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ON THE $C^{2,\alpha}$-REGULARITY OF THE COMPLEX MONGE–AMPÈRE EQUATION

Yu Wang

Abstract. We prove the $C^{2,\alpha}$-regularity of the solution u to the equation

$$\det(u_{\bar{k}j}) = f, \quad f^{1/n} \in C^\alpha, \quad f^{1/n} \geq \lambda,$$

under the assumption that Δu is bounded from above. Our result settles one of the regularity problems left open in the paper [12] (see also [13]). The proof is based on a reduction of the complex Monge–Ampère equation to a Bellman-type equation, to which the regularity theory of fully nonlinear uniformly elliptic equations can be applied.

1. Introduction

Schauder estimates for solutions to fully nonlinear elliptic equations with C^α right-hand side are of considerable interest in both geometry and partial differential equations theory (PDEs). In his classic paper [6], Caffarelli established sharp interior $C^{2,\alpha}$-estimates for the solutions to the real Monge–Ampère equation $\det(u_{ij}) = f$ for $f \in C^\alpha$. Sharp boundary estimates were subsequently obtained by Trudinger and Wang [15]. However, neither type of estimates is available at the present time for the complex Monge–Ampère equation $\det(u_{\bar{k}j}) = f$ for $f \in C^\alpha$. Some important partial results are the interior $C^{2,\alpha}$-estimates of Evans and Krylov [8, 11] (see also Siu [14]) when f is of class C^2 and of Blocki [2] when f is Lipschitz. The difficulty with $f \in C^\alpha$ resides in the fact that we cannot differentiate the equation. In particular, the well-known techniques of Yau [18] cannot be applied.

In this note, we prove

Theorem 1.1. *Let $u \in C^2(B_1)$ be a plurisubharmonic function that solves the equation*

$$\det(u_{\bar{k}j}) = f \quad \text{in } B_1. \tag{1.1}$$

Suppose that for some $0 < \alpha < 1$, $f^{1/n} \in C^\alpha(\overline{B}_1)$ and

$$\sup_{B_1} \Delta u \leq \Lambda, \quad \inf_{B_1} f^{1/n} \geq \lambda > 0. \tag{1.2}$$

Then, there exists a constant $\beta \in (0, \alpha)$ depending only on α, λ, Λ and n such that $u \in C^{2,\beta}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\beta}(\overline{B}_{1/2})} \leq C,$$

where C depends only on $n, \lambda, \Lambda, \alpha, \|f\|_{C^\alpha(\overline{B}_1)}$ and $\|u\|_{L^\infty(B_1)}$.

Combining a recent result of Dinew et al. [13], one can improve the Hölder exponent of D^2u to that of $f^{1/n}$. Precisely speaking,

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Theorem 1.2. *Let $u \in C^2(B_1)$ be a plurisubharmonic function that solves equation* (1.1). *Suppose that for some $0 < \alpha < 1$, $f^{1/n} \in C^\alpha(\overline{B}_1)$ and*

$$\sup_{B_1} \Delta u \leq \Lambda, \quad \inf_{B_1} f^{1/n} \geq \lambda > 0. \tag{1.3}$$

Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C,$$

where C depends only on $n, \lambda, \Lambda, \alpha, \|f\|_{C^\alpha(\overline{B}_1)}$ and $\|u\|_{L^\infty(B_1)}$.

The above theorems provides one of the key estimates left open in the paper of Chen and Tian [12]. In that work, under certain geometric conditions, a Kähler matric with Hölder potential u has been constructed and Δu has been proved to be bounded in L^∞ (see page 98–100 of [12]). Theorem 1.2 then can be applied to conclude that u is $C^{2,\alpha}$.

Before the present work, several authors had considered this problem. The best previous result was obtained recently by Dinew et al. [13]. Following the techniques developed by Trudinger and Wang [15, 16], they proved the $C^{2,\alpha}$-estimate with constants depending on the norm $\|D^2u\|_{L^\infty}$ instead of the norm $\|\Delta u\|_{L^\infty}$. The dependence on $\|D^2u\|_{L^\infty}$ instead of $\|\Delta u\|_{L^\infty}$ is a significant restriction, since $\|\Delta u\|_{L^\infty}$ can often be controlled by the techniques of Yau's C^2-estimates [18], while it is not the case for $\|D^2u\|_{L^\infty}$.

Our approach to the problem is rather different from the method employed in [13] and many other earlier works in the study of complex Monge–Ampère equations. Indeed, we shall convert the complex Monge–Ampère equation to a fully nonlinear elliptic Bellman-type equation with respect to the real Hessian of u.

