REPRESENTING HOMOLOGY CLASSES BY SYMPLECTIC SURFACES

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ABSTRACT. We derive an obstruction to representing a homology class of a symplectic 4-manifold by an embedded, possibly disconnected, symplectic surface.

A natural question concerning symplectic 4-manifolds is the following: given a closed symplectic 4-manifold (M, ω) and a homology class $B \in H_2(M; \mathbb{Z})$, determine whether there exists an embedded, possibly disconnected, closed symplectic surface representing the class B. This question has been studied by Lê and Li [8, 9]. We always assume that the orientation of a symplectic surface is the one induced by the symplectic form. One necessary condition is then, of course, that the symplectic class $[\omega]$ evaluates positively on the class B, meaning that $\langle [\omega], B \rangle > 0$. Among other things, it is shown in [9] that a class B with $\langle [\omega], B \rangle > 0$ in a symplectic 4-manifold is always represented by a symplectic *immersion* of a connected surface. It is also noted that an obstruction to representing a homology class B by an embedded *connected* symplectic surface comes from the adjunction formula: the (even) integer

$$K_M B + B^2,$$

where K_M denotes the canonical class of the symplectic 4-manifold (M, ω), has to be at least -2. This obstruction, however, disappears, if the number of components of the symplectic surface is allowed to grow large. Note that there are examples of classes in symplectic 4-manifolds, which are represented by an embedded disconnected symplectic surface, but not by a connected symplectic surface: for example in the twofold blow-up $X \# 2\overline{\mathbb{CP}}^2$ of any closed symplectic 4-manifold X the sum of the classes of the exceptional spheres is not represented by a connected embedded symplectic surface according to the adjunction formula. It is the purpose of this paper to derive an obstruction to representing a homology class by an embedded, possibly disconnected, symplectic surface.

In [9], it is also shown that for symplectic manifolds M of dimension at least six, every class in $H_2(M; \mathbb{Z})$ on which the symplectic class evaluates positively is represented by a connected embedded symplectic surface. In [8], there is a conjecture that in the case of symplectic 4-manifolds M says that if α is a class in $H_2(M; \mathbb{Z})$ on which the symplectic class evaluates positively, then there exists a positive integer N depending on α such that $N\alpha$ is represented by an embedded, not necessarily connected, symplectic surface. In the examples at the end of this paper, we give counterexamples to this conjecture in the four-dimensional case.

The non-existence of an embedded symplectic surface in the class B has the following consequence for the Seiberg–Witten invariants, which we only state in the case $b_2^+ > 1$.

Proposition 1. *Let (M, ω) be a closed symplectic 4-manifold with $b_2^+(M) > 1$ and $B \neq 0$ an integral second homology class, which cannot be represented by an embedded, possibly disconnected, symplectic surface. Then the Seiberg–Witten invariant of the $Spin^c$-structure*

$$s_0 \otimes PD(B)$$

is zero, where s_0 denotes the canonical $Spin^c$-structure with determinant line bundle K_M^{-1} induced by a compatible almost complex structure.

Here PD denotes the Poincaré dual of a homology class. Note that the first Chern class of the $Spin^c$-structure $s_0 \otimes PD(B)$ is equal to $-K_M + 2PD(B)$. Proposition 1 is a consequence of a theorem of Taubes, relating classes with non-zero Seiberg–Witten invariants to embedded symplectic surfaces [14].

In the following, let (M, ω) denote a closed symplectic 4-manifold and $\Sigma \subset M$ an embedded, possibly disconnected, closed symplectic surface representing a class $B \in H_2(M; \mathbb{Z})$. We always assume that the orientation of M is given by the symplectic form $(\omega \wedge \omega > 0)$. If the class B is divisible by an integer $d > 1$, in the sense that there exists a class $A \in H_2(M; \mathbb{Z})$ such that $B = dA$, then there exists a d-fold cyclic ramified covering $\phi \colon \overline{M} \to M$, branched along Σ. The branched covering is again a closed symplectic 4-manifold. This is a well-known fact (the pullback of the symplectic form ω plus t times a Thom form for the preimage $\overline{\Sigma}$ of the branch locus is for small positive t a symplectic form on $\overline{M}$; see [3, 11] for a careful discussion). The invariants of $\overline{M}$ are given by the following formulas [4, p. 243], [5]:

