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RESONANT STATES FOR THE STATIC KLEIN–GORDON–MAXWELL–PROCA SYSTEM

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Abstract. We prove the existence of resonant states for the critical static Klein–Gordon–Maxwell–Proca system in the case of closed manifolds. Standing waves solutions with arbitarilly large multi-spikes amplitudes and unstable phases are constructed.

We investigate in this paper the existence of resonant states for the electrostatic Klein–Gordon–Maxwell–Proca system in closed manifolds, a massive version of the more traditional electrostatic Klein–Gordon–Maxwell system. The system provides a dualistic model for the description of the interaction between a charged relativistic matter scalar field and the electromagnetic field that it generates. The external vector field (φ, A) in the system inherits a mass and is governed by the Proca action which generalizes that of Maxwell. Let (M, g) be a closed three-dimensional Riemannian manifold. Writing the matter scalar field in polar form as $\psi(x,t) = u(x,t)\mathrm{e}^{\mathrm{i}S(x,t)}$, the full Klein–Gordon–Maxwell–Proca system is written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^5 + \left(\left(\frac{\partial S}{\partial t} + q\varphi\right)^2 - |\nabla S - qA|^2\right) u \\ \frac{\partial}{\partial t}\left(\left(\frac{\partial S}{\partial t} + q\varphi\right)u^2\right) - \nabla.\left(\left(\nabla S - qA\right)u^2\right) = 0 \\ -\nabla.\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + m_1^2\varphi + q\left(\frac{\partial S}{\partial t} + q\varphi\right)u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + m_1^2 A = q\left(\nabla S - qA\right)u^2, \end{cases} \tag{0.1}$$

where $\Delta_g = -\mathrm{div}_g\nabla$ is the Laplace–Beltrami operator, $\overline{\Delta}_g = \delta d$ is half the Laplacian acting on forms, and δ is the codifferential. In its electrostatic form we assume A and φ do not depend on the time variable. Looking for standing waves solutions $\psi(x,t) = u(x)\mathrm{e}^{\mathrm{i}\omega t}$, letting $\varphi = \omega v$, there necessarily holds that $A = 0$ and the system reduces to the two following equations:

$$\begin{cases} \Delta_g u + m_0^2 u = u^5 + \omega^2\left(qv - 1\right)^2 u, \\ \Delta_g v + \left(m_1^2 + q^2u^2\right)v = qu^2. \end{cases} \tag{0.2}$$

In the above, $m_0, m_1 > 0$ are masses (m_0 is the mass of the particle, m_1 is the Proca mass), and $q > 0$ is the electric charge of the particle. The Proca formalism comes with the assumption $m_1 > 0$. We refer to Section 1 for a discussion on the physics origin of the system. The system (0.2), in Proca form in closed manifolds, has been investigated in Druet and Hebey [5] and Hebey and Truong [8]. Existence of variational solutions and a priori bounds, which guarantee phase stability, were established in these papers. The existence of resonant states was left open. We answer the question in this paper.

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As a remark, (0.2) is critical from the Sobolev viewpoint since $5 = 2^\star - 1$ when the dimension is 3, where $2^\star$ is the usual notation for the critical Sobolev exponent associated with H^1. We consider in this paper the case of the unit 3-sphere. Our theorem is stated as follows. The θ_k's in the theorem are referred to as resonant states.

Theorem. *Let (S^3, g) be the unit 3-sphere, $m_0, m_1 > 0$, and $q > 0$. There exists a sequence $(\theta_k)_k$ of positive real numbers, satisfying that $\theta_1 = \frac{\sqrt{3}}{2}$, $\theta_k > \theta_1$ when $k \geq 2$, and $\theta_k \to +\infty$ as $k \to +\infty$, and there exists a sequence $(c_k(m_1))_k$, satisfying that $c_1(m_1) = 0$, $c_k(m_1) > 0$ for $k \geq 2$, and $c_k(m_1) \to +\infty$ as $k \to +\infty$, such that any $\omega_k \in (-m_0, m_0)$ given by $\theta_k^2 = m_0^2 - \omega_k^2$, which satisfy that $q^2\omega_k^2 \neq c_k(m_1)$, is an unstable phase for (0.2) associated with a k-spikes configuration.*

The θ_k's in the theorem are independent of m_0, m_1, and q, while the $c_k(m_1)$'s, as indicated by the notation, depend only on m_1 and k. Concerning terminology, a phase $\omega \in (-m_0, m_0)$ is said to be *unstable* (or resonant) for (0.2) if there exist a sequence $(\omega_\alpha)_\alpha$ of phases, and sequences $(u_\alpha)_\alpha$, $(v_\alpha)_\alpha$ of positive solutions of

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^5 + \omega_\alpha^2 (q v_\alpha - 1)^2 u_\alpha, \\ \Delta_g v_\alpha + (m_1^2 + q^2 u_\alpha^2) v_\alpha = q u_\alpha^2, \end{cases} \tag{0.3}$$

for all $\alpha \in \mathbb{N}$, such that $\omega_\alpha \to \omega$ as $\alpha \to +\infty$, and such that $\|u_\alpha\|_{C^2} + \|v_\alpha\|_{C^2} \to +\infty$ as $\alpha \to +\infty$. By elliptic theory, because of the structure of the equation (see Section 2), the latest turns out to be equivalent to $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. In case the u_α's blow up with precisely k singularities (see Struwe [16]), the unstable phase ω is said to be *associated with a k-spikes configuration.* An unstable phase may be associated of course with different k-spikes configurations for different k, but the more k is large, the reacher is the blowing-up structure. Conversely, a phase ω is said to be *stable* if for any sequence $(\omega_\alpha)_\alpha$ of phases, and any sequences $(u_\alpha)_\alpha$, $(v_\alpha)_\alpha$ of positive solutions of (0.3), the convergence $\omega_\alpha \to \omega$ in $\mathbb{R}$ as $\alpha \to +\infty$ implies that, up to a subsequence, the u_α's and v_α's converge in $C^2(S^3)$ as $\alpha \to +\infty$.

By the analysis in Druet and Hebey [5], any phase in $(-m_0, m_0)$ is stable when $m_0^2 < \kappa$, where $\kappa > 0$ is such that $\Delta_g + \kappa$ has a nonnegative mass at each point in the manifold. The result in Druet and Hebey [5] was stated with $\kappa = \frac{1}{8}\min_M S_g$, for which we have the positive mass theorem of Schoen and Yau [14]. It holds with any such κ. More generally, phase compensation was established in Druet and Hebey [5], and we get that any phase $\omega \in (-m_0, m_0)$ such that $m_0^2 - \omega^2 < \kappa$ is stable (thus allowing situations where m_0 can be large). In the case of S^3, the best κ possible is $\kappa = \frac{3}{4}$. Our main result states that there are resonant states for the system when we do not assume the bound on m_0. The result is sharp since these appear precisely when the Druet and Hebey [5] result stops to apply. As a remark, it is often the case in the literature that blowing-up solutions of critical equations are constructed with the help of an additional, somehow artificial, parameter which breaks the original structure of the equation. The parameter usually affects the nonlinearity, replacing $2^\star$ by $2^\star \pm \varepsilon$ in a pure power nonlinearity, or the potential term, replacing h in a Schrödinger operator $\Delta_g + h$ by $h \pm \varepsilon\psi$, where $\psi > 0$ is a suitable positive function. A main feature in the above theorem is that we do not need to add any such parameter. The multi-spikes blowing-up solutions we construct are pure solutions of our systems. They all satisfy

(0.3). The phase, which is part of the system, plays the role of the parameter. Two basic consequences of our theorem are as follows:

(i) When $m_0 > \frac{\sqrt{3}}{2}$, $\omega_1 = \sqrt{m_0^2 - \frac{3}{4}}$ and $-\omega_1$ are unstable phases for (0.2) associated with a single spike configuration, and
(ii) There exists $\varepsilon_0 = \varepsilon_0(m_1)$, $\varepsilon_0 > 0$, such that if $qm_0 < \varepsilon_0$, then for any $k \geq 2$ satisfying that $m_0^2 - \theta_k^2 \geq 0$, $\omega_k = \sqrt{m_0^2 - \theta_k^2}$ and $-\omega_k$ are unstable phases for (0.2) associated with a k-spikes configuration.

