = 0$. Therefore a C^2 geodesic connecting $0, v \in \mathcal{H}$ gives rise to a solution $u \in C^2(\overline{S} \times X)$ of a boundary value problem for this Monge–Ampère equation on $\overline{S} \times X$; furthermore $\omega + i\partial\bar\partial u \geq 0$. This latter is expressed by saying that u is ω-plurisubharmonic. By a weak, or generalized, geodesic connecting, say, $0, v \in \mathcal{H}$ one then means an ω-plurisubharmonic solution $u : \overline{S} \times X \to \mathbb{R}$ of

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the problem

$$(1.1)\qquad \begin{aligned} &(\omega + i\partial\bar\partial u)^{m+1} = 0,\\ &u(s+\sigma, x) = u(s,x), \quad \text{if } (s,x)\in \overline{S}\times X,\ \sigma\in\mathbb{R},\\ &u(s,x) = \begin{cases} 0, & \text{if } \operatorname{Im} s = 0,\\ v(x), & \text{if } \operatorname{Im} s = 1.\end{cases}\end{aligned}$$

It has to be assumed that u is sufficiently regular so that $(\omega + i\partial\bar\partial u)^{m+1}$ can be given sense; for example, according to [BT], the continuity of u more than suffices. Chen has indeed proved that for $v \in \mathcal{H}$ (1.1) admits a continuous ω-plurisubharmonic solution for which the current $\partial\bar\partial u$ is represented by a bounded form, see [C] and complements in [Bł]. In other words, any two points in $\mathcal{H}$ can be connected by a weak geodesic. One should keep in mind, though, that a weak geodesic u need not give rise to a curve in $\mathcal{H}$, first because $v_t = u(t,\cdot)$ is not necessarily C^∞, not even C^2, and second because even if v_t is C^∞, there is no reason why it should be strongly ω_0-plurisubharmonic.

In this paper we show that the regularity that Chen obtains cannot be improved: (1.1) may have a solution with $\partial\bar\partial u$ bounded, but in general it will not have a solution with $\partial\bar\partial u$ continuous.

If $\overline{Z}$ is a complex manifold, possibly with boundary, and $Z = \operatorname{int}\overline{Z}$, we define

$$C^{\partial\bar\partial}(\overline{Z}) = \{w \in C(\overline{Z})\colon \text{ the current } \partial\bar\partial(w|Z) \text{ is represented by a form continuous on } \overline{Z}\}.$$

Given $w \in C^{\partial\bar\partial}(\overline{Z})$, we will simply write $\partial\bar\partial w$ for the continuous form on $\overline{Z}$ that represents the current $\partial\bar\partial(w|Z)$, and if $z_1, z_2, \ldots,$ are local coordinates on Z, we write $w_{z_j\bar z_k}$ for the coefficient of $dz_j \wedge d\bar z_k$ in $\partial\bar\partial w$.

Clearly $C^2(\overline{Z}) \subset C^{\partial\bar\partial}(\overline{Z})$, and it is well understood in harmonic analysis that the inclusion is strict. For example, if $\overline{Z} = \{\zeta \in \mathbb{C} : |\zeta| \le 1/2\}$ and $k = 2, 3, \ldots,$ the function

$$w(\zeta) = \begin{cases} \zeta^k \log\log|\zeta|^{-2}, & \text{if } 0 < |\zeta| \le 1/2,\\ 0, & \text{if } \zeta = 0\end{cases}$$

is not in $C^k(\overline{Z})$, but $w_{\bar\zeta} \in C^{k-1}(\overline{Z})$ and $w_{\zeta\bar\zeta} \in C^{k-2}(\overline{Z})$.

Theorem 1.1. *Suppose a connected compact Kähler manifold (X, ω_0) admits a holomorphic isometry $g : X \to X$ with an isolated fixed point, and $g^2 = id_X$. Then there is a $v \in \mathcal{H}$ for which (1.1) has no ω-plurisubharmonic solution $u \in C^{\partial\bar\partial}(\overline{S} \times X)$. One can choose v to satisfy $g^*v = v$.*

The proof will show that among symmetric potentials the $v \in \mathcal{H}$ in Theorem 1.1 even form an open set.