By the standard nonlinear elliptic theory (see Section 2), Theorem 1.1 is a direct consequence of the following theorem.

Theorem 1.3. *Let $u \in C^2(B_1)$ be a plurisubharmonic function that solves equation* (1.1). *Denote by* $\mathrm{Sym}(2n)$ *the space of $2n \times 2n$ real symmetric matrices and by $\|M\|$ the standard spectral norm of the matrix M. Suppose that*

$$\inf_{B_1} f^{1/n} \geq \lambda > 0 \quad \textit{and} \quad \sup_{B_1} \Delta u \leq \Lambda. \tag{1.4}$$

Then there exists a concave function $\widetilde{F}$ on $\mathrm{Sym}(2n)$ *such that*

(i) *$\widetilde{F}$ is θ-uniformly elliptic, i.e.,*

$$\theta\|P\| \leq \widetilde{F}(M+P) - \widetilde{F}(M) \leq \theta^{-1}\|P\|, \quad \forall\, M, P \in \mathrm{Sym}(2n) \textit{ and } P \geq 0,$$

where θ only depends on λ, Λ and n.

(ii) *u satisfies the equation*

$$\widetilde{F}(D^2u) = f^{1/n} \quad \textit{in } B_1.$$

Remark 1.4. We have stated Theorems 1.1–1.3 as a priori estimates, that is, we have assumed that u is a C^2-solution of equation (1.1). In fact, as pointed by a referee of this paper, it suffices to assume that u is a weak solution to the equation (viscosity solution or pluripotential solution, see [9, 17] for the definition of viscosity solution to the complex Monge–Ampère) and $\|\Delta u\|_{L^\infty} < \Lambda$. Δu being bounded in L^∞ is sufficient to conclude that u is a viscosity solution of the uniform elliptic equation given in Theorem 1.3. The arguments of this paper can be carried through without much changes if one only assumes that u is a weak solution. In order to illustrate the idea in a more transparent fashion, we shall remain in the case that u is C^2.

The idea of this note was suggested to the author by Professor Ovidiu Savin. After the note had been posted on the ArXiv, the author learned from Professor Pengfei Guan that the technique of reducing a complex Monge–Ampère equation to a Bellman-type equation had been used in the late 1980s by Krylov [11] in order to obtain C^2 solutions to the Dirichlet problem for the homogeneous complex Monge–Ampère equation with C^3 boundary data. Theorem 1.2 is pointed out by a referee of this paper.

Besides establishing Theorem 1.1, another important purpose of this note is to provide a different point of view on complex Monge–Ampère equations. Up until now, major studies of complex Monge–Ampère equations have been based on a priori estimates [4, 18] or pluripotential theory and complex analysis (e.g., [1, 10] and references therein). On the other hand, our approach here mainly focuses on the nonlinear PDE structure of the equation. Each approach has its own advantage. We hope that the approach employed in this note will give further insight on the study of complex Monge–Ampère equations.

2. Preliminaries

In this section, we recall two important theorems from regularity theory of fully nonlinear uniformly elliptic equations. For more details, one may refer to [3].

First, we recall a result of Evans (see Section 6 of [3]).

Theorem 2.1. *[3, Theorem 6.2]. Let F be a concave θ-uniformly elliptic function on* Sym(n). *If u is a viscosity solution to $F(D^2u) = 0$ in B_1, then $u \in C^{2,\bar\beta}(\overline{B}_{1/2})$ and*

$$\|u\|_{C^{2,\bar\beta}(\overline{B}_{1/2})} \le C\left\{\|u\|_{L^\infty} + F(0)\right\},$$

where $\bar\beta$ and C are constants only depending on θ and n.

Next, we recall a result of Caffarelli (see [5] or Section 8 of [3]). Consider an equation of the form

$$F(D^2u, x) = f(x) \tag{2.1}$$

and denote

$$\omega_F(x) := \sup_{M\in \mathrm{Sym}(n)} \frac{|F(M,x) - F(M,0)|}{\|M\| + 1}.$$

Theorem 2.2. *[3, Theorem 8.1]. Let u be a viscosity solution to equation* (2.1) *in $B_1(0)$. Assume that F and f are continuous in x, F is θ-uniformly elliptic and $F(0,0) = f(0) = 0$. Suppose that the following hypotheses hold:*