$$K_{\overline{M}} = \phi^*(K_M + (d-1)PD(A))$$
$$K_{\overline{M}}^2 = d(K_M + (d-1)PD(A))^2$$
$$w_2(\overline{M}) = \phi^*(w_2(M) + (d-1)PD(A)_2)$$
$$\sigma(\overline{M}) = d\left(\sigma(M) - \frac{d^2-1}{3}A^2\right)$$

Here $PD(A)_2 \in H^2(M; \mathbb{Z}_2)$ is the mod 2 reduction of $PD(A)$. The second equation follows from the first because the branched covering map has degree d.

Suppose that the branched covering $\overline{M}$ is symplectically minimal and not a ruled surface over a curve of genus >1. Then theorems of Taubes and Liu [10, 13] imply that $K_{\overline{M}}^2 \geq 0$. With the formula above, we obtain the following obstruction on class A.

Theorem 2. *Let (M, ω) be a closed symplectic 4-manifold, $\Sigma \subset M$ an embedded, possibly disconnected, closed symplectic surface and $d > 1$ an integer such that $dA = [\Sigma]$ for a class $A \in H_2(M; \mathbb{Z})$. Consider the d-fold cyclic branched cover $\overline{M}$, branched along Σ. If $\overline{M}$ is minimal and not a ruled surface over a curve of genus > 1, then*

$$(K_M + (d-1)PD(A))^2 \geq 0.$$

It is therefore important to ensure that the branched covering $\overline{M}$ is minimal and not a ruled surface. First, we have the following lemma.

Lemma 3. *Let $\phi\colon \overline{M} \to M$ be a cyclic d-fold branched covering of closed oriented 4-manifolds. Then $b_2^+(\overline{M}) \geq b_2^+(M)$.*

Proof. With our choice of orientations, the map $\phi\colon \overline{M} \to M$ has positive degree. By Poincaré duality, the induced map $\phi^*\colon H^*(M;\mathbb{R}) \to H^*(\overline{M};\mathbb{R})$ is injective. It maps classes in the second cohomology of positive square to classes of positive square. This implies the claim. □

Proposition 4. *In the notation of Theorem 2, each of the following two conditions implies that $\overline{M}$ is minimal and has $b_2^+(\overline{M}) > 1$ and hence is not a ruled surface:*
  (a) *If d is odd assume that M is spin and if d is even assume that $PD(A)$ is characteristic. Also assume that $3\sigma(M) \neq (d^2 - 1)A^2$.*
  (b) *Assume that $b_2^+(M) \geq 2$ and there exists an integer $k \geq 2$ such that the class*

$$K_M + (d-1)PD(A)$$

  is divisible by k.

Proof. Consider the d-fold branched covering $\overline{M}$, branched along Σ. The assumptions in case (a) imply that $\overline{M}$ is spin and that the signature $\sigma(\overline{M})$ is non-zero. According to a theorem of Furuta [2], we have $b_2^+(\overline{M}) \geq 3$. Also the symplectic manifold $\overline{M}$ is minimal, because it is spin. In case (b), the lemma implies that $b_2^+(\overline{M}) \geq 2$. In addition, the symplectic manifold $\overline{M}$ is minimal, because its canonical class is divisible by k (a non-minimal symplectic 4-manifold Y contains a symplectic sphere S with $K_Y S = -1$). □

Example 5. Consider $M = K3$. Then we have $K_M = 0$. Let $d \geq 3$ be an integer and $A \in H_2(M;\mathbb{Z})$ a class with $A^2 < 0$. Theorem 2 together with Proposition 4 part (b) imply that dA is not represented by an embedded symplectic surface. Note that $K3$ contains indivisible classes of negative self-intersection which, for a suitable choice of symplectic structure, are represented by symplectic surfaces, for example symplectic (-2)-spheres. Let A be the homology class of such a sphere and $\alpha = 3A$. Then α is a counterexample to Lê's Conjecture 1.4 in [8].