In particular, the larger m_0 is, as long as qm_0 remains small, the more we find unstable phases. Point (i) corresponds to $k = 1$ in the theorem. Point (ii) is obtained by letting $\varepsilon^2 = \inf_{k\geq 2} c_k(m_1)$. When $m_0^2 = \theta_k^2$, $k \geq 2$, $\omega_k = 0$ is an unstable phase for (0.2). The same holds for $k = 1$ by taking $\omega_\alpha = 0$ for all α in (0.3) and thanks to the existence of exact solutions $U_{\varepsilon,x}$ of the equations which blow up as $\varepsilon \to 0$, see (2.5).

In terms of (0.1) the theorem rephrases as the existence of a sequence $u_\alpha(x)\mathrm{e}^{\mathrm{i}\omega_\alpha t}$ of standing waves solutions of (0.1) with purely electrostatic fields $\varphi_\alpha = \omega v_\alpha$ such that $\omega_\alpha \to \omega_k$ as $\alpha \to +\infty$, $(v_\alpha)_\alpha$ converges in L^∞ as $\alpha \to +\infty$, and $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$ with k-spikes. The convergence of the v_α's directly follows from elliptic theory and the second equation in (0.2).

The theorem and its consequence in terms of (0.1) hold true in quotients of S^3 for specific values of k, like on the projective space $\mathbb{P}^3(\mathbb{R})$ when k is even.

We discuss the physics origin of the system in Section 1. We prove our theorem in Section 2 by using the so-called localized energy method which goes through the choice of suitable approximate solutions and the use of finite-dimensional reduction.

1. The physics origin of the system

The Klein–Gordon–Maxwell–Proca system discussed in this work describes an interacting field theory model in theoretical physics. Most electromagnetic phenomena are described by conventional electrodynamics, which is a theory of the coupling of electromagnetic fields to matter fields. Of prime importance for particle physics is fermion electrodynamics in which matter is represented by spinor fields. However, one may have also boson electrodynamics in which matter is described by integer spin or bosonic fields. The simplest one is of course the complex scalar field, describing spinless particles having electric charges $\pm q$. It gives rise to scalar electrodynamics, which describes in the nonrelativistic limit the superconductivity of metals at very low temperatures. In the more general context of particle physics, a complex scalar field ψ may serve to describe scalar mesons in nuclear matter interacting via a massive vector boson field (φ, A).

The interaction in this model is described by the minimum substitution rule

$$\partial_t \to \partial_t + \mathrm{i}q\varphi \quad \text{and} \quad \nabla \to \nabla - \mathrm{i}qA,$$

in a nonlinear Klein–Gordon Lagrangian. As for the external massive vector field it is governed by the Maxwell–Proca Lagrangian. The constructions in this section follow the lines of the massless case addressed in Benci and Fortunato [2] (see also Benci and Fortunato [3]). Assuming for short that the manifold is orientable, we define the

Lagrangian densities $\mathcal{L}_{\text{NKG}}$ and $\mathcal{L}_{\text{MP}}$ of ψ, φ, and A by

$$\begin{aligned}\mathcal{L}_{\text{NKG}}(\psi,\varphi,A) &= \frac{1}{2}\left|\left(\frac{\partial}{\partial t}+\mathrm{i}q\varphi\right)\psi\right|^2-\frac{1}{2}\left|(\nabla-\mathrm{i}qA)\psi\right|^2-\frac{m_0^2}{2}|\psi|^2+\frac{1}{6}|\psi|^6,\\ \mathcal{L}_{\text{MP}}(\varphi,A) &= \frac{1}{2}\left|\frac{\partial A}{\partial t}+\nabla\varphi\right|^2-\frac{1}{2}|\nabla\times A|^2+\frac{m_1^2}{2}|\varphi|^2-\frac{m_1^2}{2}|A|^2,\end{aligned}\tag{1.1}$$

where $\nabla\times=\star d$, $\star$ is the Hodge dual, ψ represents the matter complex scalar field, m_0 its mass, q its charge, (φ,A) the electromagnetic vector field, and m_1 its mass. It can be noted that

$$\|(\varphi,A)\|_L^2=|\varphi|^2-|A|^2$$

is the square of the Lorentz norm of (φ,A) with respect to the Lorentz metric $\mathrm{diag}(1,-1,\dots,-1)$. The total action functional for ψ, ϕ, and A is then given by

$$\mathcal{S}(\psi,\varphi,A)=\int\int\left(\mathcal{L}_{\text{NKG}}+\mathcal{L}_{\text{MP}}\right)dv_g\,dt.\tag{1.2}$$

Writing ψ in polar form as $\psi(x,t)=u(x,t)\mathrm{e}^{\mathrm{i}S(x,t)}$, taking the variation of $\mathcal{S}$ with respect to u, S, φ, and A, we get four equations, which are written as

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}+\Delta_g u+m_0^2u=u^5+\left(\left(\frac{\partial S}{\partial t}+q\varphi\right)^2-|\nabla S-qA|^2\right)u\\ \frac{\partial}{\partial t}\left(\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2\right)-\nabla.\left((\nabla S-qA)\,u^2\right)=0\\ -\nabla.\left(\frac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2\varphi+q\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2=0\\ \overline{\Delta}_gA+\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2A=q\,(\nabla S-qA)\,u^2,\end{cases}\tag{1.3}$$

where $\Delta_g=-\mathrm{div}_g\nabla$ is the Laplace–Beltrami operator, $\overline{\Delta}_g=\delta d$ is half the Laplacian acting on forms, and δ is the codifferential. We refer to this system as a nonlinear Klein–Gordon–Maxwell–Proca system. There holds that $\overline{\Delta}_gA=\nabla\times(\nabla\times A)$. The above system consists in the nonlinear Klein–Gordon matter equation, the charge continuity equation and the massive modified Maxwell equations in SI units, which are hereafter explicitly written down:

$$\begin{aligned}&\nabla.E=\rho/\varepsilon_0-\mu^2\varphi,\\ &\nabla\times H=\mu_0\left(J+\varepsilon_0\frac{\partial E}{\partial t}\right)-\mu^2A,\\ &\nabla\times E+\frac{\partial H}{\partial t}=0\quad\text{and}\quad\nabla.H=0.\end{aligned}\tag{1.4}$$

Indeed, if we let $E=-\left(\frac{\partial A}{\partial t}+\nabla\varphi\right)$, $H=\nabla\times A$, $\rho=-\left(\frac{\partial S}{\partial t}+q\varphi\right)qu^2$, and $J=(\nabla S-qA)\,qu^2$, then the two last equations in (1.3) give rise to the first pair of the Maxwell–Proca equations (1.4) with $\epsilon_0=\mu_0=1$ (units are chosen such that $c=1$) and $\mu^2=m_1^2$, while the second pair of the Maxwell–Proca equations, as usual, is given for free because of the expressions of E and H. The first equation in (1.3) gives rise to the nonlinear Klein–Gordon matter equation. The second equation in (1.3) gives rise to the charge continuity equation $\frac{\partial\rho}{\partial t}+\nabla.J=0$ which, thanks to (1.4), is equivalent to the Lorentz condition $\nabla.A+\frac{\partial\varphi}{\partial t}=0$. The massive Maxwell equations (1.4), as modified to Proca form, appear to have been first written in modern format by Schrödinger [15]. The Proca formalism a priori breaks Gauge invariance. Gauge invariance can be restaured by the Stueckelberg trick, as pointed out by Pauli [11], and then by the Higgs mechanism. We refer to Goldhaber and Nieto [6,7], Luo et al. [10], and Ruegg and Ruiz–Altaba [13] for very complete references on the Proca approach.

We assume in what follows that $u(x,t) = u(x)$ does not depend on t, $S(x,t) = \omega t$ does not depend on x, and $\varphi(x,t) = \varphi(x)$, $A(x,t) = A(x)$ do not depend on t. In other words, we look for standing waves solutions of (1.3) and assume that we are in the static case of the system where (φ, A) depends on the sole spatial variable. By the fourth equation in (1.3) we then obtain that

$$\overline{\Delta}_g A + (q^2u^2 + m_1^2)A = 0.$$

This clearly implies that, and is equivalent to, $A \equiv 0$ since $\int (\overline{\Delta}_g A, A) = \int |dA|^2$. As a remark, assuming that $A \equiv 0$, the Lorentz condition for the external Proca field (φ, A) would make φ dependent on the sole spatial variables. As for the second equation in (1.3) it reduces to $\frac{\partial^2 S}{\partial t^2} = 0$. It is automatically satisfied when $S(t) = \omega t$, and we are thus left with the first and third equations in (1.3). Letting $S = -\omega t$, and $\varphi = \omega v$, we recover our original system

$$\begin{cases} \Delta_g u + m_0^2 u = u^5 + \omega^2 \left(qv - 1\right)^2 u, \\ \Delta_g v + \left(m_1^2 + q^2u^2\right) v = qu^2. \end{cases}$$

In other words, our original system (0.2) corresponds to looking for standing waves solutions of the Klein–Gordon–Maxwell–Proca system (1.3) in static form. The theorem we prove then provide the existence of resonant states for the static Klein–Gordon–Maxwell–Proca system (1.3).

