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HEIGHT AND GIT WEIGHT

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Abstract. In this paper, we first establish a connection between the weight in the geometric invariant theory and the height introduced by Cornalba and Harris [CH] and Zhang [Z]. Then we give two applications. First, it provides a converse of Cornalba and Harris's result, which can be treated as a function field analog of Zhang's theorem over number field. In particular, this connection gives a numerical interpretation of the moral *stability = positivity* that was advocated by Viehweg [Vi]. Second, we relate these to the study the positivity of CM-line bundle introduced by Tian [PT] and the determinant line bundle introduced by Donaldson [Do0].

1. Introduction

In 1987, Harris and Cornalba [CH] proved the following result:

Theorem. *Let $\mathcal{X}$ be separated scheme of dimension $n+1$ and B be a smooth projective curve. Let*

$$\pi : \mathcal{X} \longrightarrow B$$

be a flat proper morphism. Let $\mathcal{L} \to \mathcal{X}$ be a line bundle such that $\mathcal{E} := \pi_*\mathcal{L}$ is locally free of rank $N+1$. Suppose that the following conditions are satisfied: (1) If $b \in B$ be generic point, then $\mathcal{E}|_{b\in B} \subset H^0(\mathcal{X}_b, \mathcal{L}|_{\mathcal{X}_b})$ is base point free, very ample and yields a semi-stable embedding of $\mathcal{X}_b$; (2) $\mathcal{L}$ is relatively ample. Then

$$(N+1)\pi_* c_1(\mathcal{L})^{n+1} - (n+1)\deg \mathcal{X}_b c_1(\mathcal{E}) \geq 0, \tag{1.1}$$

that is, it is semi-positive.

A natural question is that if the positivity of (1.1) is also sufficient to guarantee the semi-stability. More precisely, whether or not the positivity of any flat family (e.g., coming from a test configuration introduced by Donaldson [Do0]) containing X as the general fiber would guarantee the semi-stability of the corresponding Hilbert point. For arithmetic varieties over a number field, Zhang has proved a converse to the Cornalba and Harris's result in [Z] using Arakelov intersection theory. Our first main result is a converse to the above theorem, which can also be viewed as a function field analogue of Zhang's result.

Theorem (Corollary 14). *Let $(X, \mathcal{O}_X(1))$ be a n-dimensional projective manifold polarized by a very ample line bundle $O_X(1)$. Suppose that for any flat proper family $\pi : (\mathcal{X}, \mathcal{L}) \to B$ over a smooth curve B containing $(X, \mathcal{O}_X(1))$ as a general fiber, that is $(\mathcal{X}_b, \mathcal{L}|_{\mathcal{X}_b}) \cong (X, \mathcal{O}_X(1))$ for all $b \in B\backslash S$, where $S \subset B$ is a set of finite number of*

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points and $\mathcal{L} \to \mathcal{X}$ is a relatively very ample line bundle with $\mathcal{E} := \pi_*\mathcal{L}$ being locally free of rank $N+1$, we always have

$$\deg \pi_* \left\{ ((N+1)c_1(\mathcal{L}) - \pi^* c_1(\mathcal{E}))^{n+1} \cap [\mathcal{X}] \right\} \geq 0.$$

Then X is Chow semi-stable with respect to the polarization L.

The rationale behind this Theorem is the localization formula for equivariant cohomology, which equates the intersection number to the weight. The simplest toy model is the following:

Example 1. Let $\mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1$ be the hyperplane bundle and $\omega \in H^2(\mathbb{P}^1, \mathbb{Z})$ be the first Chern class of $\mathcal{O}_{\mathbb{P}^1}(1)$. Consider a $\mathbb{C}^\times$-action on $\mathbb{C}^2$ given by $t \cdot [z_0, z_1] = [t^\lambda z_0, t^\mu z_1]$, this induces an action on total space of $\mathcal{O}_{\mathbb{P}^1}(1)$ covering a $\mathbb{C}^\times$-action on $\mathbb{P}^1$. Then the localization formula reads

$$\int_{\mathbb{P}^1} \omega = \frac{\mu}{\mu - \lambda} - \frac{\lambda}{\mu - \lambda},$$

where the LHS is the intersection number and RHS are weights contribution from $0, \infty \in \mathbb{P}^1$. In particular, if $\lambda = 0$ and $\mu = 1$ then the weight μ is exactly the intersection number $[\omega] \cdot [\mathbb{P}^1]$.

Now let us turn to the second motivation of this paper, let $\mathcal{X} \to B$ be a flat family as introduced above, Paul and Tian [PT] introduced the CM-line bundle $\Lambda^{CM}(\mathcal{X}) \to B$ (cf. Section 3) for a proper flat family in their study of K-stability of polarized manifold. The notion K-stability of a polarized manifold, first introduced by Tian [T] and generalized by Donaldson in [Do0] is conjecturally to be a necessary and sufficient condition to the existence of the constant scalar curvature Kähler (cscK) metric [Do0]. This was originally motivated by Yau's [SY] conjecture that the existence of cscK metric should be related to certain geometric invariant theory (GIT) stability. In [PT], Paul and Tian have shown that K-stability can be interpreted as GIT stability provided the "polarization" over the Hilbert scheme was CM-line. However, this is NOT a real polarization in the sense that the ampleness of CM-line bundle is not guaranteed. So it has been a major concern to determine the positive locus of the CM-line inside the Hilbert scheme. Fine and Ross made the first step toward this problem [FR], they found that CM-line is semi-positive over the asymptotic Hilbert semi-stable locus, and at the same time they were able to construct examples such that the CM-line fails to be non-negative for some Hilbert unstable family. A consequence of the our theorem above is the following (cf. Section 3).

Proposition 17. *Suppose $(X, \mathcal{O}_X(1))$ is K-unstable (cf.* [Do0] *and* [PT]*), then there is a flat family $(\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1$ containing X as a general fiber such that CM-line bundle $\Lambda^{CM}(\mathcal{X}) \to \mathbb{P}^1$ is negative.*

Now we outline the organization of the paper. In Section 2, we will associate to any family GIT problem with a line bundle called height over the base and prove its positivity under the assumption of semi-stability. Moreover, we will show that there is a natural proper family such that the height is identified with the GIT weight (Theorem 7). This is the heart of the paper. In the following two sections, applications of the theory developed in Section 2 are presented. In Section 3, we apply Section 2 to Chow scheme, and in particular, we are able to extend the result of [FR] by given a more precise description of the positivity of CM-line for general families. In Section 4, we apply the theory to the study of vector bundles and establish a connection between height and Donaldson's determinant line bundle introduced in [Do1].

2. GIT height

In this section, we will first introduce a notion called height for a family of GIT problem. Then we establish a precise relationship between the height and the weight in the sense of GIT.

First, let us review some basics of GIT. Let $(Z, \mathcal{O}_Z(1))$ be a projective manifold with an action of a reductive algebraic group G, and the polarization $\mathcal{O}_Z(1)$ is a G-linearized ample line bundle on Z, that is, there is a G-action on the total space of $\mathcal{O}_Z(1)$ that covers its action on Z.

Definition 2. Let $z \in Z$ and $\lambda : \mathbb{C}^\times \to G$ be a one parameter subgroup. Let

$$z_0 := \lim_{t \to 0} \lambda(t) \cdot z.$$

Then the λ-**weight** of z, which will be denoted by $w_z(\lambda) \in \mathbb{Z}$, is the weight of $\mathbb{C}^\times$-action on $\mathcal{O}_Z(1)|_{z_0} \cong \mathbb{C}$. A point z is called **(semi-)stable** with respect to the G-linearization of $\mathcal{O}_Z(1)$ if $w_z(\lambda)(\geq) > 0$ for all one parameter subgroups $\lambda : \mathbb{C}^\times \to G$.

A **family GIT** problem over a smooth curve B consists of the following data:

- Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold with an action of a connected reductive algebraic group $\tilde{G}$, and $\mathcal{O}_Z(1)$ is **very ample** and also $\tilde{G}$-linearized. We assume the induced representation

$$\rho : \tilde{G} \longrightarrow GL(H^0(Z, \mathcal{O}_Z(1))) \tag{2.1}$$

  satisfying

$$\mathrm{Im}\rho \supset \mathbb{C}^\times \cdot I, \tag{2.2}$$

  where I is the identity in $GL(H^0(Z, \mathcal{O}_Z(1)))$. Let

$$G := \ker(\det \circ \rho), \tag{2.3}$$

  we study the GIT problem for the G-action on $(Z, \mathcal{O}_Z(1))$.
- Let

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & Fr^{\tilde{G}} \\ & \pi & \downarrow \\ & & B \end{array}$$

  be a principal $\tilde{G}$-bundle over B.