Example 6. Let X be a closed symplectic spin 4-manifold with $b_2^+ > 1$ and M the blow-up $X \# \overline{\mathbb{CP}}^2$. Let E denote the class of the exceptional sphere in M. We have $K_M = K_X + PD(E)$. For every positive even integer d with $d^2 > K_X^2$, the class dE is not represented by a symplectic surface. Taking for example the blow-up of the $K3$ surface and $\alpha = 2E$, we obtain another counterexample to Lê's conjecture.

Note that with this method it is impossible to find a counterexample to Lê's conjecture under the additional assumption that $\alpha^2 > 0$.

In light of the second example, the following conjecture seems natural.

Conjecture. *Let M be the blow-up $X \# \overline{\mathbb{CP}}^2$ of a closed symplectic 4-manifold X and E the class of the exceptional sphere. Then dE is not represented by an embedded symplectic surface for all integers $d \geq 2$.*

This conjecture holds by a similar argument as above for X the $K3$ surface and the 4-torus T^4. Moreover, using positivity of intersections, the conjecture holds in the complex category for the blow-up of a complex surface and embedded complex curves. In fact, in this category the result holds not only for the exceptional curve in

a blow-up, but for multiples of the class of any connected embedded complex curve with negative self-intersection in a complex surface.

Remark 7. Branched covering arguments have been used in the past to find lower bounds on the genus of a connected surface representing a divisible homology class in a closed 4-manifold; see [1, 6, 7, 12].

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References

[1] J. Bryan, *Seiberg–Witten à la Furuta and genus bounds for classes with divisibility*, Turkish J. Math. **21** (1997), 55–59.

[2] M. Furuta, *Monopole equation and the $\frac{11}{8}$-conjecture*, Math. Res. Lett. **8**(3) (2001), 279–291.

[3] R.E. Gompf, *Symplectically aspherical manifolds with non-trivial π_2*, Math. Res. Lett. **5**(5), (1998), 599–603.

[4] R.E. Gompf and A.I. Stipsicz, 4-*manifolds and Kirby calculus, number 20 in Graduate Studies in Mathematics*, American Mathematical Society, 1999.

[5] F. Hirzebruch, *The signature of ramified coverings*, in Global analysis (Papers in Honor of K. Kodaira), 253–265, University of Tokyo Press, 1969.

[6] W.C. Hsiang and R.H. Szczarba, *On embedding surfaces in four-manifolds*, Proc. Symp. Pure Math. **22** (1971), 97–103.

[7] D. Kotschick and G. Matić, *Embedded surfaces in four-manifolds, branched covers, and* SO(3)-*invariants*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 275–286.

[8] H.-V. Lê, *Realizing homology classes by symplectic submanifolds*, Preprint 61, Max-Planck-Institute for Mathematics in the Sciences, 2004.

[9] T.-J. Li, *Existence of symplectic surfaces*, in Geometry and topology of manifolds, Vol. 47 of *Fields Inst. Commun.*, 203–217, American Mathematical Society, 2005.

[10] A.-K. Liu, *Some new applications of general wall crossing formula, Gompf's conjecture and its applications*, Math. Res. Lett. **3**(5) (1996), 569–585.

[11] F. Pasquotto, *On the geography of symplectic manifolds*, PhD thesis, Leiden University, 2004.

[12] V.A. Rohlin, *Two-dimensional submanifolds of four-dimensional manifolds*, Func. Analysis Appl. **6** (1971), 39–48.

[13] C.H. Taubes, *The Seiberg–Witten and Gromov invariants*, Math. Res. Lett. **2**(2) (1995), 221–238.

[14] ———, *SW ⇒ Gr: from the Seiberg–Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), 845–918.

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