Hypothesis 1: *there are constants* $0 < \bar{\beta} < 1$ *and* $c_\epsilon > 0$ *such that for every symmetric matrix* M *with* $F(M, 0) = 0$ *and and* $w_0 \in C(\partial B_1)$, *there exists a* $w \in C^2(B_1) \cap C(\overline{B}_1) \cap C^{2,\bar{\beta}(B_{1/2})}$ *which satisfies*

$$\begin{cases} F(D^2 w(x) + M, 0) = 0 & \text{in } B_1, \\ w = w_0 & \text{on } \partial B_1 \end{cases}$$

and

$$\|w\|_{C^{,\bar{\beta}}(\overline{B}_{3/4})} \le c_\epsilon \|w\|_{L^\infty(B_1)}. \tag{2.2}$$

Hypothesis 2: F *and* f *satisfy*

$$\left(⨍_{B_r} \omega_F^n \right)^{1/n} \le C_1 r_0^{-\beta} r^\beta, \quad \forall r \le r_0 \tag{2.3}$$

and

$$\left(⨍_{B_r} |f|^n \right)^{1/n} \le C_2 r_0^{-\beta} r^\beta, \quad \forall r \le r_0, \tag{2.4}$$

for some $0 < \beta < \bar{\beta}, r_0 > 0, C_1 > 0, C_2 > 0.$

Then u *is* $C^{2,\beta}$ *at origin; that is, there is a polynomial* P *of degree* 2 *such that*

$$\|u - P\|_{L^\infty(B_r(0))} \le C_3 r_0^{-(2+\beta)} r^{2+\beta}, \quad \forall r \le r_1, \tag{2.5}$$

$$r_0 \, |DP(0)| + r_0^2 \|D^2 P\| \le C_3, \tag{2.6}$$

$$C_3 \le C_0(\|u\|_{L^\infty(B_{r_0}(0))} + r_0^2(C_2 + 1)), \tag{2.7}$$

and

$$r_1 = C_0^{-1} r_0, \tag{2.8}$$

where $C_0 > 1$ *depends only on* $n, \theta, c_\epsilon, \bar{\beta}, \beta$ *and* C_1.

The above theorem is the nonlinear version of the classical Schauder estimate. Theorem 2.2 states that: given a fully nonlinear uniformly elliptic equation $F(D^2u, x) = f(x)$ with continuous coefficients, by freezing the coefficients of the equation, one obtains fully nonlinear equation of the form $F(D^2u, x_0) = f(x_0)$. If one has the $C^{2,\overline{\beta}}$-estimates for the equation $F(D^2u, x_0) = f(x_0)$, then under mild restriction one can obtain corresponding estimates for $F(D^2u, x) = f(x)$.

For our discussion, the following corollary is sufficient.

Corollary 2.3. *Let* F *be a concave and* θ*-uniformly elliptic function on* $\mathrm{Sym}(n)$ *and* $f \in C^\alpha(\overline{B}_1)$. *If* $u \in C^2(B_1)$ *satisfies* $F(D^2u) = f(x)$ *in* B_1, *then there is a constant* $\beta \in (0, \alpha)$ *depending only on* n, θ, α *such that* $u \in C^{2,\beta}(\overline{B}_{1/2})$ *and*

$$\|u\|_{C^{2,\beta}(\overline{B}_{1/2})} \le C,$$

where C *depends only on* $n, \theta, \alpha, |F(0)|, \|f\|_{C^\alpha(\overline{B}_1)}$ *and* $\|u\|_{L^\infty(B_1)}$.

Proof. Without lose of generality, we may assume $F(0) = f(0) = 0$. Since otherwise, we may replace F and f by

$$G(M) := F(M + t_0 I) - f(0), \quad \text{and} \quad g := f - f(0),$$

where I is the $n \times n$ identity matrix and $t_0 \in \mathbb{R}$ is chosen so that $G(0) = 0$. Then the function $v := u - t\,|x|^2/2$ satisfies

$$G(D^2 v) = g \quad \text{in } B_1.$$

By uniform ellipticity, we have

$$|t| \le |F(0) - f(0)|/\theta.$$

Thus, it suffices to estimate v.

Since F is uniformly elliptic, the existence of viscosity solution to the equation $F(D^2 w + M) = 0$ is given by Perron's method (see [7]). Since F is concave, Theorem 2.1 implies that every solution w of $F(D^2 w + M) = 0$ satisfies the estimate (2.2) with $\bar{\beta}$ and c_ϵ depending only on n, θ. Thus, Hypothesis 1 of Theorem 2.2 is satisfied.