We introduce a locally trivial fibration associated to $Fr^{\tilde{G}}$

$$\pi_{\mathcal{Z}} : \mathcal{Z} := Fr^{\tilde{G}} \times_{\tilde{G}} Z \longrightarrow B.$$

By our assumption that $\mathcal{O}_Z(1)$ is very ample and $\tilde{G}$-linearized, we have

$$\mathcal{O}_{\mathcal{Z}}(1) := Fr^{\tilde{G}} \times_{\tilde{G}} \mathcal{O}_Z(1) \tag{2.4}$$

is π-**relative very ample.** And for a fixed $b \in B$, there is an isomorphism $\iota_b : (\mathcal{Z}_b = \pi_{\mathcal{Z}}^{-1}(b), \mathcal{O}_{\mathcal{Z}}(1)|_{\mathcal{Z}_b}) \cong (Z, \mathcal{O}_Z(1))$. Moreover, there is a bundle morphism

$$\begin{array}{ccccc} & \mathcal{Z} & \longrightarrow & \mathbb{P}\mathcal{E} & \\ \pi_{\mathcal{Z}} & \downarrow & & \downarrow & \pi \\ & B & \xrightarrow{id} & B & \end{array} .$$

with $\mathcal{E} = (\pi_{\mathcal{Z}})_* \mathcal{O}_{\mathcal{Z}}(1) = Fr^{\tilde{G}} \times_{\tilde{G}} H^0(Z, \mathcal{O}_Z(1))^\vee \to B$. Over $\mathcal{Z}$, there is a natural vector bundle $\pi_{\mathcal{Z}}^* \mathcal{E}^\vee(1) = \pi_{\mathcal{Z}}^* \mathcal{E}^\vee \otimes \mathcal{O}_{\mathcal{Z}}(1) = \mathcal{H}om(\pi_{\mathcal{Z}}^* \mathcal{E}, \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|_{\mathcal{Z}})$.

Definition 3. We define the **height** $h_{(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1))}$ for the family $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1))$ to be the determinant line bundle $\det \pi_{\mathcal{Z}}^* \mathcal{E}^\vee(1) \in Pic(\mathcal{Z})$, where $Pic(\mathcal{Z})$ is the Picard variety of $\mathcal{Z}$. For any cross-section s of the fibration $\pi : (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1)) \to B$, we define **height of** s to be

$$h_{(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1))}(s) := c_1(s^* \det \pi_{\mathcal{Z}}^* \mathcal{E}^\vee(1)) \in NS(B),$$

where $NS(B)$ is the Neron–Severi group of B. In particular,

$$h_{(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1))}(s)[B] = (N+1) s^* c_1(\mathcal{O}_{\mathcal{Z}}(1))[B] - \deg((\pi_{\mathcal{Z}})_* \mathcal{O}_{\mathcal{Z}}(1)),$$

with $N+1$ being the rank of $\mathcal{E} = (\pi_{\mathcal{Z}})_* \mathcal{O}_{\mathcal{Z}}(1) \to B$.

Remark 4. The name height is borrowed from [Z], and as we will see later that $h_{(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1))}(s)$ can be thought as a cohomological invariant for the section s of the family $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1))$.

Let s be a section of the fibration $\pi_{\mathcal{Z}} : (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(1)) \to B$, it can be viewed as a $\tilde{G}$-equivariant Z-valued function of $Fr^{\tilde{G}}$, i.e.,

$$s \in Map_{hol.}(Fr^{\tilde{G}}, Z)^{\tilde{G}} := \left\{ s : Fr^{\tilde{G}} \to Z | \bar{\partial} s = 0 \quad \text{and} \quad s(\cdot g) = g^{-1} \cdot s(\cdot), \forall g \in \tilde{G} \right\}.$$

For any $k \geq 1$, suppose that $\sigma \in H^0(Z, \mathcal{O}_Z(k))^G$ is a G-invariant section, then it induces a map

$$\sigma \circ s : Fr^{\tilde{G}} \xrightarrow{s} Z \xrightarrow{\sigma} \mathcal{O}_Z(k).$$

By our assumption (2.2), for any $g \in \tilde{G}$, there is a $h \in \tilde{G}$ such that $\rho(h) = \lambda I$ and $h^{-1} \cdot g \in G$. These imply

$$\det \rho(g) = \det \rho(h) = \lambda^{N+1} \quad \text{and} \quad \rho_k(g) \cdot \sigma = \rho_k(h) \cdot \sigma = \lambda^k \sigma,$$

where $\rho_k : \tilde{G} \to GL(H^0(Z, \mathcal{O}_Z(k)))$ induced from the linearization of $\tilde{G}$ on $\mathcal{O}_Z(1)$. Hence for any $x \in Fr^{\tilde{G}}$,

$$\sigma \circ s(x \cdot g) = \sigma(g^{-1} \cdot s(x)) = \rho_k(g)^{-1} (\sigma \circ s)(x) = \lambda^{-k} \sigma \circ s(x) \in \mathcal{O}_Z(k)|_{s(x)} \cong \mathbb{C}.$$

In particular, if $k = (N+1)k'$ then $\sigma \circ s(x \cdot g) = (\det \rho(g))^{-k'} \cdot \sigma \circ s(x)$. So $\sigma \circ s$ defines a holomorphic section of the line bundle $(\det \mathcal{E})^{\otimes(-k')} \otimes s^* \mathcal{O}_{\mathcal{Z}}(k) = \det(s^*(\pi_{\mathcal{Z}}^* \mathcal{E}))^{\otimes(-k')} \otimes s^* \mathcal{O}_{\mathcal{Z}}(k)$ since $\pi_{\mathcal{Z}} \circ s = id$, that is,

$$\sigma \circ s \in H^0 \left(B, s^* \left(\det \left(\pi_{\mathcal{Z}}^* \mathcal{E}^\vee \otimes \mathcal{O}_{\mathcal{Z}}(1) \right) \right)^{\otimes k'} \right) = H^0 \left(B, s^* \left(\det \left(\pi_{\mathcal{Z}}^* \mathcal{E}^\vee(1) \right) \right)^{\otimes k'} \right), \tag{2.5}$$

from which we deduce

Theorem 5. *Let B be a smooth projective curve. Suppose that for a general $b \in B$ (in the sense of analytic topology), $s(b) \in \mathcal{Z}_b$ is semi-stable. Then $h_{(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}(1))}(s) \geq 0$.*

Proof. It follows from GIT[MFK] that if $s(b) \in (\mathcal{Z}_b, \mathcal{O}_{\mathcal{Z}}(1)|_{\mathcal{Z}_b}) \cong (Z, \mathcal{O}_Z(1))$ is semi-stable with respect the G-action then there is a $\sigma \in H^0(Z, \mathcal{O}_Z(k))^G$ for some $k \gg 1$ such that $\sigma(s(b)) \neq 0$. Without loss of generality, we may assume that $k = (N+1)k'$.

By the construction above (c.f. (2.5)), we have

$$\sigma \circ s \in H^0\left(B, s^*\left(\det(\pi^*\mathcal{E}^\vee(1))\right)^{\otimes k'}\right);$$

such that $\sigma \circ s(b) \neq 0$, this implies

$$\begin{aligned} 0 &\leq c_1\left(s^*\left(\det\left(\pi^*\mathcal{E}^\vee(1)\right)\right)^{\otimes k'}\right) \\ &= k'\left((N+1)c_1\left(s^*\mathcal{O}_{\mathcal{Z}}(1)\right) - c_1\left(\mathcal{E}\right)\right) \\ &= k'\left[(N+1)c_1\left(s^*\mathcal{O}_{\mathcal{Z}}(1)\right) - c_1\left(\det \pi_*\mathcal{O}_{\mathcal{Z}}(1)\right)\right], \end{aligned}$$

which is exactly what we want to prove. □

Remark 6. The proof above is essentially due to Cornalba and Harris [CH].