Theorem 1.1 corresponds to [LV, Theorem 1.2], but the C^3 regularity from [LV] has been lowered. The proofs here and in [LV] are similar in that, denoting by $x_0 \in X$ an isolated fixed point of g, in both proofs we analyze the behavior of a regular solution u in a neighborhood of $\overline{S} \times \{x_0\}$. The upshot of the analysis is a condition on the Hessian of the boundary value at x_0, a condition that not all $v \in \mathcal{H}$ satisfy. In [LV] the analysis involved the Monge–Ampère foliation associated with a $u \in C^3(\overline{S} \times X)$, and it was crucial that the foliation was of class C^1. The foliation method is not available when u is only $C^{\partial\bar\partial}$, and we will have to be thriftier with our tools, but in spite of this, we will recover the same condition on the Hessian as in [LV] when $m = 1$. When $m > 1$, the present condition is even slightly stronger than the one in [LV].

2. Generalities

In this section, we collect a few simple facts concerning currents and the homogeneous Monge–Ampère equation.

Proposition 2.1. *Let $f\colon Y \to Z$ be a holomorphic map of complex manifolds, φ and ψ continuous forms on Z satisfying $\partial\bar\partial\varphi = \psi$ as currents. Then $\partial\bar\partial f^*\varphi = f^*\psi$ as currents.*

Proof. We can assume Z is an open subset of some $\mathbb{C}^n$. Regularizing φ and ψ by convolutions gives rise to sequences of smooth forms φ_k and $\psi_k = \partial\bar\partial\varphi_k$ that converge locally uniformly to φ, resp. ψ. Therefore $f^*\varphi_k \to f^*\varphi$ and $\partial\bar\partial f^*\varphi_k = f^*\partial\bar\partial\varphi_k \to f^*\psi$ locally uniformly, whence the claim follows from the continuity of $\partial\bar\partial$ in the space of currents. ☐

Next consider a complex manifold Z and a plurisubharmonic $U \in C^{\partial\bar\partial}(Z)$. Suppose $Y \subset Z$ is a one-dimensional, not necessarily closed complex submanifold and $TY \subset \operatorname{Ker}\partial\bar\partial U$. The normal bundle $NY = (T^{1,0}Z|Y)/T^{1,0}Y$ is a holomorphic vector bundle and $\partial\bar\partial U$ induces a possibly degenerate Hermitian metric h on it. With $p\colon T^{1,0}Z|Y \to NY$ the canonical projection, the metric is

$$h(p\zeta) = \partial\bar\partial U(\zeta, \bar\zeta) \geq 0, \quad \zeta \in T^{1,0}Z|Y.$$

Thus h is continuous, but can degenerate, i.e., vanish on nonzero vectors as well.

Proposition 2.2. *The metric h is seminegatively curved in the sense that* $\log h \circ \sigma$ *is subharmonic for any holomorphic section σ of NY over some open $Y' \subset Y$. (Here it is convenient not to exclude from subharmonic functions those that are identically $-\infty$ on some component of Y'.) Further, on the line bundle* $\det NY$ *the metric induced by h is also seminegatively curved.*

When h is smooth and nondegenerate, and moreover $(\partial\bar\partial U)^{\dim Z} = 0$, the seminegativity of $\det NY$ was first proved by Bedford and Burns in [BB, Proposition 4.1], and [CT, Theorem 4.2.8] gives the seminegativity of NY itself. For possibly degenerate h [BF, Lemma] represents an equivalent result, albeit without the curvature interpretation, and under the assumption that U is C^2. Our proof is a variant of the proof in [BF].