Since F does not depend on x, (2.3) of Hypothesis 2 is automatically satisfied. Since $f(0) = 0$ and f is Hölder, (2.4) of Hypothesis 2 is satisfied with $C_2 = \|f\|_{C^\alpha(\overline{B}_1)}/n$, $r_0 = 1/2$ and $\beta = \min\{\overline{\beta}/2, \alpha\}$.

Thus, we may apply Theorem 2.2 to conclude that the solution u is $C^{2,\beta}$ at 0 with bounds given by (2.5)–(2.8). By a translation of coordinates, we can conclude that u is $C^{2,\alpha}$ at every $x \in \overline{B}_{1/2}$ with bounds given by (2.5)–(2.8).

By a standard covering argument, one can show that a function u that is $C^{2,\alpha}$ at all points $x \in \overline{B}_{1/2}$ with bounds given by (2.5)–(2.8) belongs to $C^{2,\alpha}(\overline{B}_{1/2})$ and

$$\|u\|_{C^{2,\beta}\overline{B}_{1/2}} \le C,$$

where C depends on θ, C_0, C_2, r_0 (see Remark 3 on page 74 of [3]).

Recalling that in our case, $r_0 = 1/2$, $C_2 = \|f\|_{C^\alpha(\overline{B}_1)}/n$, we complete the proof. □

Theorem 1.1 follows immediately from Corollary 2.3 and Theorem 1.3.

3. The proof of Theorem 1.3

Let $\mathrm{Sym}(2n)$ be the space of $2n \times 2n$ real symmetric matrices and $\mathrm{Herm}(n)$ be the space of $n \times n$ complex Hermitian matrices.

On $\mathbb{R}^{2n}$ we fix the following canonical complex structure:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad I_n \text{ is the } n \times n \text{ identity matrix.}$$

Then $\mathrm{Herm}(n)$ can be identified with the subspace

$$\{M \in \mathrm{Sym}(2n) | [M, J] = MJ - JM = 0\},$$

by the map

$$\imath : H = A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

In the rest of this note, we always view $\mathrm{Herm}(n)$ as a subspace of $\mathrm{Sym}(2n)$ through $\imath$. There is also a canonical projection

$$\mathrm{p} : \mathrm{Sym}(2n) \to \mathrm{Herm}(n), \quad M \mapsto \frac{M + J^t M J}{2}.$$

The following diagram summarizes the above relations:

$$\begin{array}{ccc} \mathrm{Herm}(n) & \xrightarrow{\quad\imath\quad} & \mathrm{Sym}(2n) \\ & \overset{\sim}{\searrow} & \big\downarrow \mathrm{p} \\ & & \{M \in \mathrm{Sym}(2n) : [M, J] = MJ - JM = 0\} \end{array}$$

The complex determinant $\det_{\mathbb{C}}$ on $\mathrm{Herm}(n)$ is related to the real determinant $\det_{\mathbb{R}}$ by

$$\det_{\mathbb{R}}^{1/2n}[M] = \det_{\mathbb{C}}^{1/n}[H] \quad \text{if } \imath(H) = M, \quad H \in \mathrm{Herm}(n), \quad M \in \mathrm{Sym}(2n).$$

Let $F : \mathrm{Sym}(2n) \to \mathbb{R}$ be defined by

$$F(M) := \det_{\mathbb{R}}^{1/2n}[\mathrm{p}(M)]. \tag{3.1}$$

By the Minkowski inequality, F is a concave function on the set

$$\{M \in \mathrm{Sym}(2n) : \mathrm{p}(M) > 0\}.$$

Now we give the construction of $\widetilde{F}$.

Definition 3.1. Given $\theta \in (0, 1]$, let $\mathcal{E}_\theta \subset \mathrm{Sym}(2n)$ consist of matrices N such that

$$\theta I_{2n} \leq \mathrm{p}(N) \leq \theta^{-1} I_{2n}.$$

The function $\widetilde{F} : \mathrm{Sym}(2n) \to \mathbb{R}$ is defined as follows: for $M \in \mathrm{Sym}(2n)$,

$$\widetilde{F}(M) := \inf \left\{ \frac{1}{2\mathrm{n}} \mathrm{tr}[\mathrm{p}(N)M] + c \mid N \in \mathcal{E}_\theta, c \in \mathbb{R} \text{ s.t.} \right.$$
$$\left. \frac{1}{2\mathrm{n}} \mathrm{tr}[\mathrm{p}(N)X] + c \geq F(X) \text{ for all } X \in \mathcal{E}_\theta \right\}.$$