To get the necessity of positivity in the above theorem, let

$$\begin{array}{ccc} \mathbb{C}^\times & \longrightarrow & Fr(\mathcal{O}_{\mathbb{P}^1}(1)) \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

be the frame bundle associated to $\mathcal{O}_{\mathbb{P}^1}(1)$. For any one parameter subgroup $\lambda : \mathbb{C}^\times \to G$, where G is defined in the first we introduce an associated locally trivial fibration as follows

$$\mathcal{Z}_\lambda := Fr\left(\mathcal{O}_{\mathbb{P}^1}(1)\right) \times_\lambda Z = \frac{\{(p,z) \in Fr\left(\mathcal{O}_{\mathbb{P}^1}(1) \times Z\right)\}}{(p,z) \sim (pt, \lambda(t^{-1})z)\ \forall t \in \mathbb{C}^\times},$$

so we have

$$\begin{array}{ccc} z \in Z = \pi^{-1}(1) & \subset & \mathcal{Z}_\lambda \\ \downarrow & & \pi \downarrow \\ \{1\} & \in & \mathbb{P}^1 \end{array}.$$

Since $\mathcal{O}_Z(1)$ is G-linearized, this allows us to define the line bundle

$$\mathcal{O}_{\mathcal{Z}_\lambda}(1) := Fr(\mathcal{O}_{\mathbb{P}^1}(1)) \times_\lambda \mathcal{O}_Z(1) \longrightarrow \mathcal{Z}_\lambda.$$

To fit the family $\mathcal{Z}_\lambda$ into the framework we developed in the beginning of this section, we let $G_0 = \lambda \subset G$ and $\tilde{G}_0 = \lambda \times \mathbb{C}^\times$. We define the $\tilde{G}_0$-action on $\mathcal{O}_Z(1)$ as follows, for $(t,s) \in \tilde{G}_0$ and $\hat{z} \in \mathcal{O}_Z(1)|_{z \in Z}$, we define $(t,s) \cdot \hat{z} = s(\lambda(t) \cdot \hat{z})$, that is, the action of λ followed by a rescaling. Then it is easy to see that the family $\mathcal{Z}_\lambda$ satisfies the assumptions since $\rho(G) \subset SL(H^0(\mathcal{O}_Z(1)))$ by (2.3).

Next we introduce a $\mathbb{C}^\times$-action on $Fr(\mathcal{O}_{\mathbb{P}^1}(1))$ as follows. Let $\Delta_0 := \{|t| < 10\} \subset \mathbb{C}$ and $\Delta_\infty := \{|1/t| < 10\} \subset \mathbb{C}$ be the chart of $\mathbb{P}^1$ at 0 and ∞ respectively, then $Fr(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{C}^\times \times \Delta_0 \bigcup_f \mathbb{C}^\times \times \Delta_\infty$ with the transition function given by

$$\begin{array}{ccc} f : \mathbb{C}^\times \times \Delta_0 & \longrightarrow & \mathbb{C}^\times \times \Delta_\infty \\ (t \cdot z, t) & \longmapsto & (z, 1/t) \end{array},$$

so

$$\mathcal{Z}_\lambda = Z \times \Delta_0 \bigcup_{\lambda\circ f} Z \times \Delta_\infty .$$

Now for any one parameter subgroup $\mu : \mathbb{C}^\times \to G$ commuting with λ, it induces an action on $\mathcal{Z}_\lambda$ as follows

$$\tau \cdot (z,t) := (\mu(\tau)\cdot z, \tau t) \in Z \times \Delta_0 \quad \text{and} \quad \tau \cdot \left(z, \frac{1}{t}\right) := \left(\mu(\tau)\cdot z, \frac{1}{\tau t}\right) \in Z \times \Delta_\infty ,$$

this action has a natural lifting to $\mathcal{O}_{\mathcal{Z}_\lambda}(1) \to \mathcal{Z}_\lambda$ since $\mathcal{O}_Z(1)$ is G-linearized. Moreover, we have $\mathbb{C}^\times$-equivariant bundle morphism

$$\begin{array}{ccccc} & \mathcal{Z} & \longrightarrow & \mathbb{P}\mathcal{E} & \\ \pi_{\mathcal{Z}} & \downarrow & & \downarrow & \pi \\ & \mathbb{P}^1 & \xrightarrow{id} & \mathbb{P}^1 & \end{array}$$

with $\mathcal{E} = \pi_* \mathcal{O}_{\mathcal{Z}}(1)$. Let $s_\mu(z) \subset \mathcal{Z}_\lambda$ be the closure of the μ-orbit through the point $z \in \pi^{-1}(1) \cong Z$; hence it is a section over $\mathbb{P}^1$ with

$$\lim_{t\to 0} \mu(t)\cdot z = s_\mu(z)(0) \in Z \quad \text{and} \quad \lim_{t\to\infty} \mu(t)\cdot z = s_\mu(z)(\infty) \in Z.$$

Then our main result of this section is the following formula.

Theorem 7. *Let $z \in Z$ and $\lambda, \mu : \mathbb{C}^\times \to G$ be two commuting one parameter subgroups. Then we have*

$$w_z(\mu\circ\lambda) + w_z(\mu^{-1}) = \frac{1}{rk(\pi_*\mathcal{O}_{\mathcal{Z}_\lambda}(1))} h_{(\mathcal{Z}_\lambda, \mathcal{O}_{\mathcal{Z}_\lambda}(1))}(s_\mu(z)). \tag{2.6}$$

In particular, if $\mu = \lambda^{-1}$ then

$$w_z(\lambda) = \frac{1}{rk(\pi_*\mathcal{O}_{\mathcal{Z}_\lambda}(1))} h_{(\mathcal{Z}_\lambda, \mathcal{O}_{\mathcal{Z}_\lambda}(1))}(s_{\lambda^{-1}}(z)).$$

Proof. First, by our assumption $\det \rho\circ\lambda = 1$, this implies $c_1(\det \mathcal{E}) = 0$. So to prove (2.6) all we need is

$$\frac{1}{rk(\pi_*\mathcal{O}_{\mathcal{Z}_\lambda}(1))} h_{(\mathcal{Z}_\lambda, \mathcal{O}_{\mathcal{Z}_\lambda}(1))}(s_\mu(z)) = \int_{s_\mu(z)} c_1(\mathcal{O}_{\mathcal{Z}_\lambda}(1)) = w_z(\mu\circ\lambda) + w_z(\mu^{-1}).$$

The section $s_\mu(z)$ give rise to an $\mathbb{C}^\times$-equivariant morphism $i : \mathbb{P}^1 \longrightarrow s_\mu(z) \subset \mathcal{Z} \subset \mathbb{P}\mathcal{E}$, which implies

$$\int_{s_\mu(z)} c_1(\mathcal{O}_{\mathcal{Z}_\lambda}(1)) = \int_{\mathbb{P}^1} i^* c_1(\mathcal{O}_{\mathcal{Z}_\lambda}(1)).$$

By equivariant localization formula, we have

$$\int_{\mathbb{P}^1} i^* c_1(\mathcal{O}_{\mathcal{Z}_\lambda}(1)) = \int_{\mathbb{P}^1} i^* \tilde{c}_1(\mathcal{O}_{\mathcal{Z}_\lambda}(1)) = \frac{w_0(\mu)}{\tilde{e}(N_{\{0\}/\mathbb{P}^1})} + \frac{w_\infty(\mu)}{\tilde{e}(N_{\{\infty\}/\mathbb{P}^1})},$$

with $\tilde{e}(N_{\{0\}/\mathbb{P}^1})$ and $\tilde{e}(N_{\{\infty\}/\mathbb{P}^1})$ being the equivariant Euler classes of normal bundle of the fixed point at 0 and ∞, respectively, $\tilde{c}_1(\mathcal{O}_{\mathcal{Z}_\lambda}(1))$ being the equivariant extension of $c_1(\mathcal{O}_{\mathcal{Z}_\lambda}(1))$, and $w_0(\mu)$ and $w_\infty(\mu)$ being the weights of μ acting on $\mathcal{O}_{\mathcal{Z}_\lambda}(1)|_0$ and