Proof. For the first statement we only need to prove that $\log h \circ \sigma$ has the submeanvalue property, and this at points where $h \circ \sigma \neq 0$. To do so, we can assume $Z \subset \mathbb{C}^n$ is the unit polydisc, $Y = Y' = \{z \in Z : z_2 = \cdots = z_n = 0\}$, and that $\sigma(z_1, 0, 0, \ldots) = p(\partial/\partial z_2)$. Thus

$$(h \circ \sigma)(z_1, 0, \ldots) = U_{z_2\bar z_2}(z_1, 0, \ldots). \tag{2.1}$$

Green's formula implies for $0 < r < 1$

$$\frac{1}{r^2}\int_0^1 \big(U(z_1, re^{2\pi it}, 0, \dots) - U(z_1, 0, 0, \dots)\big)\,dt$$
$$= \frac{i}{\pi r^2}\int_{|z_2|\le r} (\log r - \log|z_2|) U_{z_2\bar z_2}(z_1, z_2, 0, \dots)\, dz_2 \wedge d\bar z_2, \tag{2.2}$$

certainly if U is C^2, but then upon regularizing by convolutions, whenever U and $\partial\bar\partial U$ are continuous — as in our case. Proposition 2.1, with f the embedding $Y \to Z$, implies $\partial\bar\partial(U|Y) = (\partial\bar\partial U)|Y = 0$. Hence, the left hand side of (2.2) is a subharmonic function of z_1, and so is the right-hand side. As $r \to 0$, these functions converge locally uniformly to $U_{z_2\bar z_2}(z_1, 0, \dots)$; in light of (2.1) $h \circ \sigma$ is therefore subharmonic.

If $\varphi \in \mathcal{O}(Y)$ and σ is replaced by $e^{\varphi/2}\sigma$, we obtain that $e^{\operatorname{Re}\varphi} h \circ \sigma$ is also subharmonic. Therefore, it satisfies the maximum principle, and so does $\operatorname{Re}\varphi + \log h \circ \sigma$; knowing this for all $\varphi \in \mathcal{O}(Y)$ is equivalent to the subharmonicity of $\log h \circ \sigma$, see, e.g., [H, Theorem 1.6.3].

Now given any holomorphic vector bundle $E \to Y$ of rank r, endowed with a seminegatively curved, possibly degenerate continuous Hermitian metric h, the induced metric on the line bundle $\det E$ is also seminegatively curved. Indeed, denoting by $h(e, e')$ the inner product of $e, e' \in E_y$, $y \in Y$, so that $h(e) = h(e, e)$, for (local) sections $\sigma_1, \dots, \sigma_r$ of E the induced metric is given by

$$h^{\det}(\sigma_1 \wedge \dots \wedge \sigma_r) = \det\big(h(\sigma_j, \sigma_k)\big). \tag{2.3}$$

If h is smooth and nondegenerate and $y \in Y$, any nonzero holomorphic section of $\det E$ in a neighborhood of y can be written as $\sigma_1 \wedge \dots \wedge \sigma_r$, where the σ_j are holomorphic sections of E near y, and $h(\sigma_j, \sigma_k)$ vanish to second order at y for $j \ne k$. Thus $\det\big(h(\sigma_j, \sigma_k)\big) - \prod_{j=1}^r h(\sigma_j, \sigma_j)$ vanishes to *fourth* order at y. By virtue of (2.3) this implies that at y

$$i\partial\bar\partial \log h^{\det}(\sigma_1 \wedge \dots \wedge \sigma_r) = i\partial\bar\partial \log \prod_{j=1}^r h(\sigma_j, \sigma_j) \ge 0.$$

Therefore $h^{\det}$ is seminegatively curved when h is smooth and nondegenerate. To prove for a general h we can assume $Y \subset \mathbb{C}$ is connected, $E = Y \times \mathbb{C}^r$ is holomorphically trivial, and $h^{\det}$ degenerates nowhere. We can regularize h by convolutions, and obtain $h^{\det}$ as the locally uniform limit of seminegatively curved metrics, hence itself seminegatively curved. □

Lastly, we record a uniqueness result and its corollary:

Proposition 2.3. *Given a compact Kähler manifold (X, ω_0) and $v \in \mathcal{H}$, the equation (1.1) has at most one ω-plurisubharmonic solution $u \in C^{\partial\bar\partial}(\overline{S} \times X)$.*