2. Proof of the theorem

Formally, solutions of (0.2) are critical points of the functional S defined by

$$\begin{aligned} S(u,v) = &\frac{1}{2}\int_M |\nabla u|^2 dv_g - \frac{\omega^2}{2}\int_M |\nabla v|^2 dv_g + \frac{m_0^2}{2}\int_M u^2 dv_g \\ &- \frac{\omega^2 m_1^2}{2}\int_M v^2 dv_g - \frac{1}{p}\int_M u^p dv_g - \frac{\omega^2}{2}\int_M u^2(1-qv)^2 dv_g. \end{aligned} \tag{2.1}$$

The functional S is strongly indefinite because of the competition between u and v. Following a very nice idea going back to Benci–Fortunato [2], we introduce the auxiliary functional Φ given by

$$\Delta_g \Phi(u) + (m_1^2 + q^2u^2)\Phi(u) = qu^2, \tag{2.2}$$

and then consider that u in (0.2) can be seen as a critical point of

$$\begin{aligned} I(u) = &\frac{1}{2}\int_M |\nabla u|^2 dv_g + \frac{m_0^2}{2}\int_M u^2 dv_g - \frac{1}{6}\int_M (u^+)^6 dv_g \\ &- \frac{\omega^2}{2}\int_M \left(1 - q\Phi(u)\right) u^2 dv_g, \end{aligned} \tag{2.3}$$

where $u^+ = \max(u,0)$ is the nonnegative part of u. Let $F_\Phi : H^1 \to \mathbb{R}$ be defined by $F_\Phi(u) = \frac{1}{2}\int_M (1 - q\Phi(u))u^2 dv_g$. As is easily checked, $\Phi : H^1 \to H^1$ is uniquely defined, it satisfies that $0 \leq \Phi(u) \leq \frac{1}{q}$ for all $u \in H^1$, Φ and F_Φ are C^1, and

$$DF_\Phi(u).(\varphi) = \int_M (1 - q\Phi(u))^2 u\varphi dv_g,$$

for all $u, \varphi \in H^1$. Now the goal is to construct blowing-up multi-spikes solutions to (0.2) when ω is close to resonant frequencies ω_k. To each ω_k is associated a sequence of n_k-spikes solutions with $n_k \to +\infty$ as $k \to +\infty$. This can be considered as bifurcation from infinity (see Bahri [1]). More precisely we use here the so-called localized energy method (see Del Pino et al. [4], Rey and Wei [12], and Wei [17]) which goes through the choice of suitable approximate solutions and the use of finite-dimensional reduction. The proof we present here follows closely the lines of Hebey and Wei [9].

Let $P_1 = (1, 0, 0, 0)$ in S^3 and $k \in \mathbb{N}$, $k \geq 1$. We define the P_i's, $i = 1, \ldots, k$, by $P_i = (\mathrm{e}^{\mathrm{i}\theta_i}, 0) \in S^3 \subset \mathbb{R}^2 \times \mathbb{R}^2$, where $\theta_i = \frac{2\pi(i-1)}{k}$. Let G_k be the maximal isometry group of (S^3, g), which leaves globally invariant the set $\{P_1, \ldots, P_k\}$. Let also $\Sigma_k \subset S^3$ be the slice

$$\Sigma_k = \left\{ \left(r\mathrm{e}^{\mathrm{i}\theta}, z \right), r > 0, z \in \mathbb{C}, r^2 + |z|^2 = 1, -\frac{\pi}{k} \leq \theta \leq \frac{\pi}{k} \right\}. \tag{2.4}$$

The Yamabe equation in S^3 is written as

$$\Delta_g u + \frac{3}{4} u = u^5.$$

Its solutions are given by

$$U_{\varepsilon, x_0} = \frac{3^{1/4}}{\sqrt{2}} \left(\frac{\varepsilon}{\varepsilon^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{1/2}, \tag{2.5}$$

where $\varepsilon \in (0, 1)$, $r = d_g(x_0, \cdot)$, and $x_0 \in S^3$ is arbitrary. Given $\theta > 0$, we let G_θ be the Green's function of $\Delta_g + \theta^2$. Then

$$G_\theta(x, y) = \frac{\sinh(\mu_\theta(\pi - r))}{4\pi \sinh(\mu_\theta \pi) \sin r}, \tag{2.6}$$

for all $x, y \in S^3$, $x \neq y$, where $r = d_g(x, y)$ and $\mu_\theta = \sqrt{\theta^2 - 1}$. We define R_θ to be given by

$$G_\theta = G_{\frac{\sqrt{3}}{2}} + R_\theta. \tag{2.7}$$

The following lemma holds true.

Lemma 2.1. *Let G_θ and R_θ be as above. Given $k \in \mathbb{N}$, $k \geq 1$, define*

$$\eta_k(\theta) = R_\theta(P_1, P_1) + \sum_{i=2}^{k} G_\theta(P_1, P_i), \tag{2.8}$$

where the second term in the right hand side of (2.8) is zero, if $k = 1$. There exists a unique $\theta_k > 0$ such that $\eta_k(\theta_k) = 0$. There holds $\eta_k(\theta) > 0$, when $\theta < \theta_k$, $\eta_k(\theta) < 0$ when $\theta > \theta_k$, $\theta_1 = \frac{\sqrt{3}}{2}$, $\theta_k \to +\infty$ as $k \to +\infty$, and $\theta_k > 1 > \theta_1$ for all $k \geq 2$.

Proof of Lemma 2.1. There holds that

$$R_\theta(P_1, P_1) = -\frac{1}{4\pi}\mu_\theta \coth(\mu_\theta \pi),$$

so that $\eta_1(\theta) = 0$ if and only if $\theta^2 = \frac{3}{4}$, while $\eta_1'(\frac{\sqrt{3}}{2}) < 0$. It is easily checked that $\eta_k(\theta) \to -\infty$ as $\theta^2 \to +\infty$, while $\eta_k(1) > 0$ for $k \geq 2$. There also holds that $\frac{d}{d\mu}(\mu \coth(\mu\pi)) > 0$ while, by the maximum principle, $G_\theta \leq G_{\theta_0}$, if $\theta^2 \geq \theta_0^2$. Hence there exists a unique $\theta_k > 0$ such that $\eta_k(\theta_k) = 0$. Then $\eta_k(\theta) > 0$ if $\theta < \theta_k$ and $\eta_k(\theta) < 0$ if $\theta > \theta_k$. Since $\sinh(tx)/\sin(x) \geq t$ for $x \in (-\pi, \pi)$, there holds that $\theta_k \to +\infty$ as $k \to +\infty$. There also holds that $\theta_k > 1$ for $k \geq 2$ since $\eta_k(1) > 0$ for $k \geq 2$, and we have that $\theta_1 = \frac{\sqrt{3}}{2} < 1$. This ends the proof of the lemma. $\square$

Letting $R_{\theta,P_1} = R_\theta(P_1, \cdot)$, where R_θ is given by (2.7), we can check

$$R_{\theta,P_1} = -\frac{\mu_\theta \coth(\mu_\theta \pi)}{4\pi} + \frac{1}{8\pi}\left(\theta^2 - \frac{3}{4}\right) r + O(r^2), \tag{2.9}$$

where $r = d_g(P_1, \cdot)$. Given $\varepsilon > 0$, we define the projections $\mathcal{U}_{\varepsilon,P_i}$, $i = 1, \ldots, k$, by

$$\Delta_g \mathcal{U}_{\varepsilon,P_i} + \theta^2 \mathcal{U}_{\varepsilon,P_i} = U_{\varepsilon,P_i}^5 \tag{2.10}$$

and we define $\varphi_{\varepsilon,P_i}$ and $\mathcal{W}_\varepsilon$ to be given by

$$\mathcal{U}_{\varepsilon,P_i} = U_{\varepsilon,P_i} + \varphi_{\varepsilon,P_i} \quad \text{and} \quad \mathcal{W}_\varepsilon = \sum_{i=1}^{k} \mathcal{U}_{\varepsilon,P_i}, \tag{2.11}$$

where U_{ε,P_i} is as in (2.5). The $\mathcal{W}_\varepsilon$'s are G_k-invariant. As shown in Hebey and Wei [9], the following lemma holds true.