$\mathcal{O}_{\mathcal{Z}_\lambda}(1)|_\infty$. To find the weight, we notice that $\mathbb{P}\mathcal{E}$ is obtained by gluing $\mathbb{P}^N \times \Delta_0$ and $\mathbb{P}^N \times \Delta_\infty$ via transition function

$$\begin{array}{ccc} f_\lambda : \mathbb{P}^N \times \Delta_0 & \longrightarrow & \mathbb{P}^N \times \Delta_\infty, \\ (\lambda(t) \cdot [x_0, \ldots, x_N], t) & \longmapsto & ([x_0, \ldots, x_N], 1/t). \end{array}.$$

Then the μ-action on $\mathbb{P}\mathcal{E}$ is given by

$$\tau \circ (\lambda(t) \cdot z, t) := (\mu(\tau) \cdot \lambda(\tau t) \cdot z, \tau t) \in \mathbb{P}^N \times \Delta_0,$$

and

$$\tau \circ \left(z, \frac{1}{t}\right) := \left(\mu(\tau) \cdot z, \frac{1}{\tau t}\right) \in \mathbb{P}^N \times \Delta_\infty,$$

and the section $s_\mu(z)$ is obtained via gluing

$$\begin{array}{ccc} f_\lambda : \mathbb{P}^N \times \Delta_0 & \longrightarrow & \mathbb{P}^N \times \Delta_\infty, \\ (\mu(\tau) \cdot \lambda(\tau) \cdot z, \tau) & \longmapsto & (\mu(\tau) \cdot z, 1/\tau). \end{array}.$$

So we have

$$w_0(\mu) = w_z(\mu \circ \lambda), \quad w_\infty(\mu) = -w_z(\mu^{-1}),$$

since $\lim_{\tau\to\infty} \mu(\tau) \cdot z = \lim_{\tau\to 0} \mu(\tau^{-1}) \cdot z = \lim_{\tau \to 0} \mu^{-1}(\tau) \cdot z$. Our statement follows from the fact the μ-action on $\mathbb{P}^1$ has $\tilde{e}(N_{\{0\}/\mathbb{P}^1}) = 1$, $\tilde{e}(N_{\{\infty\}/\mathbb{P}^1}) = -1$. □

Example 8. Let $Z = \mathbb{P}^N$ with $\mathbb{C}^\times$-action defined by

$$t \cdot [x_0, \ldots, x_n] := [t^{w_0} x_0, \ldots, t^{w_N} x_N] \quad \text{with} \sum_{i=0}^{N} w_i = 0.$$

Then $\mathcal{Z}_w = \mathbb{P}\mathcal{E}$ with $\mathcal{E} = \bigoplus_{i=0}^N \mathcal{O}(w_i)$ and $H^*(\mathbb{P}\mathcal{E}) = \mathbb{Z}[\xi, \eta]/(\xi^{N+1}, \xi^N \eta - 1, \eta^2)$, where $\xi = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ and $\eta = [\pi^{-1}(1)]$. Let $e_i := [0, \ldots, \overset{i\text{th}}{1}, \ldots, 0]$, then

$$[s(e_i)] = \prod_{k \neq i} (\xi - w_k \eta) \in H^{2N}(\mathbb{P}\mathcal{E})$$

and

$$\int_{s(e_i)} \xi = \xi \prod_{k\neq i} (\xi - w_k \eta) = -\sum_{k \neq i} w_k = w_i.$$

On the other hand, if we apply the above proposition we obtain

$$\begin{aligned} \int_{s(e_i)} \xi &= w_{e_i}(\mu) + w_{e_i}(\lambda) + w_{e_i}(\mu^{-1}) \\ &= w_{e_i}(\mu) + w_{e_i}(\lambda) - w_{e_i}(\mu) \\ &= w_{e_i}(\lambda). \end{aligned}$$

Corollary 9. *If $z \in (Z, \mathcal{O}_Z(1))$ be a un-stable point and $\lambda : \mathbb{C}^\times \to G$ be the destablizing one parameter subgroup, that is $w_z(\lambda) < 0$. Then*

$$w_z(\lambda) = \frac{1}{rk(\pi_* \mathcal{O}_{\mathcal{Z}_\lambda}(1))} h_{(\mathcal{Z}_\lambda, \mathcal{O}_{\mathcal{Z}_\lambda}(1))}(s_{\lambda^{-1}}(z)) < 0.$$

3. Geometric height and positivity of CM-line

Our first application of the result in the previous section is to the study a proper flat family of varieties.

3.1. Geometric height. First, let us briefly summarize the construction of Chow section due to Mumford (for details please see [MFK] Section 5.4, [Z] Section 1.3 or [BGS] Section 4.3). Let B be an integral scheme $\mathcal{E}$ be a locally free sheaf of rank $N+1$ over B, and $\mathcal{X}$ be an effective cycle of $\mathbb{P}\mathcal{E} := \mathrm{Proj}(\mathrm{Sym}^*\mathcal{E})$, the projective space over B, whose components are flat and of dimension n over B. Thus, we have diagram

$$\begin{array}{ccccc} & \mathcal{X} & \longrightarrow & \mathbb{P}\mathcal{E} & \\ \pi & \downarrow & & \downarrow & \pi \\ & B & \longrightarrow & B & \end{array} .$$

Let $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ and $\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1)$ denote the hyperplane line bundle of $\mathbb{P}\mathcal{E}$ and $\mathbb{P}\mathcal{E}^\vee$ respectively. Then the canonical section of $\mathcal{E}\otimes\mathcal{E}^\vee$, which is dual to the canonical pairing $\mathcal{E}^\vee\otimes\mathcal{E}\to\mathcal{O}_B$, gives a section Δ of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)\boxtimes\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1)$ over $\mathbb{P}\mathcal{E}\times\mathbb{P}\mathcal{E}^\vee$. Let π_i denote the ith projection

$$\pi_i : (\mathbb{P}\mathcal{E}^\vee)^{n+1} \longrightarrow \mathbb{P}\mathcal{E}^\vee,$$

and Δ_i's be the corresponding sections of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)\otimes\pi_i^*\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1)$ on $\mathbb{P}\mathcal{E}\otimes(\mathbb{P}\mathcal{E}^\vee)^{n+1}$. Let

$$\Gamma := \bigcap_{i=0}^{n} \Delta_i^{-1}(0).$$

If we regard the points of $\mathbb{P}\mathcal{E}^\vee$ as hyperplanes of $\mathbb{P}\mathcal{E}$ then

$$\Gamma = \left\{(x, H_0, \ldots, H_n) \in \mathbb{P}\mathcal{E}\otimes(\mathbb{P}\mathcal{E}^\vee)^{n+1} |\ x\in H_i, \forall i\right\}.$$

And if we regard Γ as a correspondence from $\mathbb{P}\mathcal{E}$ to $(\mathbb{P}\mathcal{E}^\vee)^{n+1}$

$$\begin{array}{ccccc} & \Gamma\subset & \mathbb{P}\mathcal{E}\otimes(\mathbb{P}\mathcal{E}^\vee)^{n+1} & & \\ p_1 & \swarrow & & \searrow & p_2 \\ \mathbb{P}\mathcal{E} & & & & (\mathbb{P}\mathcal{E}^\vee)^{n+1} \end{array} ,$$

then

$$Y(\mathcal{X}) = \Gamma_*(\mathcal{X}) := p_{2*}(p_1^*(\mathcal{X})\cap\Gamma) \subset (\mathbb{P}\mathcal{E}^\vee)^{n+1}$$

will be a divisor of degree $(d,\ldots,d)$ of $(\mathbb{P}\mathcal{E}^\vee)^{n+1}$ whose components are flat (cf. [BGS] Lemma 4.3.1) over B, where d is the degree of $\mathcal{X}_b\subset\mathbb{P}\mathcal{E}_b := \pi^{-1}(b)$ for general $b\in B$.