The result follows from [Bł, Proposition 2.2 or Theorem 2.3] or from [PS, p. 144], once one checks that for ω-plurisubharmonic $u \in C^{\partial\bar\partial}(\overline{S} \times X)$ the Monge–Ampère measure $(\omega + i\partial\bar\partial u)^{m+1}$, as defined, e.g., in [BT], agrees with what is obtained by taking the exterior power of the continuous form $\omega + i\partial\bar\partial u$. Alternatively, the more elementary arguments for [D, Lemma 6] and the first paragraph of the proof of [LV, Proposition 2.3] also give uniqueness, provided one first checks the following: if Z is a complex manifold and $w \in C^{\partial\bar\partial}(Z)$ is real valued, then $i\partial\bar\partial w \ge 0$ at any local minimum point of w. Because of Proposition 2.1, it suffices to verify this latter when $\dim Z = 1$, and then it is straightforward: if $i\partial\bar\partial w < 0$ at a point, then $i\partial\bar\partial w < 0$ in a neighborhood, whence w is strongly superharmonic there, and has no local minimum.

Corollary 2.1. *Suppose $v \in \mathcal{H}$ satisfies $g^*v = v$, and $u \in C^{\partial\bar\partial}(\overline{S} \times X)$ is an ω-plurisubharmonic solution of (1.1). Then $u(s,x) = u(s, g(x))$.*

3. Proof of Theorem 1.1

Let X, ω_0, ω, and g be as in Theorem 1.1, and let $x_0 \in X$ be an isolated fixed point of g. Using [LV, Proposition 2.2] we choose local coordinates $z_1, \dots, z_m$ in a neighborhood $V \subset X$ of x_0 in which g is expressed as $(z_j) \mapsto (-z_j)$.

Proposition 3.1. *If an ω-plurisubharmonic $u \in C^{\partial\bar\partial}(\overline{S} \times V)$ is a solution of (1.1) and $u(s,x) = u(s,g(x))$, then $u(s,x_0) = a \operatorname{Im} s$ for $s \in \overline{S}$, with some $a \in \mathbb{R}$.*

Proof (essentially taken over from [BF, Proposition]). From our symmetry assumption it follows that $u_{s\bar z_j}(s,x_0) = 0$, and so

$$0 = (-i\omega + \partial\bar\partial u)^{m+1} = (-i\omega + \partial_X\overline{\partial}_X u)^m \wedge \ \partial_S\overline{\partial}_S u \tag{3.1}$$

at points of $\overline{S} \times \{x_0\}$. Hence, for any $s \in \overline{S}$ either $(-i\omega + \partial_X\overline{\partial}_X u)^m$ or $\partial_S\overline{\partial}_S u$ vanishes at (s, x_0). The goal is to show that it is always the latter that vanishes.

We claim that on $S \times \{x_0\}$

$$\lambda = \log \left(-i\omega + \partial_X\overline{\partial}_X u\right)^m \left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar z_1} \wedge \cdots \wedge \ \frac{\partial}{\partial z_m} \wedge \frac{\partial}{\partial \bar z_m}\right)$$

is subharmonic and not identically $-\infty$. Indeed, by Proposition 2.1 $(\partial\bar\partial u)|\{s\} \times X = \partial\bar\partial(u|\{s\} \times X)$ for $s \in S$, and by the continuity of $\partial\bar\partial$, also for $s \in \overline{S}$. But $u(0,\cdot) = 0$ is strongly ω_0-plurisubharmonic, hence $\lambda(s,x_0) > -\infty$ when $s = 0$, and also when $s \in \overline{S}$ is near 0. As to subharmonicity, it suffices to verify it on the open set

$$S_0 = \{s \in S : \lambda(s,x_0) > -\infty\}.$$

Choose a smooth w_0 in a neighborhood of $x_0 \in X$ such that $\omega_0 = i\partial\bar\partial w_0$, let $w(s,x) = w_0(x)$ and $U = u + w$. By what has been observed above, $U_{s\bar z_j}(s,x_0) = U_{s\bar s}(s,x_0) = 0$ if $s \in S_0$; in other words, $S_0 \times \{x_0\}$ is tangential to $\operatorname{Ker} \partial\bar\partial U$. By virtue of Proposition 2.2 λ is subharmonic on $S_0 \times \{x_0\}$, hence on $S \times \{x_0\}$, as claimed.