Lemma 2.2 (**Hebey and Wei [9]**). *There holds that*

$$\begin{aligned} \varphi_{\varepsilon,P_1} &= A\sqrt{\varepsilon} R_{\theta,P_1} + B_\theta \varepsilon^{3/2} \psi\left(\frac{r}{\varepsilon}\right) + o\left(\varepsilon^{3/2}\right) \quad \text{and} \\ \mathcal{W}_\varepsilon &= U_{\varepsilon,P_1} + A\sqrt{\varepsilon}\left(R_{\theta,P_1} + \sum_{i=2}^{k} G_{\theta,P_i}\right) + B_\theta \varepsilon^{3/2} \psi\left(\frac{r}{\varepsilon}\right) + o\left(\varepsilon^{3/2}\right) \end{aligned} \tag{2.12}$$

in Σ_k, where $r = d_g(P_1, \cdot)$, $G_{\theta,P_i} = G_\theta(P_i, \cdot)$, $A = 4\pi 3^{1/4}\sqrt{2}$, $B_\theta = \frac{A}{4\pi}\left(\frac{3}{4} - \theta^2\right)$, and ψ is the solution of $\Delta\psi = \frac{1}{\sqrt{4+|x|^2}} - \frac{1}{|x|}$ in $\mathbb{R}^3$.

As a remark, there holds that $|\psi(x)| \leq C\frac{\ln(2+|x|)}{1+|x|}$ and $|\nabla\psi(x)| \leq C\frac{\ln(2+|x|)}{(1+|x|)^2}$ as $|x| \to +\infty$. In the equation for ψ, $\Delta = -\sum_i \partial_i^2$. Now we prove the following.

Lemma 2.3. *Let $k \in \mathbb{N}$, $k \geq 1$. Let $\mathcal{W}_\varepsilon$ be as in (2.11), and $\Phi : H^1 \to H^1$ be as in (2.2). Then $\frac{1}{\varepsilon}\Phi(\mathcal{W}_\varepsilon) \to q\Phi_{k,\theta}$ in H^1, where $\Phi_{k,\theta}$ solves*

$$\Delta_g \Phi_{k,\theta} + m_1^2 \Phi_{k,\theta} = A^2 G^2,$$

$G = \sum_{i=1}^k G_{\theta,P_i}$, $G_{\theta,P_i} = G_\theta(P_i, \cdot)$ for all i, and G_θ is the Green's function of $\Delta_g + \theta^2$ given by (2.6).

Proof of Lemma 2.3. Let $v_\varepsilon = \frac{1}{\varepsilon}\Phi(\mathcal{W}_\varepsilon)$. By the definition of $\Phi(\mathcal{W}_\varepsilon)$ there holds that

$$\Delta_g v_\varepsilon + (m_1^2 + q^2\mathcal{W}_\varepsilon^2)v_\varepsilon = q\left(\frac{\mathcal{W}_\varepsilon}{\sqrt{\varepsilon}}\right)^2. \tag{2.13}$$

By (2.12) in Lemma 2.2,

$$\frac{\mathcal{W}_\varepsilon}{\sqrt{\varepsilon}} \le C\left(\sin\frac{r_i}{2}\right)^{-1}$$

around P_i, while $\frac{\mathcal{W}_\varepsilon}{\sqrt{\varepsilon}} \le C$ when standing far from the P_i's, where $r_i = d_g(P_i,\cdot)$. Hence, the family $(\mathcal{W}_\varepsilon/\sqrt{\varepsilon})_\varepsilon$ is bounded in L^p for all $p < 3$. It clearly follows, when multiplying (2.13) by v_ε and integrating over S^3, that $(v_\varepsilon)_\varepsilon$ is bounded in H^1. We use for this Hölder's inequality and note that $\frac{12}{5} < 3$. There also holds that $\left(\mathcal{W}_\varepsilon^2 v_\varepsilon\right)_\varepsilon$ is bounded in L^p for $p \le 2$. By (2.13), we then obtain that

$$\Delta_g v_\varepsilon + m_1^2 v_\varepsilon = f_\varepsilon,$$

where $(f_\varepsilon)_\varepsilon$ is bounded in L^p for all $p < \frac{3}{2}$. By elliptic theory this implies that $(v_\varepsilon)_\varepsilon$ is bounded in $H^{2,p}$ for all $p < \frac{3}{2}$. In particular, since $H^{2,p} \subset H^1$ is compact for p close to $\frac{3}{2}$, there exists Φ such that, up to a subsequence, $v_\varepsilon \to \Phi$ in H^1 as $\varepsilon \to 0$. As is easily checked, it follows from Hölder's inequality and (2.12) that $\int \mathcal{W}_\varepsilon^2 v_\varepsilon \varphi \to 0$ as $\varepsilon \to 0$ for all $\varphi \in H^1$. By (2.12) and (2.13), Φ solves

$$\Delta_g \Phi + m_1^2 \Phi = qA^2G^2.$$

In particular, Φ is unique. This ends the proof of the lemma. $\square$

It follows from Lemma 2.3 that $\Phi(\mathcal{W}_\varepsilon) = O(\varepsilon^\sigma)$ for all $\sigma \in (0,1)$. Indeed there holds that for any $\delta \in (0,1)$,

$$\Delta_g(\varepsilon^{\delta-1}\Phi(\mathcal{W}_\varepsilon)) + m_1^2(\varepsilon^{\delta-1}\Phi(\mathcal{W}_\varepsilon)) = q\Psi(\mathcal{W}_\varepsilon)\varepsilon^\delta\left(\frac{\mathcal{W}_\varepsilon}{\sqrt{\varepsilon}}\right)^2, \tag{2.14}$$

where $0 \le \Psi \le 1$ is given by $\Psi(u) = 1 - q\Phi(u)$. We have that

$$\varepsilon^\delta\left(\frac{\mathcal{W}_\varepsilon}{\sqrt{\varepsilon}}\right)^2 \le \frac{C\varepsilon^\delta}{\varepsilon^2 + r_i^2},$$

around P_i, while $\varepsilon^\delta(\frac{\mathcal{W}_\varepsilon}{\sqrt{\varepsilon}})^2 \le C\varepsilon^\delta$ when standing far from the P_i's. Then

$$\begin{aligned}\int_{S^3}\left(\varepsilon^\delta\left(\frac{\mathcal{W}_\varepsilon}{\sqrt{\varepsilon}}\right)^2\right)^p dv_g &\le C_1\int_0^1\left(\frac{\varepsilon^\delta}{\varepsilon^2+r^2}\right)^p r^2dr + C_2 \\ &\le C_1\varepsilon^{3-(2-\delta)p}\int_0^{+\infty}\left(\frac{1}{1+r^2}\right)^p r^2dr + C_2 \le C_3,\end{aligned}$$

for $p = p_\delta = \frac{3}{2-\delta} > \frac{3}{2}$. By (2.14) we then obtain that $(\varepsilon^{\delta-1}\Phi(\mathcal{W}_\varepsilon))_\varepsilon$ is bounded in H^1 since $p_\delta > \frac{5}{6}$. Then the family is also bounded in H^{2,p_δ}, and since by Sobolev $H^{2,p_\delta} \subset L^\infty$, we obtain that $\Phi(\mathcal{W}_\varepsilon) \le C_\delta\varepsilon^{1-\delta}$, $\delta \in (0,1)$. Letting $\sigma = \delta - 1$, this proves the bound. Now we prove that the following asymptotic development for $I(\mathcal{W}_\varepsilon)$ holds true.

Lemma 2.4. *Let I be given by (2.3) and $\mathcal{W}_\varepsilon$ be given by (2.11). There holds that*

$$I(\mathcal{W}_\varepsilon) = A_{0,k} + A_{1,k}\varepsilon\eta_k(\theta) + A_{2,k}(\omega)q^2\varepsilon^2 + A_{3,k}(\theta)\varepsilon^2 + O(\eta_k(\theta)^2\varepsilon^2) + o(\varepsilon^2), \tag{2.15}$$

where $\theta^2 = m_0^2 - \omega^2$, $A_{0,k} = \frac{k}{3}(\frac{3}{4})^{3/2}\int_{\mathbb{R}^3} U_0^6 dx$, $A_{1,k} = -\frac{kA}{2}(\frac{3}{4})^{5/4}\int_{\mathbb{R}^3} U_0^5 dx$,

$$\begin{aligned} A_{2,k}(\omega) &= \frac{\omega^2}{2}\int_{S^3}\left(|\nabla\Phi_{k,\theta}|^2 + m_1^2\Phi_{k,\theta}^2\right)dv_g, \\ A_{3,k}(\theta) &= -16\pi k\sqrt{3}\left(\theta^2 - \frac{3}{4}\right)\int_0^{+\infty}\frac{dr}{4+r^2}, \end{aligned} \tag{2.16}$$

the function U_0 is given by $U_0(x) = \left(1 + \frac{|x|^2}{4}\right)^{-1/2}$ for $x \in \mathbb{R}^3$, and $\Phi_{k,\theta}$ is as in Lemma 2.3.