Let $\mathcal{O}_{\mathfrak{P}^{K_d}}(1)$ be the hyperplane bundle of

$$\mathfrak{P}^{K_d} := \mathbb{P}[(\mathrm{Sym}^d\mathcal{E})^{\otimes(n+1)}] \longrightarrow B$$

with

$$K_d + 1 := \dim(\mathrm{Sym}^d\mathbb{C}^{N+1})^{\otimes(n+1)}.$$

Then the canonical pairing of $(\mathrm{Sym}^d\mathcal{E})^{\otimes(n+1)}\otimes(\mathrm{Sym}^d\mathcal{E}^\vee)^{\otimes(n+1)}$ give rise to a section Δ' of the line bundle $\mathcal{O}_{\mathfrak{P}^{K_d}}(1)\otimes\pi_0^*\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(d)\otimes\cdots\otimes\pi_n^*\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(d)$ on $\mathbb{P}[(\mathrm{Sym}^d\mathcal{E})^{\otimes(n+1)}]\times(\mathbb{P}\mathcal{E}^\vee)^{n+1}$. By viewing points of $\mathbb{P}[(\mathrm{Sym}^d\mathcal{E})^{\otimes(n+1)}]$ as hypersurface of $(\mathbb{P}\mathcal{E}^\vee)^{n+1}$ of degree $(d,\ldots,d)$, we can regard

$$\Gamma' = \{\Delta' = 0\} = \{(H, y_0,\ldots,y_n) | (y_0,\ldots,y_n)\in H\}.$$

as a correspondence

$$\begin{array}{ccc} & \Gamma' \subset \quad \mathfrak{P}^{K_d} \otimes (\mathbb{P}\mathcal{E}^\vee)^{n+1} & \\ p_1 \ \swarrow & & \searrow \ p_2 \\ \mathfrak{P}^{K_d} & & (\mathbb{P}\mathcal{E}^\vee)^{n+1} \end{array},$$

and section

$$s_{\mathcal{X}} = p_{1*}(p_2^* Y(\mathcal{X}) \cap \Gamma') \tag{3.1}$$

of $\mathfrak{P}^{K_d}$ over B corresponds to $Y(\mathcal{X})$ via Γ' is the *Chow section* for $\mathcal{X}$. Furthermore, Zhang have show in [Z] the following:

Proposition 10.

$$\mathcal{L}^{\langle n+1\rangle}(\mathcal{X}/B) := \langle \mathcal{L}, \ldots, \mathcal{L}\rangle(\mathcal{X}/B) \simeq s_{\mathcal{X}}^*(\mathcal{O}_{\mathfrak{P}^{K_d}}(1)) \in \mathrm{Pic}(B),$$

where $\mathcal{L}^{\langle n+1\rangle} = \langle \mathcal{L}, \ldots, \mathcal{L}\rangle$ *is the Deligne pairing (c.f.* [De]*) of* $\mathcal{L}$ *over* B*. In particular,* $c_1(\mathcal{L}^{\langle n+1\rangle}) = \pi_* c_1(\mathcal{L})^{n+1}$*.*

Now suppose that $\mathcal{X}$ is a separated scheme of dimension $n+1$ and B is a smooth curve. Let

$$\pi : \mathcal{X} \longrightarrow B$$

be a flat proper morphism. Let $\mathcal{L} \to \mathcal{X}$ be a line bundle such that $\mathcal{E} := \pi_*\mathcal{L}$ is locally free of rank $N+1$. Suppose that the following conditions are satisfied:

(1) $\mathcal{L}$ is relative very ample and $H^0(\mathcal{X}_b, \mathcal{L}_b) = \mathcal{E}_b$ for every $b \in B$,
(2) There is a $b \in B$ such that $H^0(\mathcal{X}_b, \mathcal{L}_b)$ yields a Chow semi-stable embedding of $\mathcal{X}_b$.

From the above assumption, we may view $\mathcal{X}$ as an effective cycle of $\mathbb{P}\mathcal{E}$

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{P}\mathcal{E} \\ \downarrow & & \downarrow \\ B & \xrightarrow{id} & B \end{array}$$

whose components are flat and of dimension n over B. Applying Mumford's construction of Chow section outlined above, we obtain section $s_{\mathcal{X}}$

$$\begin{array}{cccc} & & \mathfrak{P}^{K_d} & \\ s_{\mathcal{X}} & \nearrow & \downarrow & \Pi, \\ B & \xrightarrow{id} & B & \end{array}$$

in particular, for general point $b \in B$, $s_{\mathcal{X}}(b)$ maps to the Chow point of $(\mathcal{X}_b, \mathcal{L}|_{\mathcal{X}_b})$. By Theorem 5, we deduce

$$c_1(s_{\mathcal{X}}^* \det((\Pi_* \mathcal{O}_{\mathfrak{P}^{K_d}}(1))^\vee(1))) \geq 0.$$

To unwind its meaning in terms of the geometry of $\mathcal{L}$ and $\mathcal{X}$, we need the following

Proposition 11. *For the family* $(\mathfrak{P}^{K_d}, \mathcal{O}_{\mathfrak{P}^{K_d}}(1)) \to B$ *with section* $s_{\mathcal{X}}$ *we have*

(1)

$$\begin{aligned} \Lambda^{\mathrm{Chow}}(\mathcal{L}) &:= \left(\det\left(s_{\mathcal{X}}^*(\Pi_* \mathcal{O}_{\mathfrak{P}^{K_d}}(1))^\vee(1)\right)\right)^{\otimes N+1} \\ &= \left\{(\mathcal{L}^{\langle n+1\rangle})^{\otimes(N+1)} \otimes (\det \mathcal{E}^\vee)^{\otimes(n+1)d}\right\}^{\otimes(K_d+1)}. \end{aligned}$$

(2)

$$h_{(\mathfrak{P}^{K_d},\mathcal{O}_{\mathfrak{P}^{K_d}}(1))}(s_{\mathcal{X}}) = \frac{K_d+1}{(N+1)^n}\pi_*\left\{(N+1)c_1(\mathcal{L}) - \pi^* c_1(\pi_*\mathcal{L})\right\}^{n+1},$$

and we call

$$h_{\mathcal{E}}(\mathcal{X}) := \frac{1}{(N+1)^{n+1}}\pi_*\left\{(N+1)c_1(\mathcal{L}) - \pi^* c_1(\pi_*\mathcal{L})\right\}^{n+1} \in NS(B),$$

the ***geometric height****.*

Proof. For the first identity, we notice that the natural embedding

$$\rho : GL(N+1) \longrightarrow GL\left(\left(\mathrm{Sym}^d\mathbb{C}^{N+1}\right)^{\otimes n+1}\right) = GL(K_d+1),$$

implies

$$\det(\rho(g))^{N+1} = (\det g)^{d(n+1)(K_d+1)}.$$

This together with the fact $s_{\mathcal{X}}^*\mathcal{O}_{\mathfrak{P}^{K_d}}(1) \cong \mathcal{L}^{\langle n+1\rangle}$ (by Proposition 10) and

$$s_{\mathcal{X}}^*(\Pi_*\mathcal{O}_{\mathfrak{P}^{K_d}}(1)) \cong (\mathrm{Sym}^d\mathcal{E})^{\otimes n+1}$$

imply

$$\begin{aligned}
&\left[s_{\mathcal{X}}^* \det\left(\Pi_*\mathcal{O}_{\mathfrak{P}^{K_d}}(1)\right)^{\vee}(1)\right]^{\otimes(N+1)} \\
&\quad= \det\mathcal{H}\mathrm{om}\left(\left(\mathrm{Sym}^d\mathcal{E}\right)^{\otimes(n+1)}, s_{\mathcal{X}}^*\mathcal{O}_{\mathfrak{P}^{K_d}}(1)\right)^{\otimes(K_d+1)} \\
&\quad= \left\{\mathcal{H}\mathrm{om}\left((\det\mathcal{E})^{\otimes d(n+1)}, \left(\mathcal{L}^{\langle n+1\rangle}\right)^{\otimes(N+1)}\right)\right\}^{\otimes(K_d+1)} \\
&\quad= \left\{\left(\mathcal{L}^{\langle n+1\rangle}\right)^{\otimes(N+1)} \otimes (\det\mathcal{E}^{\vee})^{\otimes d(n+1)}\right\}^{\otimes(K_d+1)}.
\end{aligned}$$

For the second identity, we have

$$\begin{aligned}
(3.2)\qquad &(N+1)h_{(\mathfrak{P}^{K_d},\mathcal{O}_{\mathfrak{P}^{K_d}}(1))}(s_{\mathcal{X}}) \\
&= (N+1)c_1(s_{\mathcal{X}}^*\det(\Pi_*\mathcal{O}_{\mathfrak{P}^{K_d}}(1))^{\vee}(1)) \\
&= (K_d+1)\{(N+1)c_1(\mathcal{L}^{\langle n+1\rangle}) - (n+1)dc_1(\mathcal{E})\} \\
&= (K_d+1)\left\{(N+1)\pi_*c_1(\mathcal{L})^{n+1} - (n+1)dc_1(\mathcal{E})\right\} \\
&= \frac{K_d+1}{(N+1)^n}\pi_*\left((N+1)c_1(\mathcal{L}) - \pi^*c_1(\mathcal{E})\right)^{n+1},
\end{aligned}$$

where we have used formula $c_1(\mathcal{L}^{\langle n+1\rangle}) = \pi_*c_1(\mathcal{L})^{n+1}$ in the third identity. □