Remark 3.2. $\widetilde{F}$ is the concave envelope of F over the set $\mathcal{E}_\theta$. Moreover, $\widetilde{F}$ has the same invariant property as the complex determinant, i.e.,

$$\widetilde{F}(M + Z) = \widetilde{F}(M) \quad \text{for all } Z \in \mathrm{Sym}(n) \text{ s.t. } \mathrm{p}(Z) = 0.$$

Remark 3.3. The above construction is suggested by Prof. Ovidiu Savin. The author's original approach is to extend the level sets of F outside $\mathcal{E}_\theta$. Although it gives essentially the same function as above, the construction in Definition.3.1 is more direct and more transparent.

The following lemma is the main ingredient in proving Theorem 1.3.

Lemma 3.4. *$\widetilde{F}$ is concave and uniformly elliptic in* $\mathrm{Sym}(2n)$, *i.e., there exists $\tilde{\theta} > 0$ only depending on θ and n such that*

$$\tilde{\theta}\|P\| \leq \tilde{F}(M + P) - \tilde{F}(M) \leq \tilde{\theta}^{-1}\|P\|, \quad \forall M,\ P \in \mathrm{Sym}(2n) \quad \text{and } P \geq 0. \tag{3.2}$$

Moreover, $\widetilde{F}(M) = F(M)$ for all $M \in \mathcal{E}_\theta$.

Proof. The concavity of $\widetilde{F}$ and the equality that $\widetilde{F}(M) = F(M)$ on $\mathcal{E}_\theta$ follow directly from the construction. We only need to check the ellipticity of $\widetilde{F}$. Given $M, P \in \mathrm{Sym}(2n), P \geq 0$, by definition, there exists $N_1, N_2 \in \mathcal{E}_\theta$ such that

$$\widetilde{F}(M + P) = \frac{1}{2\mathrm{n}} \mathrm{tr}[\mathrm{p}(N_1)(M + P)] + c_1$$

and

$$\widetilde{F}(M) = \frac{1}{2\text{n}}\text{tr}[\text{p}(N_2)(M)] + c_2.$$

By the minimality, we have

$$\frac{1}{2\text{n}}\text{tr}[\text{p}(N_1)M] + c_1 \geq \frac{1}{2\text{n}}\text{tr}[\text{p}(N_2)M] + c_2$$

and

$$\frac{1}{2\text{n}}\text{tr}[\text{p}(N_1)(M+P)] + c_1 \leq \frac{1}{2\text{n}}\text{tr}[\text{p}(N_2)(M+P)] + c_2.$$

Then combine the above inequalities, we have

$$\widetilde{F}(M+P) - \widetilde{F}(M) \geq \frac{1}{2\text{n}}\text{tr}[\text{p}(N_1)(M+P)] - \frac{1}{2\text{n}}\text{tr}[\text{p}(N_1)(M)] \geq \frac{\theta}{2n}\|P\|$$

and

$$\widetilde{F}(M+P) - \widetilde{F}(M) \leq \frac{1}{2\text{n}}\text{tr}[\text{p}(N_2)(M+P)] - \frac{1}{2\text{n}}\text{tr}[\text{p}(N_2)M] \leq \theta^{-1}\|P\|.$$

This completes the proof of the lemma. □

Proof of Theorem 1.3. Since $u \in C^2(B_1)$ satisfies (1.4), we have

$$\frac{\lambda^n}{\Lambda^{n-1}} I \leq \text{p}(D^2u)(x) \leq \Lambda I, \quad \forall\, x \in B_1.$$

In turn, by taking $\theta = \min\{\lambda^n/\Lambda^{n-1}, \Lambda^{-1}, 1\}$, we have

$$D^2u(x) \in \mathcal{E}_\theta, \quad \forall\, x \in B_1. \tag{3.3}$$

Now consider $\widetilde{F}$ given by Definition 3.1 with respect to $\mathcal{E}_\theta$. By (3.1), (3.3) and Lemma 3.4

$$\widetilde{F}(D^2u(x)) = F(D^2u(x)) = f^{1/n}(x), \quad x \in B_{1/2}.$$

The uniform ellipticity and the concavity of $\widetilde{F}$ have been proved in Lemma 3.4. This completes the proof of Theorem 1.3. □

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Department of Mathematics, Columbia University, Room 509, MC 4406, 2990 Broadway, New York, NY 10027, USA

E-mail address: `yuwang@math.columbia.edu`