Proof of Lemma 2.4. We proceed as in Hebey and Wei [9]. By (2.12) in Lemma 2.2 we have that

$$\begin{aligned} &\int_{S^3}|\nabla\mathcal{W}_\varepsilon|^2 dv_g + \theta^2\int_{S^3}\mathcal{W}_\varepsilon^2 dv_g \\ &= k\left(\frac{3}{4}\right)^{3/2}\int_{\mathbb{R}^3}U_0^6 dx + k\left(\frac{3}{4}\right)^{5/4}A\varepsilon\eta_k(\theta)\int_{\mathbb{R}^3}U_0^5 dx \\ &\quad + k\left(\frac{3}{4}\right)^{5/4}\frac{A}{8\pi}\left(\theta^2 - \frac{3}{4}\right)\varepsilon^2\int_{\mathbb{R}^3}U_0^5 r dx \\ &\quad + k\left(\frac{3}{4}\right)^{5/4}B_\theta\varepsilon^2\int_{\mathbb{R}^3}U_0^5\psi dx + o\left(\varepsilon^2\right), \end{aligned} \tag{2.17}$$

where B_θ and ψ are as in (2.12). Still by (2.12), we have that

$$\begin{aligned} \int_{S^3}\mathcal{W}_\varepsilon^6 dv_g &= k\left(\frac{3}{4}\right)^{3/2}\int_{\mathbb{R}^3}U_0^6 dx + 6k\left(\frac{3}{4}\right)^{5/4}A\varepsilon\eta_k(\theta)\int_{\mathbb{R}^3}U_0^5 dx \\ &\quad + 6k\left(\frac{3}{4}\right)^{5/4}\frac{A}{8\pi}\left(\theta^2 - \frac{3}{4}\right)\varepsilon^2\int_{\mathbb{R}^3}U_0^5 r dx \\ &\quad + 6k\left(\frac{3}{4}\right)^{5/4}B_\theta\varepsilon^2\int_{\mathbb{R}^3}U_0^5\psi dx \\ &\quad + O(\varepsilon^2\eta_k(\omega)^2) + o(\varepsilon^2). \end{aligned} \tag{2.18}$$

Now we use the equation satisfied by $\Phi(\mathcal{W}_\varepsilon)$ to write that

$$\begin{aligned} \frac{q\omega^2}{2}\int_{S^3}\Phi(\mathcal{W}_\varepsilon)\mathcal{W}_\varepsilon^2 dv_g &= \frac{\omega^2}{2}\int_{S^3}(|\nabla\Phi(\mathcal{W}_\varepsilon)|^2 + m_1^2\Phi(\mathcal{W}_\varepsilon)^2)dv_g \\ &\quad + \frac{\omega^2}{2}q^2\int_{S^3}\mathcal{W}_\varepsilon^2\Phi(\mathcal{W}_\varepsilon)^2 dv_g. \end{aligned}$$

By (2.12), $\int\mathcal{W}_\varepsilon^2 = O(\varepsilon)$, while we have seen that $\Phi(\mathcal{W}_\varepsilon) = O(\varepsilon^\sigma)$ for all $\sigma \in (0,1)$. Picking $\sigma < 1$ sufficiently close to 1, it follows that $\int\mathcal{W}_\varepsilon^2\Phi(\mathcal{W}_\varepsilon)^2 = o(\varepsilon^2)$, and by

Lemma 2.3 we obtain that

$$\begin{aligned}(2.19)\qquad &\frac{q\omega^2}{2}\int_{S^3}\Phi(\mathcal{W}_\varepsilon)\mathcal{W}_\varepsilon^2 dv_g\\ &= \frac{\omega^2\varepsilon^2}{2}\int_{S^3}\left(\left|\nabla\frac{\Phi(\mathcal{W}_\varepsilon)}{\varepsilon}\right|^2 + m_1^2\left(\frac{\Phi(\mathcal{W}_\varepsilon)}{\varepsilon}\right)^2\right)dv_g\\ &= \frac{\omega^2q^2\varepsilon^2}{2}\int_{S^3}\left(|\nabla\Phi(\mathcal{W}_{k,\theta})|^2 + m_1^2\Phi(\mathcal{W}_{k,\theta})^2\right)dv_g + o(\varepsilon^2).\end{aligned}$$

Combining (2.17)–(2.19), the lemma follows with

$$\begin{aligned}A_{3,k}(\theta) &= -\frac{k}{2}\left(\frac{3}{4}\right)^{5/4}\left(\frac{A}{8\pi}\left(\theta^2-\frac{3}{4}\right)\int_{\mathbb{R}^3}U_0^5 r dx + B_\theta\int_{\mathbb{R}^3}U_0^5\psi dx\right)\\ &= -\frac{k}{2}\left(\frac{3}{4}\right)^{3/2}\left(\theta^2-\frac{3}{4}\right)\int_{\mathbb{R}^3}U_0^5(r-2\psi)dx.\end{aligned}$$

Integrating by parts, since $\Delta U_0 = \frac{3}{4}U_0^5$, we obtain that

$$A_{3,k}(\theta) = -16\pi k\sqrt{3}\left(\theta^2-\frac{3}{4}\right)\int_0^{+\infty}\frac{dr}{4+r^2}.$$

This ends the proof of Lemma 2.4. □

Let us write that $A_{2,k}(\omega) = \omega^2 B_{2,k}(\theta)$. Then, $B_{2,k}(\theta) > 0$. Let θ_k be given by Lemma 2.1. The function $\Phi_{k,\theta}$ in Lemma 2.3 is G_k-invariant. By Hölder's inequalities we can write that

$$\begin{aligned}\int_{S^3}(|\nabla\Phi_{k,\theta}|^2 + m_1^2\Phi_{k,\theta}^2)dv_g &\le C\sum_{i=1}^k\int_{S^3}G_{\theta,P_i}^2\Phi_{k,\theta}dv_g\\ &\le Ck\int_{S^3}G_{\omega,P_1}^2\Phi_{k,\theta}dv_g \le Ck\|G_{\theta,P_1}\|_{L^{12/5}}\|\Phi_{k,\theta}\|_{L^6}.\end{aligned}$$

By the maximum principle, $G_{\theta',P_1} \le G_{\theta,P_1}$ for all $\theta' \ge \theta$. Since $\theta_k \to +\infty$, it follows that $B_{2,k}(\theta_k) \le Ck^2$, where $C > 0$ is independent of k. On the other hand, by the definition of θ_k, $\mu_k\coth(\mu_k\pi) \ge CG_{\theta_k,P_1}(P_2)$, where $\mu_k \le C\theta_k$, and we thus obtain that $\theta_k \ge Ck$, where $C > 0$ is independent of k. As a consequence we obtain that $|A_{3,k}(\theta_k)| \ge CkB_{2,k}(\theta_k)$. Then

$$\lim_{k\to+\infty}\frac{A_{3,k}(\theta_k)}{B_{2,k}(\theta_k)} = -\infty. \tag{2.20}$$

While $\frac{A_{3,k}(\theta_k)}{B_{2,k}(\theta_k)} = 0$ when $k = 1$, the quotient can be made arbitrarily large in absolute value and negative in specific situations. Now we turn our attention to the finite-dimensional reduction part of the proof. We let Θ_k be given by

$$\Theta_k = q^2A_{2,k}(\omega_k) + A_{3,k}(\theta_k), \tag{2.21}$$

where $A_{2,k}(\omega), A_{3,k}(\theta)$ are as in Lemma 2.4, $\theta_k^2 = m_0^2 - \omega_k^2$, and the θ_k's are as in Lemma 2.1. Then we let $\varepsilon = \Lambda\tilde{\varepsilon}$, where $\frac{1}{C} \le \Lambda \le C$ for $C \gg 1$, and we define $\tilde{\varepsilon} = \eta_k(\theta)$ for $\theta \in (\theta_k - \delta, \theta_k)$ with $\delta > 0$ small in case $\Theta_k > 0$, and $\tilde{\varepsilon} = -\eta_k(\theta)$ for