Remark 12. Note that the second part of the above Proposition together with Theorem 5 imply Theorem 1. And the approach we adapt here is slightly different from [CH] in the sense that we use the Chow scheme instead of Hilbert scheme, the advantage here is the geometric height is *exactly* the GIT weight (cf. Proposition 13)

Now Theorem 1 is a direct consequence of Theorem 5. To get the converse, let

$$\lambda(t) = \begin{bmatrix} t^{\lambda_0} & & \\ & \ddots & \\ & & t^{\lambda_N} \end{bmatrix} \subset SL(N+1) \text{ with } \lambda_i \in \mathbb{Z}$$

be a 1-parameter subgroup and

$$\mathcal{E}_\lambda = \bigoplus_{i=0}^{N} \mathcal{O}(\lambda_i) \longrightarrow \mathbb{P}^1.$$

Then there is a natural lifting of the $\mathbb{C}^\times$-action on $\mathbb{P}^1$ to $\mathbb{P}\mathcal{E}_\lambda$ as explained in Section 2. If we embed $X \subset \pi^{-1}(1) \cong \mathbb{P}^N$ as a subvariety of $\mathbb{P}\mathcal{E}_\lambda$ lying over $1 \in \mathbb{P}^1$, and let $\mathcal{X}_\lambda \subset \mathbb{P}\mathcal{E}_\lambda$ be the effective $\mathbb{C}^\times$-invariant cycle obtained via taking the closure of the λ^{-1}-orbit through $X \subset \pi^{-1}(1)$, i.e.,

$$\begin{array}{ccc} \mathcal{X}_\lambda & \longrightarrow & \mathbb{P}\mathcal{E}_\lambda \\ \downarrow & & \downarrow \rho \\ \mathbb{P}^1 & \xrightarrow{id} & \mathbb{P}^1 \end{array} . \tag{3.3}$$

Then the family $\mathcal{X}_\lambda \to B$ is flat over $\mathbb{P}^1$ (cf. [Mum] Section 2) and by the construction in the beginning of this subsection (cf. (3.1)) we obtain the Chow section $s_{\mathcal{X}_\lambda}$ of

$$\mathfrak{P}^{K_d} := \mathbb{P}\left[\left(\mathrm{Sym}^d \mathcal{E}_\lambda\right)^{\otimes(n+1)}\right] \to \mathbb{P}^1.$$

Note that it follows from Mumford's construction that $s_{\mathcal{X}_\lambda} : \mathbb{P}^1 \to \mathfrak{P}^{K_d}$ is $\mathbb{C}^\times$-equivariant, since Γ and Γ' both are $\mathbb{C}^\times$-invariant cycle (see also [MFK] Section 5.4). As a consequence of Proposition 7, we have the following:

Proposition 13. *Let $X \subset \mathbb{P}^N$ be a n-dimensional subvariety of degree d and*

$$\lambda(t) = \begin{bmatrix} t^{\lambda_0} & & \\ & \ddots & \\ & & t^{\lambda_N} \end{bmatrix} \subset SL(N+1)$$

be a one parameter subgroup. Then we have

$$h_{\mathcal{E}_\lambda}(\mathcal{X}_\lambda) = \frac{1}{(N+1)^{n+1}} \pi_* \left\{(N+1)c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) - \pi^* c_1(\mathcal{E})\right\}^{n+1} = w_{\mathrm{Chow}(X)}(\lambda),$$

for $\mathcal{E}_\lambda := \bigoplus_{i=0}^{N} \mathcal{O}(\lambda_i)$.

In particular, it implies the following converse to Cornalba–Harris's result.

Corollary 14. *Let $(X, \mathcal{O}_X(1))$ be a projective variety polarized by a very ample line bundle $\mathcal{O}_X(1)$. Suppose that for any flat proper family $(\mathcal{X}, \mathcal{L}) \to B$ over a smooth curve B with generic $(\mathcal{X}_b, \mathcal{L}|_{\mathcal{X}_b}) \cong (X, \mathcal{O}_{\mathbb{P}^N}(1)|_X)$ for all but finite $b \in B$ and $\mathcal{L} \to \mathcal{X}$ be a relative very ample line bundle such that $\mathcal{E} := \pi_* \mathcal{L}$ is locally free of rank $N+1$, we always have*

$$\pi_* \left\{((N+1)c_1(\mathcal{L}) - \pi^* c_1(\mathcal{E}))^{n+1} \cap [\mathcal{X}]\right\} \geq 0.$$

Then X is Chow semi-stable.

Now to relate the geometric height with the CM-line bundle introduced by Tian, we need to replace $\mathcal{L}$ by $\mathcal{L}^k$ and then let $k \to \infty$. First we shall note that the one parameter subgroup

$$\lambda_k : \mathbb{C}^\times \longrightarrow GL(H^0(X_0, \mathcal{O}_{X_0}(k)))$$

induced from λ might not lie in $SL(N_k+1)$ with $N_k + 1 := \dim H^0(X_0, \mathcal{O}_{X_0}(k))$. So we introduce the normalization

$$\tilde{\lambda}_k := t^{-w_k/(N_k+1)} \begin{bmatrix} t^{\lambda_0} & & \\ & \ddots & \\ & & t^{\lambda_{N_k}} \end{bmatrix} \subset SL(N_k+1)$$

with w_k being the λ-weight of $\wedge^{\mathrm{top}} H^0(X_0, \mathcal{O}_{X_0}(k))$. Let us define vector bundle

$$\mathcal{E}_{\lambda_k} := \bigoplus_{i=0}^{N_k} \mathcal{O}(\lambda_i) \longrightarrow \mathbb{P}^1$$

and $\mathbb{Q}$-bundle

$$\mathcal{E}_{\tilde{\lambda}_k} := \mathcal{E}_{\lambda_k} \otimes \mathcal{O}(-w_k/(N_k+1)) = \bigoplus_{i=0}^{N} \mathcal{O}\left(\lambda_i - \frac{w_k}{N_k+1}\right) \longrightarrow \mathbb{P}^1.$$

By following the construction in the previous subsection, we obtain a flat family $\mathcal{X}_{\tilde{\lambda}_k} \to \mathbb{P}^1$ with corresponding Chow section $s_{\mathcal{X}_{\tilde{\lambda}_k}}$ of $\mathfrak{P}^{K_{d,k}} := \mathbb{P}((\mathrm{Sym}^{dk^n} \mathcal{E})^{\otimes n+1})$ over $\mathbb{P}^1$. Then Proposition 13 implies the following:

Corollary 15. *Let $\tilde{\lambda}_k$ be as above and $\mathcal{X}_k \to \mathbb{P}^1$ be the induced family, then*

$$w_{\mathrm{Chow}_k(X)}(\tilde{\lambda}_k) = \frac{1}{(N_k+1)^{n+1}} \pi_* \left((N_k+1) c_1 \left(\mathcal{O}_{\mathbb{P}\mathcal{E}_{\lambda_k}}(1) \right) - \pi^* c_1(\mathcal{E}_{\lambda_k}) \right)^{n+1}. \tag{3.4}$$

Proof. We only need to notice the fact that

$$h_{\mathcal{E}_{\tilde{\lambda}_k}}(\mathcal{X}_{\tilde{\lambda}_k}) = h_{\mathcal{E}_{\lambda_k}}(\mathcal{X}_{\lambda_k})$$

since $\mathcal{O}_{\mathbb{P}\mathcal{E}_{\tilde{\lambda}_k}}(1) = \mathcal{O}_{\mathbb{P}\mathcal{E}_{\lambda_k}}(1) \otimes \pi^* \mathcal{O}(-w_k/(N_k+1))$. □