Once we know $\lambda|S \times \{x_0\}$ is subharmonic, it follows that S_0 is dense in S; since by (3.1) $u_{s\bar s}$ vanishes on $S_0 \times \{x_0\}$, it vanishes on all of $S \times \{x_0\}$. The Proposition now follows, because a harmonic function on S that depends only on $\operatorname{Im} s$ must be a linear function of $\operatorname{Im} s$. □

In the proof of the next proposition we will make use of the Poisson integral representation of harmonic functions in a strip. If ψ is harmonic in S, continuous and bounded in $\overline{S}$, then we have the following integral representation (for more on this see [W]):

$$\psi(\xi + i\eta) = \int_{-\infty}^{+\infty} P(t-\xi, \eta)\psi(t)dt + \int_{-\infty}^{+\infty} P(t-\xi, 1-\eta)\psi(t+i)dt, \tag{3.2}$$

where P is the following Poisson kernel:

$$P(\xi,\eta) = \frac{\sin \pi\eta}{2(\cosh \pi\xi - \cos \pi\eta)}.$$

As expected, the above integral representation formula also gives a recipe to generate bounded continuous harmonic functions in $\overline{S}$ given bounded continuous boundary data on ∂S.

Let $\omega_0 = \sum_{j,k=1}^m \omega_{jk} dz_j \wedge d\bar{z}_k$ on $V \subset X$.

Proposition 3.2. *Suppose $a \in \mathbb{R}$ and u is a bounded continuous ω-plurisubharmonic function in $\overline{S} \times V$ satisfying*

$$\begin{aligned} u(s,x) &= 0, && \text{if } (s,x) \in \mathbb{R} \times V, \\ u(s,x_0) &= a \operatorname{Im} s, && \text{if } s \in \overline{S}. \end{aligned}$$

If $v = u(i,\cdot)$ is twice differentiable at x_0 and $dv = 0$ there, then

$$\left| \sum_{j,k=1}^m v_{z_j z_k}(x_0)\xi_j\xi_k \right| \le \sum_{j,k=1}^m \big(2\,\omega_{jk}(x_0) + v_{z_j\bar{z}_k}(x_0)\big)\xi_j\bar{\xi}_k \quad \text{for } \xi_j \in \mathbb{C}, \tag{3.3}$$

and this estimate is sharp.

Proof. We will assume $a = 0$ (otherwise we replace $u(s,x)$ by $u(s,x) - a \operatorname{Im} s$). Thus $u(s,x_0) = v(x_0) = 0$. By passing to a slice, the proof is reduced to the case $m = 1$. We will denote the local coordinate on V by $z = z_1$; it identifies V and x_0 with a neighborhood of $0 \in \mathbb{C}$ and with $0 \in \mathbb{C}$. Since $m = 1$, we need to verify

$$|v_{zz}(0)| \le 2\,\omega_{11}(0) + v_{z\bar{z}}(0). \tag{3.4}$$

Suppose $f : \overline{S} \to \mathbb{C}$ is bounded and holomorphic with $f(\alpha) = 0$, for some $\alpha \in S$. Let $q = v_{zz}(0)$, and choose real numbers $p > v_{z\bar{z}}(0)$ and $r > \omega_{11}(0)$. With a neighborhood $V' \subset V$ of 0 we will have

$$v(z) \le p|z|^2 + \operatorname{Re} qz^2 \ \text{ and } \ \omega < i\partial\overline{\partial} r|z|^2$$

for all $z \in V'$. Clearly, this implies that the function $U(s,z) = r|z|^2 + u(s,z)$ is plurisubharmonic in $S \times V'$, and if $\zeta \in \mathbb{C}$ is sufficiently small, then

$$\phi(s) = U(s,\zeta f(s)) = r|\zeta f(s)|^2 + u(s,\zeta f(s))$$

is a subharmonic function of $s \in S$. On the boundary of $\overline{S}$ we have the following estimates:

$$\phi(s) \begin{cases} = r|\zeta f(s)|^2, & \text{if } \operatorname{Im} s = 0, \\ \le (p+r)|\zeta f(s)|^2 + \operatorname{Re} q\zeta^2 f(s)^2, & \text{if } \operatorname{Im} s = 1. \end{cases}$$