$\theta \in (\theta_k, \theta_k + \delta)$ with $\delta > 0$ small in case $\Theta_k < 0$. In the above constructions, $\tilde{\varepsilon} > 0$ and $\tilde{\varepsilon} \to 0$ as $\theta \to \theta_k$. We let

$$f_{\tilde{\varepsilon}} : \frac{1}{\tilde{\varepsilon}} S^3 \to S^3$$

be the map given by $f_{\tilde{\varepsilon}}(x) = \tilde{\varepsilon} x$. If $g_{\tilde{\varepsilon}}$ is the standard metric on $\frac{1}{\tilde{\varepsilon}} S^3$, induced from the Euclidean metric, then $f_{\tilde{\varepsilon}}^\star g = \tilde{\varepsilon}^2 g_{\tilde{\varepsilon}}$. Given $u : S^3 \to \mathbb{R}$, we define the $\sim$-procedure which, to u, associate $\tilde{u} : \frac{1}{\tilde{\varepsilon}} S^3 \to \mathbb{R}$, where

$$\tilde{u} = \sqrt{\tilde{\varepsilon}} u \circ f_{\tilde{\varepsilon}}.$$

We let $\tilde{Y} = \frac{\partial \tilde{\mathcal{W}}_\varepsilon}{\partial \Lambda}$, where $\tilde{\mathcal{W}}_\varepsilon$ is obtained from $\mathcal{W}_\varepsilon$ in (2.11) by the $\sim$-procedure, and we define

$$\tilde{Z} = \Delta_{g_{\tilde{\varepsilon}}} \tilde{Y} + \tilde{\varepsilon}^2 (m_0^2 - \omega^2) \tilde{Y}. \tag{2.22}$$

There holds that $\langle \tilde{Y}, \tilde{Z} \rangle = \gamma_0 + o(1)$, where $\gamma_0 > 0$ and $\langle \cdot, \cdot \rangle$ is the L^2-scalar product with respect to $g_{\tilde{\varepsilon}}$. We say in what follows that a function $\tilde{u}$ in $\frac{1}{\tilde{\varepsilon}} S^3$ is G_k-invariant if u is G_k-invariant in S^3. In particular $\tilde{Y}$ and $\tilde{Z}$ are G_k-invariant. Let Ψ be given by $\Psi(u) = \omega^2 \Phi(u)(2 - q\Phi(u))$. By the $\sim$-procedure, the equation

$$\Delta_g u + (m_0^2 - \omega^2) u + q \Psi(u) u = u^5$$

in S^3, which is the equation associated to I, is equivalent to

$$\Delta_{g_{\tilde{\varepsilon}}} \tilde{u} + \tilde{\varepsilon}^2 (m_0^2 - \omega^2) \tilde{u} + q \tilde{\varepsilon}^2 \overline{\Psi(u)} \tilde{u} = \tilde{u}^5$$

in $\frac{1}{\tilde{\varepsilon}} S^3$, where $\overline{\Psi(u)} = \Psi(u) \circ f_{\tilde{\varepsilon}}$. Now we define the norms $\| \cdot \|_{\star,\sigma}$ and $\| \cdot \|_{\star\star,\sigma}$ by

$$\begin{aligned} \|u\|_{\star,\sigma} &= \sup_{x \in \frac{1}{\tilde{\varepsilon}} S^3} \left(\min_{i=1,\dots,k} \left(1 + d_{g_{\tilde{\varepsilon}}}(\tilde{P}_i, x)\right)^\sigma \right) |u(x)|, \\ \|u\|_{\star\star,\sigma} &= \sup_{x \in \frac{1}{\tilde{\varepsilon}} S^3} \left(\min_{i=1,\dots,k} \left(1 + d_{g_{\tilde{\varepsilon}}}(\tilde{P}_i, x)\right)^{2+\sigma} \right) |u(x)| \end{aligned} \tag{2.23}$$

for $u \in L^\infty(\frac{1}{\tilde{\varepsilon}} S^3)$, where $0 < \sigma < 1$ and $f_{\tilde{\varepsilon}}(\tilde{P}_i) = P_i$, $i = 1, \dots, k$. Given a function $h \in L^\infty \left(\frac{1}{\tilde{\varepsilon}} S^3\right)$ we consider the problem

$$\begin{cases} \Delta_{g_{\tilde{\varepsilon}}} \phi + \tilde{\varepsilon}^2 (m_0^2 - \omega^2) \phi - 5 \tilde{W}_\varepsilon^4 \phi = h + c_0 \tilde{Z} \\ \int_{\frac{1}{\tilde{\varepsilon}} S^3} \tilde{Z} \phi dv_{g_{\tilde{\varepsilon}}} = 0, \end{cases} \tag{2.24}$$

where $c_0 \in \mathbb{R}$, and $\tilde{Z}$ is as in (2.22). Following Hebey and Wei [9], Del Pino et al. [4], and Rey and Wei [12], we obtain that there exist $\tilde{\varepsilon}_0 > 0$ and $C > 0$ such that for any $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$ and any G_k-invariant function $h \in L^\infty(\frac{1}{\tilde{\varepsilon}} S^3)$, (2.24) has a unique G_k-invariant solution $\phi = \mathcal{L}_{\tilde{\varepsilon}}(h)$ with $\|\phi\|_{\star,\sigma} \le C \|h\|_{\star\star,\sigma}$. Moreover, the map $\mathcal{L}_{\tilde{\varepsilon}}$ is C^1 w.r.t. Λ and $\|D_\Lambda \mathcal{L}_{\tilde{\varepsilon}}(h)\|_{\star,\sigma} \le C \|h\|_{\star\star,\sigma}$. Now we prove the following estimates on the Ψ functional.

Lemma 2.5. *Let $\mathcal{W}_\varepsilon$ be as in (2.11). Let $\Psi(u) = \omega^2 \Phi(u)(2 - q\Phi(u))$ be as above. There exists $C > 0$, independent of ε, such that for any u, u_1, u_2 in the $\tilde{\varepsilon}$-ball $B_{\tilde{\varepsilon}} = \{u \in H^1 \cap L^\infty$ s.t. $\|\tilde{u}\|_{\star,\sigma} \le \tilde{\varepsilon}\}$, there holds that*

$$\begin{aligned} &\|\Psi_{\tilde{\varepsilon}}(u)\|_{\star\star,\sigma} \le C(\tilde{\varepsilon}^\sigma + \tilde{\varepsilon}^{1-\sigma}) \|\tilde{u}\|_{\star,\sigma}, \quad \textit{and} \\ &\|\Psi_{\tilde{\varepsilon}}(u_2) - \Psi_{\tilde{\varepsilon}}(u_1)\|_{\star\star,\sigma} \le C(\tilde{\varepsilon}^\sigma + \tilde{\varepsilon}^{1-\sigma}) \|\tilde{u}_2 - \tilde{u}_1\|_{\star,\sigma} \end{aligned} \tag{2.25}$$

where $\Psi_{\tilde\varepsilon}(u) = \tilde\varepsilon^2\left(\overline{\Psi(\mathcal{W}_\varepsilon + u)}(\tilde{\mathcal{W}}_\varepsilon + \tilde u) - \overline{\Psi(\mathcal{W}_\varepsilon)}\tilde{\mathcal{W}}_\varepsilon\right)$, *and the norms* $\|\cdot\|_{\star,\sigma}$ *and* $\|\cdot\|_{\star\star,\sigma}$ *are as in* (2.23).