3.2. Positivity of CM-line bundle. Now we are ready to address our application to K-stability introduced by Tian and Donaldson. To set the scene, let $\mathcal{X}$ be a separated scheme of pure dimension $n+1$ and B be a smooth curve. Let

$$\pi : \mathcal{X} \longrightarrow B$$

be a flat proper morphism and suppose that $\mathcal{X}$ over B has a relative canonical line bundle $K_{\mathcal{X}/B}$ whose first Chern class will be denoted by $-c_1(\mathcal{X}/B)$. Let $\mathcal{L} \to \mathcal{X}$ be a line bundle such that $\mathcal{E} := \pi_* \mathcal{L}$ is locally free and relatively very ample. The CM**-line** $\Lambda^{CM}(\mathcal{X}) \to B$ for the family $\mathcal{X} \to \mathcal{B}$ was introduced by Paul and Tian [PT] in order to give a GIT interpretation of K-stability defined by Tian and extended by Donaldson [T, Do0]. We are referring to [PT, RT] for the precise definition of the K-stability, as the definition itself is not used in the following discussion. One important fact we need is that

$$c_1(\Lambda^{CM}(\mathcal{X})) := \pi_* \left(n c_1(\mathcal{L})^{n+1} \mu - (n+1) c_1(\mathcal{L})^n c_1(\mathcal{X}/B) \right) \in H^2(B)$$

with

$$\mu = \frac{\pi_* \left(c_1(\mathcal{L})^{n-1} c_1(\mathcal{X}/B)\right)}{\pi_* c_1(\mathcal{L})^n}.$$

The introduction of CM-line is motivated by the fact that the distortion of the metric on the CM-line is directly related to the Mabuchi functional, which play an essential role in the problem of finding cscK metric.

As was observed by Zhang [Z1], and Fine and Ross [FR] independently that the leading term of height resulting from the re-embedding is proportional to Λ^{CM}. More precisely, we have following asymptotic formula:

Proposition 16. *(Zhang, Fine and Ross, cf.* [FR] *Section 4) For $k \gg 1$, there is a constant $a_0 > 0$ such that*

$$\Lambda^{\mathrm{Chow}}(\mathcal{L}^k) = (\Lambda^{CM})^{\otimes a_0 k^{2n}} \otimes \cdots (\textit{terms with lower exponents on } k)$$

and

$$\begin{aligned}(N_k+1) h_{\mathcal{F}_\lambda}(\mathcal{X}) &= \frac{1}{(N_k+1)^n} \pi_* \left\{ (N_k+1) c_1(\mathcal{L}^k) - \pi^* c_1(\pi_* \mathcal{L}^k) \right\}^{n+1} \\ &= \frac{\pi_* c_1(\mathcal{L})^n}{2n!} \pi_* \left\{ n c_1(\mathcal{L})^{n+1} \mu - (n+1) c_1(\mathcal{L})^n c_1(\mathcal{X}/B) \right\} k^{2n} + O(k^{2n-1})\end{aligned}$$

with

$$\mu = \frac{\pi_*(c_1(\mathcal{L})^{n-1} c_1(\mathcal{X}/B))}{\pi_* c_1(\mathcal{L})^n}.$$

With those understood, we have the following:

Proposition 17. *Let λ_k, $\mathcal{F}_{\lambda_k}$ be as in the previous subsection,*

(1) *Let $\pi : \mathcal{X}_\lambda \to B$ be the flat family we constructed in (3.3). Then*

$$-F(\lambda) = \lim_{k\to\infty} \frac{h_{\mathcal{F}_{\lambda_k}}(\mathcal{X}_\lambda)}{N_k+1},$$

where $F(\lambda)$ is the generalized Futaki invariant defined in [Do0, RT] *and* $\deg X := \int_X c_1(\mathcal{L})^n$. *In particular, if we assume $K_{\mathcal{X}/B}$ is $\mathbb{Q}$-Cartier then*

$$-F(\lambda) = \frac{n!}{2\deg X} \pi_* \left\{ n c_1(\mathcal{L})^{n+1} \mu - (n+1) c_1(\mathcal{L})^n c_1(\mathcal{X}_\lambda/\mathbb{P}^1) \right\} [\mathbb{P}^1].$$

(2) *Suppose $X^n \subset \mathbb{P}^N$ is K-unstable (for the definition see* [Do0, PT]*), then there is a flat family $\mathcal{X} \to \mathbb{P}^1$ such that the CM line bundle $\Lambda^{CM}(\mathcal{X}) \to \mathbb{P}^1$ is negative.*

Proof. 1. First, it follows from Corollary 15 that

$$\lim_{k\to\infty} \frac{h_{\mathcal{F}_{\lambda_k}}(X)}{N_k+1} = \lim_{k\to\infty} \frac{h_{\mathcal{F}_{\tilde\lambda_k}}(X)}{N_k+1} = \lim_{k\to\infty} \frac{w_{\mathrm{Chow}_k(X)}(\tilde\lambda_k)}{N_k+1} = -F(\lambda),$$

where the last identity follows from Theorem 38 in [W] or Proposition 4.2 in [R]. For the second identity, we have

$$\begin{aligned}\lim_{k\to\infty} \frac{h_{\mathcal{F}_{\lambda_k}}(X)}{N_k+1} &= \lim_{k\to\infty} \frac{(N_k+1) h_{\mathcal{F}_{\lambda_k}}(X)}{(N_k+1)^2} \\ &= \frac{n!}{2\deg X} \pi_* \left\{ n c_1(\mathcal{L})^{n+1} \mu - (n+1) c_1(\mathcal{L})^n c_1(\mathcal{X}_\lambda/B) \right\}.\end{aligned}$$

2. Suppose $X \subset \mathbb{P}^N$ is K-unstable with respect to a 1-ps $\lambda : \mathbb{C}^\times \to SL(N+1)$ such that $F(\lambda) > 0$, which is the λ-weight of Hilbert point of X with respect to the refined CM polarization Λ^{CM} on the Hilbert scheme $\mathrm{Hilb}(\chi, \mathbb{P}^n)$, where $\chi(k) := \dim H^0(X, \mathcal{O}_X(k))$ (cf. Theorem 1, [PT]). We let $\mathcal{X}_\lambda \to \mathbb{P}^1$ to be the flat family obtained by completion of $\mathbb{C}^\times$-orbit in $\mathrm{Hilb}(\chi, \mathbb{P}^n)$ of λ^{-1}. Then the height of corresponding Chow section give rise to the Chow weight of λ by Theorem 7. Apply re-embedding as we did in the previous subsection we get kth height is the Chow weight for kth embedding. Since $F(\lambda) > 0$, it follows from the first part that kth Chow weight and hence kth height is negative for $k \gg 1$. Our statement $c_1(\Lambda^{CM})[\mathbb{P}^1] < 0$ then follows from Proposition 16. □

Remark 18. Notice that the above corollary implies that the CM-line is **strictly positive** along the closure of one parameter subgroup through a K-stable point.

4. Height for vector bundles

Our second application is to the study of a family of vector bundles over a fixed polarized projective manifold $(X, \mathcal{O}_X(1))$.

4.1. Height for vector bundles and Donaldson's line bundle. Let $(X, \mathcal{O}_X(1))$ be a projective manifold with polarization $\mathcal{O}_X(1)$ and

$$\mathcal{F} \longrightarrow X \times B \xrightarrow{\pi} B$$

be a rank r coherent sheaf over $X \times B$ that is flat over B. We fix a $m \gg 1$ such that $R^i\pi_*\mathcal{F}(m) = 0$ for $i > 0$ and the evaluation map $\rho : \pi^*(\pi_*\mathcal{F}(m)) \to \mathcal{F}(m)$ is surjective. By our assumption, $V := \pi_*\mathcal{F}(m)$ is a rank $p(m)$ vector bundle over B with $p(m)$ being the **Hilbert polynomial** of $\mathcal{F}|_{X\times\{b\}}$ for general $b \in B$ and we have diagram

$$\begin{array}{ccccc} \pi^*V & \xrightarrow{\rho} & \mathcal{F}(m) & \longrightarrow & 0 \\ & & \downarrow & & \\ & & X \times B & \xrightarrow{\pi_X} & X \\ & & \downarrow & \pi & \\ & & B & & \end{array} .$$