We take ζ such that $q\zeta^2$ is nonnegative. Then $\operatorname{Re} q\zeta^2 f(s)^2 = |q\zeta^2| \operatorname{Re} f(s)^2$.

Let ψ_1,ψ_2 and ψ_3 be bounded, continuous and harmonic functions on $\overline{S}$ defined by the following boundary data:

$$\psi_1(s) = |f(s)|^2, \ \text{ if } s \in \partial S,$$

$$\psi_2(s) = 0, \text{ if Im } s = 0 \text{ and } \psi_2(s) = |f(s)|^2, \text{ if Im } s = 1,$$
$$\psi_3(s) = 0, \text{ if Im } s = 0 \text{ and } \psi_3(s) = \text{Re } f(s)^2, \text{ if Im } s = 1.$$

Since ψ_1, ψ_2, ψ_3 and ϕ are all bounded on $\overline{S}$, by the maximum principle we obtain

$$0 = u(\alpha, 0) = \phi(\alpha) \leq r|\zeta|^2\psi_1(\alpha) + p|\zeta|^2\psi_2(\alpha) + |q||\zeta|^2\psi_3(\alpha). \tag{3.5}$$

We will show that $\psi_2(\alpha)/\psi_1(\alpha)$ can be chosen arbitrarily close to $1/2$ and that $\psi_3(\alpha)/\psi_1(\alpha)$ can be chosen arbitrarily close to $-1/2$. We can work with any $\alpha \in S$, but if $\alpha = \xi + i\eta = i/2$, Poisson's formula (3.2) simplifies and gives $\psi_1(i/2) = (I+J)/2$, $\psi_2(i/2) = J/2$ and $\psi_3(i/2) = K/2$, where

$$I = I(f) = \int_{-\infty}^{+\infty} \frac{|f(t)|^2}{\cosh \pi t} dt, \quad J = J(f) = \int_{-\infty}^{+\infty} \frac{|f(t+i)|^2}{\cosh \pi t} dt,$$
$$K = K(f) = \int_{-\infty}^{+\infty} \frac{\text{Re } f(t+i)^2}{\cosh \pi t} dt.$$

We need to choose f so that $I \approx J \approx -K$. No matter what f, clearly $|K| \leq J$; to achieve $J \approx -K$, the integrands in J and K must be negatives of each other, at least approximately and for most $t \in \mathbb{R}$ that make the integrands large. This means that $f(t+i)$ must be close to imaginary. If also $|f(t)| \approx |f(t+i)|$, then $I \approx J$. Now $f(s) = e^{\pi s/2} - e^{\pi i/4}$ satisfies both conditions and vanishes at $i/2$, but it is unbounded. Instead, with a large $\lambda \in \mathbb{R}$ we let

$$f_\lambda(s) = \frac{e^{\pi s/2} - e^{\pi i/4}}{1 + e^{\pi(s-\lambda)/2}}.$$

We claim that $I(f_\lambda) \sim J(f_\lambda) \sim 2\lambda$ and $K(f_\lambda) \sim -2\lambda$ as $\lambda \to \infty$. This will be verified only for $J(f_\lambda)$, the other two are treated similarly. We have

$$J(f_\lambda) = \left(\int_{-\infty}^{0} + \int_{0}^{\lambda} + \int_{\lambda}^{+\infty} \right) \frac{|ie^{\pi t/2} - e^{\pi i/4}|^2}{|1 + ie^{\pi(t-\lambda)/2}|^2 \cosh \pi t} dt.$$