Proof of Lemma 2.5. Let $F(u) = q(1 - q\Phi(u))$. There holds

$$\Delta_g\Phi(u) + m_1^2\Phi(u) = F(u)u^2$$

and we can write that

$$\begin{aligned}\Delta_g(\Phi(\mathcal{W}_\varepsilon + u_2) - \Phi(\mathcal{W}_\varepsilon + u_1)) &+ m_1^2(\Phi(\mathcal{W}_\varepsilon + u_2) - \Phi(\mathcal{W}_\varepsilon + u_1))\\ &= -q^2(\Phi(\mathcal{W}_\varepsilon + u_2) - \Phi(\mathcal{W}_\varepsilon + u_1))(\mathcal{W}_\varepsilon + u_2)^2\\ &\quad + F(\mathcal{W}_\varepsilon + u_1)(u_1 + u_2 + 2\mathcal{W}_\varepsilon)(u_2 - u_1).\end{aligned} \tag{2.26}$$

Since $\|\tilde u\|_{\star,\sigma} \le \tilde\varepsilon$ implies $\|u\|_{L^\infty} \le \sqrt{\tilde\varepsilon}$, we have by (2.12) that $\|\mathcal{W}_\varepsilon + u_2\|_{L^4} = o(1)$. Hence,

$$\begin{aligned}&\left\|\left(\Phi(\mathcal{W}_\varepsilon + u_2) - \Phi(\mathcal{W}_\varepsilon + u_1)\right)(\mathcal{W}_\varepsilon + u_2)^2\right\|_{L^2}\\ &\quad = o\left(\Phi(\mathcal{W}_\varepsilon + u_2) - \Phi(\mathcal{W}_\varepsilon + u_1)\right).\end{aligned} \tag{2.27}$$

Since $|F| \le 1$, and $\int \mathcal{W}_\varepsilon^2 = O(\varepsilon)$, there also holds that

$$\|F(\mathcal{W}_\varepsilon + u_1)(u_1 + u_2 + 2\mathcal{W}_\varepsilon)(u_2 - u_1)\|_{L^2} \le C\sqrt{\tilde\varepsilon}\|u_2 - u_1\|_{L^\infty}. \tag{2.28}$$

Combining (2.26)–(2.28), by standard elliptic theory, and since $H^2 \subset L^\infty$, we obtain that

$$\|\Phi\left(\mathcal{W}_\varepsilon + u_2\right) - \Phi\left(\mathcal{W}_\varepsilon + u_1\right)\|_{L^\infty} \le C\sqrt{\tilde\varepsilon}\|u_2 - u_1\|_{L^\infty}. \tag{2.29}$$

Noting that $\sqrt{\tilde\varepsilon}\|u\|_{L^\infty} \le \|\tilde u\|_{\star,\sigma}$, $\|\tilde u\|_{\star\star,\sigma} \le \tilde\varepsilon^{-2}\|\tilde u\|_{\star,\sigma}$, and $\|\tilde{\mathcal{W}}_\varepsilon\|_{\star\star,\sigma} \le C\tilde\varepsilon^{-1-\sigma}$, we obtain by (2.29) that

$$\begin{aligned}&\left\|\overline{\Phi(\mathcal{W}_\varepsilon + u_2)}(\tilde{\mathcal{W}}_\varepsilon + \tilde u_2) - \overline{\Phi(\mathcal{W}_\varepsilon + u_1)}\left(\tilde{\mathcal{W}}_\varepsilon + \tilde u_1\right)\right\|_{\star\star,\sigma}\\ &\quad \le C\left(\tilde\varepsilon^{-1-\sigma} + \tilde\varepsilon^{-2+\sigma}\right)\|\tilde u_2 - \tilde u_1\|_{\star,\sigma}.\end{aligned} \tag{2.30}$$

Since $|\Phi| \le \frac{1}{q}$, we easily deduce (2.25) from (2.29) and (2.30). This ends the proof of the Lemma. □

At this point we define $R_{1,\tilde\varepsilon}$, $R_{2,\tilde\varepsilon}$, and $R_{\tilde\varepsilon}$ by

$$\begin{aligned}R_{1,\tilde\varepsilon} &= \tilde{\mathcal{W}}_\varepsilon^5 - \Delta_{g_{\tilde\varepsilon}}\tilde{\mathcal{W}}_\varepsilon - (m_0^2 - \omega^2)\tilde\varepsilon^2\tilde{\mathcal{W}}_\varepsilon,\\ R_{2,\tilde\varepsilon} &= -\tilde\varepsilon^2\overline{\Psi(\mathcal{W}_\varepsilon)}\tilde{\mathcal{W}}_\varepsilon, \quad R_{\tilde\varepsilon} = R_{1,\tilde\varepsilon} + R_{2,\tilde\varepsilon},\end{aligned} \tag{2.31}$$

and we consider the problem

$$\begin{cases}\Delta_{g_{\tilde\varepsilon}}(\hat W_\varepsilon + \tilde\phi) + \tilde\varepsilon^2(m_0^2 - \omega^2)(\hat W_\varepsilon + \tilde\phi)\\ \quad + q\tilde\varepsilon^2\overline{\Psi(\mathcal{W}_\varepsilon + K_\varepsilon + \phi)}(\hat W_\varepsilon + \tilde\phi) = (\hat W_\varepsilon + \tilde\phi)^5 + c_0\tilde Z,\\ \int_{\frac{1}{\tilde\varepsilon}S^3}\tilde Z\tilde\phi dv_{g_{\tilde\varepsilon}} = 0,\end{cases} \tag{2.32}$$

where $\hat W_\varepsilon = \tilde{\mathcal{W}}_\varepsilon + \mathcal{L}_{\tilde\varepsilon}(R_{\tilde\varepsilon})$, $c_0 \in \mathbb{R}$,

$$\overline{\Phi(\mathcal{W}_\varepsilon + K_\varepsilon + \phi)} = \Phi(\mathcal{W}_\varepsilon + K_\varepsilon + \phi) \circ f_{\tilde\varepsilon},$$

and $\tilde{K}_\varepsilon = \mathcal{L}_{\tilde{\varepsilon}}(R_{\tilde{\varepsilon}})$. Thanks to Lemma 2.5 we can apply the fixed point argument as in Del Pino et al. [4], and Rey and Wei [12]. Noting that

$$\|R_{i,\tilde{\varepsilon}}\|_{\star\star,\sigma} \leq C\tilde{\varepsilon} \quad \text{and} \quad \|D_\Lambda R_{i,\tilde{\varepsilon}}\|_{\star\star,\sigma} \leq C\tilde{\varepsilon},$$

for all $i = 1, 2$, we obtain that there exist $\tilde{\varepsilon}_0 > 0$ and $C > 0$ such that for any $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$, (2.32) has a unique G_k-invariant solution $\tilde{\phi} = \tilde{\phi}_{\tilde{\varepsilon}}$ with $\|\tilde{\phi}_{\tilde{\varepsilon}}\|_{\star,\sigma} \leq C\tilde{\varepsilon}$ and $\|D_\Lambda \tilde{\phi}_{\tilde{\varepsilon}}\|_{\star,\sigma} \leq C\tilde{\varepsilon}$. Now we let

$$\hat{\mathcal{U}}_\varepsilon = \tilde{\mathcal{W}}_\varepsilon + \mathcal{L}_{\tilde{\varepsilon}}(R_{\tilde{\varepsilon}}) + \tilde{\phi}_{\tilde{\varepsilon}}. \tag{2.33}$$

There holds that $\|\mathcal{L}_{\tilde{\varepsilon}}(R_{\tilde{\varepsilon}})\|_{\star,\sigma} \leq C\tilde{\varepsilon}$. Thus $\hat{\mathcal{U}}_\varepsilon > 0$. We define $\rho : \mathbb{R}^+ \to \mathbb{R}$ by

$$\begin{aligned} \rho(\Lambda) = &\frac{1}{2}\int_{\frac{1}{\tilde{\varepsilon}}S^3} |\nabla \hat{\mathcal{U}}_\varepsilon|^2 dv_{g_{\tilde{\varepsilon}}} + \frac{(m_0^2 - \omega^2)\tilde{\varepsilon}^2}{2}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \hat{\mathcal{U}}_\varepsilon^2 dv_{g_{\tilde{\varepsilon}}} \\ &+ \frac{q\omega^2\tilde{\varepsilon}^2}{2}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \overline{\Phi(\mathcal{U}_\varepsilon)}\hat{\mathcal{U}}_\varepsilon^2 dv_{g_{\tilde{\varepsilon}}} - \frac{1}{6}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \hat{\mathcal{U}}_\varepsilon^6 dv_{g_{\tilde{\varepsilon}}}, \end{aligned} \tag{2.34}$$

where $\mathcal{U}_\varepsilon$ is such that $\tilde{\mathcal{U}}_\varepsilon = \hat{\mathcal{U}}_\varepsilon$, namely such that $\hat{\mathcal{U}}_\varepsilon$ is obtained from $\mathcal{U}_\varepsilon$ by the $\sim$-procedure. In other words, $\mathcal{U}_\varepsilon = \mathcal{W}_\varepsilon + K_\varepsilon + \phi_{\tilde{\varepsilon}}$. The following holds true.

Lemma 2.6. *The function* $\mathcal{U}_\varepsilon > 0$ *is a solution of*

$$\Delta_g U + (m_0^2 - \omega^2)U + q\Psi(U)U = U^5 \tag{2.35}$$

in S^3 *if and only if* Λ *is a critical point of* ρ.