Let us briefly recall Gieseker's construction (cf. [HL]) of Gieseker section $s_{\mathcal{F}(m)}$ of $\mathbb{P}\mathcal{H}\mathrm{om}(\wedge^r V, \pi_*(\pi_X^* Q(m))^\vee)$ $\mathbb{P}(\mathcal{H}\mathrm{om}(\wedge^r V, \pi_*(\pi_X^* Q(m)))^\vee)$ over B. For simplicity, let us assume

$$\det \mathcal{F}|_{X\times\{b\}} = Q \in Pic(X),$$

which implies that

$$\det \mathcal{F} = \pi^* L \otimes \pi_X^* Q \text{ for some } L \in Pic(B).$$

From the universal quotient $\rho : \pi^*V \to \mathcal{F}(m)$, we obtain homomorphisms $\wedge^r\rho : \wedge^r \pi^* V \to \det(\mathcal{F} \otimes \pi_X^* \mathcal{O}_X(m))$. By applying π_*, we have

$$\pi_* \circ \wedge^r \rho : \wedge^r V \otimes \mathcal{O}_B \longrightarrow \pi_* \det(\mathcal{F}(m)) = L \otimes \pi_*(\pi_X^* Q(rm)),$$

which is adjoint to

$$\tilde{s} : \mathcal{H}\mathrm{om}\left(\wedge^r V, \pi_* \left(\pi_X^* Q(rm)\right)\right) \longrightarrow L.$$

Note that $\tilde{s}$ is everywhere surjective (cf. [HL] Section 4.A) and therefore it defines the Gieseker section

$$\begin{array}{ccc} & & \mathfrak{P}^{K_m} \\ s_{\mathcal{F}(m)} & \nearrow & \downarrow \Pi \\ B & \xrightarrow{id} & B \end{array}$$

with

$$\begin{aligned} \mathfrak{P}^{K_m} &:= \mathbb{P}\left(\mathcal{H}\mathrm{om}\left(\wedge^r V, \pi_*(\pi_X^* Q(m))\right)^\vee\right) \\ &= \mathbb{P}\left(\mathcal{H}\mathrm{om}\left(\det \pi_* \mathcal{F}(m), \pi_* \det \mathcal{F}(m)\right)^\vee\right) \end{aligned}$$

and

$$K_m + 1 := \dim \mathrm{Hom}\left(\wedge H^0\left(\mathcal{F}(m)|_{X\times\{b\}}\right), H^0(Q(rm))\right),$$

that is, $s_{\mathcal{F}(m)}(b)$ is the Gieseker point of $\mathcal{F}(m)_b|_{X\times\{b\}}$ for $b \in B$. By Theorem 5, if we assume that for generic $b \in B$, $\mathcal{F}|_{X\times\{b\}}$ is semi-stable then we have

$$c_1\left(\det\left(s_{\mathcal{F}(m)}^*(\Pi_* \mathcal{O}_{\mathfrak{P}^{K_m}}(1))^\vee(1)\right)\right) \geq 0.$$

Similar to the variety case we have the following Proposition, whose proof is parallel to what we did for the variety case.

Proposition 19. *For the family* $(\mathfrak{P}^{Km}, \mathcal{O}_{\mathfrak{P}^{Km}}(1)) \to B$ *with section* $s_{\mathcal{F}(m)}$ *we have*

(1)
$$\left(\det\left(s_{\mathcal{F}(m)}^*(\Pi_* \mathcal{O}_{\mathfrak{P}^{K_m}}(1))^\vee(1)\right)\right) = \left\{\det\left(\pi_* \mathcal{F}(m)\right)^{-\frac{r}{p(m)}} \otimes L\right\}^{\chi(rm)}$$

where $\chi(m) := \dim H^0(X, Q(m))$ *for* $m \gg 1$.

(2)
$$\begin{aligned} & c_1\left(h_{(\mathfrak{P}^{Km}, \mathcal{O}_{\mathfrak{P}^{Km}}(1))}(s_{\mathcal{F}(m)})\right) \\ &\quad = \frac{1}{p(m)} \pi_* \left\{\left(p(m) c_1(\mathcal{F}(m)) - r c_1(\pi_* \mathcal{F}(m))\right) \omega^n\right\} \chi(rm), \end{aligned}$$

where $\omega := \pi_X^* c_1(\mathcal{O}_X(1))$. *And we call*

$$h(\mathcal{F}(m)) := p(m) \pi_* \left(c_1(\mathcal{F}(m)) \omega^n\right) - r_{\mathcal{F}} c_1(\pi_* \mathcal{F}(m)) \pi_* \omega^n \in NS(B)$$

the ***height*** *of the sheaf* $\mathcal{F}(m)$.

(3) *Suppose that*

$$\det \mathcal{F}|_{X\times\{b\}} = Q \in Pic(X) \textit{ for all } b \in B.$$

Then

$$h(\mathcal{F}(m)) = \pi_* \left(\mu c_1(\mathcal{F}) \omega^n - r_{\mathcal{F}}(ch_2(\mathcal{F}) \omega^{n-1})\right) \frac{m^{n-1}}{(n-1)!} + O(m^{n-2})$$

with

$$\mu = \frac{\pi_* \left((c_1(Q) + rc_1(X)/2)\, \omega^{n-1} \right)}{\pi_* \omega^n},$$

which is the exactly the first Chern class of the Donaldson's determinant line bundle [Do1].

4.2. Positivity of Donaldson's line bundle. To understand the geometric meaning of the class

$$\pi_* \left(\mu c_1(\mathcal{F}) \omega^n - rch_2(\mathcal{F}) \omega^{n-1} \right) \in NS(B),$$

let (X, H) be a projective surface polarized by an ample divisor H and $K(X)$ be the K-group of coherent sheaves on X. By setting $h := \mathcal{O}_H$, we introduce the class

$$u_1(c) := -rh + \chi(c \cdot h)[\mathcal{O}_x] \in K(X)$$

with $c := \mathcal{F}|_{X \times \{b\}} \in K(X)_{num}$, the numerical equivalence class of $[\mathcal{F}|_{X \times \{b\}}]$, and $\chi(c \cdot h)$ being the Euler characteristic of the element $c \otimes \mathcal{O}_H \in K(X)$. Following [HL], chapter 8, we define

$$\Lambda_{\mathcal{F}}(u_1(c)) := \det \pi! \left(\mathcal{F} \otimes u_1(c) \right) \in Pic(B).$$

Then a direct calculation similar to Proposition 8.3.1 in [HL] give rise to the following Proposition whose proof are omitted.

Proposition 20. *Let $\mathcal{F}$ be a B-flat family of sheaves on (X, H) of rank r, determinant Q and Chern class $c_1, c_2 \in H^*(X)$. Then*

$$c_1 \left(\Lambda_{\mathcal{F}}(u_1(c)) \right) = \pi_* \left(\mu c_1(\mathcal{F}) H^2 - rch_2(\mathcal{F}) H \right) \in H^*(B)_{\mathbb{Q}}$$

with $\mu = (c_1 - K_X/2) \cdot H/H^2$.

As a quick consequence of the above Proposition and Theorem 7, we have

Corollary 21. *Let $F \to (X, \mathcal{O}_X(1))$ be vector bundle with Hilbert Polynomial $p(m)$ and $[F] = c \in K(X)_{num}$. Then F is Mumford semi-stable if and only if for any $\mathcal{F}$ be a B-flat family of sheaves with $\mathcal{F}|_{X \times \{b\}} \cong F$ for generic $b \in B$, $c_1(\Lambda_{\mathcal{F}}(u_1(c)))$ is non-negative.*

Remark 22. Note that it was proved by Li (cf. [Li, HL]) that the line $\Lambda_{\mathcal{F}}(u_1(c))$ is nef on $M(c)$ the moduli space of Gieseker semi-stable sheaves with fixed numerical class $c \in K(X)_{\mathrm{num}}$ over a projective surface and the projective image of $\Lambda_{\mathcal{F}}(u_1(c))^{\otimes n}$ for $n \gg 1$ give rise to the Uhlenbeck Moduli space. So if one combine Donaldson's work [Do2] with the result of this section, one gets a GIT interpretation why the line bundle $\Lambda_{\mathcal{F}}(u_1(c))$ should lead to the Uhlenbeck compactification.

5. Digression

From the two applications, we have explored in the previous sections, it is natural to ask the following:

Question: If the line bundle CM-line over the Hilbert scheme give rise to a Uhlenbeck type of compactification of the moduli spaces. If it is so they will be a *smaller* compactification of the stable object. We hope to address this question in a future work.

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