Since in the first integral the numerator is bounded, and in the last it is $O(\cosh \pi t)$, both integrals have bounds independent of λ. After a change of variables $\tau = t/\lambda$ in the middle integral, we obtain

$$\int_0^\lambda \frac{|ie^{\pi t/2} - e^{\pi i/4}|^2}{|1 + ie^{\pi(t-\lambda)/2}|^2 \cosh \pi t} dt = \lambda \int_0^1 \frac{|ie^{\pi\lambda\tau/2} - e^{\pi i/4}|^2}{|1 + ie^{\pi\lambda(\tau-1)/2}|^2 \cosh(\pi\lambda\tau)} d\tau$$
$$= 2\lambda \int_0^1 \frac{|i - e^{\pi(i/4 - \lambda\tau/2)}|^2}{|1 + ie^{\pi\lambda(\tau-1)/2}|^2 (1 + e^{-2\pi\lambda\tau})} d\tau.$$

This last expression has bounded integrand, and the dominated convergence theorem implies $J(f_\lambda) \sim 2\lambda$, as claimed. Letting $\lambda \to \infty$ in (3.5) (with $\alpha = i/2$) we obtain $0 \leq p - |q| + 2r$, and letting $p \to v_{z\bar{z}}, r \to \omega_{11}$, (3.4) follows.

To prove the sharpness of estimate (3.3), suppose that $V \subset \mathbb{C}$ is the unit disc and $\omega = i\partial\overline{\partial}|z|^2$. Let

$$u(s, z) = -\frac{2 \text{ Im } s}{\varepsilon + \text{Im } s} (\text{Re } z)^2,$$

for some $\varepsilon > 0$. Clearly u is bounded and continuous on $\overline{S} \times V$, $u(s,z) = 0$ for all $(s,z) \in \mathbb{R} \times V$, and $u(s,0) = 0$ for all $s \in \overline{S}$. One verifies that u is ω-plurisubharmonic in $S \times V$ by checking that

$$\begin{vmatrix} u_{s\bar{s}} & u_{s\bar{z}} \\ u_{\bar{s}z} & 1 + u_{z\bar{z}} \end{vmatrix} = 0$$

and observing that $1 + u_{z\bar{z}} > 0$. This confirms that the Levi form of $|z|^2 + u$ is semipositive everywhere. If $\varepsilon \to 0$ then $2 + v_{z\bar{z}}(0) = 2 + u_{z\bar{z}}(i,0) \to 1$ and $|v_{zz}(0)| = |u_{zz}(i,0)| \to 1$; hence the estimate (3.3) is indeed sharp. □

Proof of Theorem 1.1. Given a g-invariant $v \in \mathcal{H}$, suppose (1.1) has an ω-plurisubharmonic solution $u \in C^{\partial\bar{\partial}}(\overline{S} \times X)$. By Corollary 2.4, $u(s,x) = u(s,g(x))$; since $dv(x_0) = 0$ is automatic for g-invariant v, by Propositions 3.1 and 3.2 v then satisfies (3.3). Conversely, if a g-invariant $v \in \mathcal{H}$ does not satisfy (3.3), then (1.1) will have no ω-plurisubharmonic solution $u \in C^{\partial\bar{\partial}}(\overline{S} \times X)$. Such v certainly exist (and form an open set among g-invariant potentials in $\mathcal{H}$), because the matrices $(v_{z_j\bar{z}_k}(x_0)) = (p_{jk})$ and $(v_{z_j z_k}(x_0)) = (q_{jk})$ can be arbitrarily prescribed for g-invariant $v \in \mathcal{H}$, as long as $(\omega_{jk}(x_0) + p_{jk})$ is positive definite; see [LV, Lemma 3.3]. □

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Department of Mathematics, Purdue University, West Lafayette, 150 N. University Street, West Lafayette, IN 47907-2067, USA

E-mail address: tdarvas@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, 150 N. University Street, West Lafayette, IN 47907-2067, USA

E-mail address: lempert@math.purdue.edu