Proof of Lemma 2.6. We define $I_{\tilde{\varepsilon}}$ by

$$\begin{aligned} I_{\tilde{\varepsilon}}(\tilde{U}) = &\frac{1}{2}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \left(|\nabla \tilde{U}|^2 + (m_0^2 - \omega^2)\tilde{\varepsilon}^2\tilde{U}^2\right) dv_{g_{\tilde{\varepsilon}}} + \frac{q\omega^2\tilde{\varepsilon}^2}{4}\int_{\frac{1}{\tilde{\varepsilon}}S^3} \overline{\Phi(U)}\tilde{U}^2 dv_{g_{\tilde{\varepsilon}}} \\ &- \frac{1}{6}\int_{\frac{1}{\tilde{\varepsilon}}S^3} (\tilde{U}^+)^6 dv_{g_{\tilde{\varepsilon}}}. \end{aligned}$$

Then $I_{\tilde{\varepsilon}}(\tilde{U}) = I(U)$, where I is as in (2.3), and there holds that $\mathcal{U}_\varepsilon$ is a solution of (2.35) if and only if $\hat{\mathcal{U}}_\varepsilon$ is a solution of

$$\Delta_{g_{\tilde{\varepsilon}}}\tilde{\mathcal{U}} + \tilde{\varepsilon}^2(m_0^2 - \omega^2)\tilde{\mathcal{U}} + q\tilde{\varepsilon}^2\overline{\Psi(\mathcal{U})}\tilde{\mathcal{U}} = \tilde{\mathcal{U}}^5.$$

This is in turn equivalent to $c_0 = 0$, where c_0 is as in (2.32), which is again equivalent to $I'_{\tilde{\varepsilon}}(\hat{\mathcal{U}}_\varepsilon).(\tilde{Y}) = 0$ since $I'_{\tilde{\varepsilon}}(\hat{\mathcal{U}}_\varepsilon).(\tilde{Y}) = c_0\langle \tilde{Y}, \tilde{Z}\rangle$ and $\langle \tilde{Y}, \tilde{Z}\rangle = \gamma_0 + o(1)$, where $\gamma_0 > 0$. Independently, there holds that $\rho'(\Lambda) = 0$, if and only if,

$$I'_{\tilde{\varepsilon}}\left(\hat{\mathcal{U}}_\varepsilon\right) . \left(\tilde{Y} + \frac{\partial \Psi_\varepsilon}{\partial \Lambda}\right) = 0,$$

where $\Psi_\varepsilon = \tilde{K}_\varepsilon + \tilde{\phi}_{\tilde{\varepsilon}}$, while if we let $y_0 = \frac{\partial \Psi_\varepsilon}{\partial \Lambda}$, then $\|y_0\|_{\star,\sigma} \leq C\varepsilon$. We write that $y_0 = y_0' + a\tilde{Y}$, where $(y_0', \tilde{Y})_{\tilde{\varepsilon}} = 0$ and $(\cdot,\cdot)_{\tilde{\varepsilon}}$ is the scalar product associated to $\Delta_{g_{\tilde{\varepsilon}}} + \tilde{\varepsilon}^2(m_0^2 - \omega^2)$. Then $\rho'(\Lambda) = 0$ if and only if $(1 + a)I'_{\tilde{\varepsilon}}(\hat{\mathcal{U}}_\varepsilon).(\tilde{Y}) = 0$ since $\langle y_0', \tilde{Z}\rangle = (y_0', \tilde{Y})_{\tilde{\varepsilon}}$. There holds that $(y_0, \tilde{Y})_{\tilde{\varepsilon}} = o(1)$ and this implies that $a = o(1)$. Hence $\rho'(\Lambda) = 0$ if and only if $I'_{\tilde{\varepsilon}}(\hat{\mathcal{U}}_\varepsilon).(\tilde{Y}) = 0$, and thus, if and only if, $\mathcal{U}_\varepsilon$ solves (2.35). This ends the proof of the lemma. □

Now we are in position to prove our theorem.

Proof of the theorem. We compute $\rho(\Lambda) = I(\mathcal{W}_\varepsilon) + o(\varepsilon^2)$. Assume now that $\Theta_k > 0$, where Θ_k is as in (2.21). Then, by Lemma 2.4,

$$\rho(\Lambda) = A_{0,k} + A_{1,k}\tilde{\varepsilon}^2\Lambda + \Theta_k\tilde{\varepsilon}^2\Lambda^2 + o(\tilde{\varepsilon}^2)\Lambda^2$$

and since $A_{1,k} < 0$ and $\Theta_k > 0$, ρ has an absolute minimum Λ_θ in $(\frac{1}{C}, C)$ for $C \gg 1$ when $\theta \in (\theta_k - \delta, \theta_k)$ and $0 < \delta \ll 1$. Let $\omega_k \in (-m_0, m_0)$ be given by $\theta_k^2 = m_0^2 - \omega_k^2$. Pick any sequence $(\omega_\alpha)_\alpha$ of phases such that $\omega_\alpha \to \omega_k$ as $\alpha \to +\infty$ and $\theta_\alpha \le \theta_k$ for all α, where $\theta_\alpha > 0$ is given by $\theta_\alpha^2 = m_0^2 - \omega_\alpha^2$. By Lemma 2.6 we then obtain that there is an associated sequence $(\mathcal{U}_\alpha, \Phi(\mathcal{U}_\alpha))$ of solutions of (0.2) with $\omega = \omega_\alpha$, where $\mathcal{U}_\alpha = \mathcal{U}_{\varepsilon_\alpha}$ and $\varepsilon_\alpha = \Lambda_{\omega_\alpha}\eta_k(\theta_\alpha)$, such that $(\mathcal{U}_\alpha)_\alpha$ is a k-spikes type solution of the first equation in (0.2). In particular, $\|\mathcal{U}_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. Similarly, if we assume that $\Theta_k < 0$, then by Lemma 2.4,

$$\rho(\Lambda) = A_{0,k} - A_{1,k}\tilde{\varepsilon}^2\Lambda + \Theta_k\tilde{\varepsilon}^2\Lambda^2 + o(\tilde{\varepsilon}^2)\Lambda^2$$

and ρ has an absolute maximum in $(\frac{1}{C}, C)$ for $C \gg 1$ when $\theta \in (\theta_k, \theta_k + \delta)$ and $0 < \delta \ll 1$. Here again let $\omega_k \in (-m_0, m_0)$ be given by $\theta_k^2 = m_0^2 - \omega_k^2$. Pick any sequence $(\omega_\alpha)_\alpha$ of phases such that $\omega_\alpha \to \omega_k$ as $\alpha \to +\infty$ and $\theta_\alpha \ge \theta_k$ for all α, where $\theta_\alpha > 0$ is given by $\theta_\alpha^2 = m_0^2 - \omega_\alpha^2$. By Lemma 2.6 we then obtain that there is an associated sequence $(\mathcal{U}_\alpha, \Phi(\mathcal{U}_\alpha))$ of solutions of (0.2) with $\omega = \omega_\alpha$, where $\mathcal{U}_\alpha = \mathcal{U}_{\varepsilon_\alpha}$ and $\varepsilon_\alpha = -\Lambda_{\omega_\alpha}\eta_k(\omega_\alpha)$, such that $(\mathcal{U}_\alpha)_\alpha$ is a k-spikes type solution of the first equation in (0.2). In particular, $\|\mathcal{U}_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. Let $A_{2,k}(\omega) = \omega^2 B_{2,k}(\theta)$. Then

$$B_{2,k}(\theta) = \frac{1}{2}\int_{S^3}\left(|\nabla\Phi_{k,\theta}|^2 + m_1^2\Phi_{k,\theta}^2\right)dv_g,$$

where $\Phi_{k,\theta}$ is as in Lemma 2.3, and there holds that

$$\Theta_k = B_{2,k}(\theta_k)\left(q^2\omega_k^2 - \frac{|A_{3,k}(\theta_k)|}{B_{2,k}(\theta_k)}\right).$$

Letting $c_k(m_1) = |A_{3,k}(\theta_k)|B_{2,k}(\theta_k)^{-1}$, we obtain that $c_k(m_1)$ depends only on k and m_1, that $c_1(m_1) = 0$, that $c_k(m_1) > 0$ for $k \ge 2$, and that $c_k(m_1) \to +\infty$ as $k \to +\infty$. Obviously, $\Theta_k \ne 0$ when $q^2\omega_k^2 \ne c_k(m_1)$. This ends the proof of the theorem. $\square$

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