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C^1-BOUNDARY REGULARITY OF PLANAR INFINITY HARMONIC FUNCTIONS

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Abstract. We prove that if $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2-boundary and $g \in C^2(\mathbb{R}^2)$, then any viscosity solution $u \in C(\overline{\Omega})$ of the infinity Laplacian equation (1.1) is $C^1(\overline{\Omega})$. The interior C^1 and $C^{1,\alpha}$-regularity of u in dimension two has been proved by Savin [20], and Evans and Savin [15], respectively. We also show that for any $n \geq 3$, if $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary and $g \in C^1(\mathbb{R}^n)$, then the solution u of equation (1.1) is differentiable on $\partial\Omega$. This can be viewed as a supplementary result to the much deeper interior differentiability theorem by Evans and Smart [16, 17].

1. Introduction

In 1960s, Aronsson [3] introduced the notion of the absolutely minimizing Lipschitz extension. Namely, $u \in W^{1,\infty}(\Omega)$ is said to be an *absolutely minimizing Lipschitz extension* in some bounded open subset $\Omega \subset \mathbb{R}^n$ if for any open set $V \subset \Omega$, we have that

$$\sup_{x \neq y \in \partial V} \frac{|u(x) - u(y)|}{|x - y|} = \sup_{x \neq y \in \overline{V}} \frac{|u(x) - u(y)|}{|x - y|}.$$

The results of Crandall et al. [13] imply that the above definition is equivalent to saying that for any open set $V \subset \Omega$ and $v \in W^{1,\infty}(V)$,

$$u|_{\partial V} = v|_{\partial V} \Rightarrow \|Du\|_{L^\infty(V)} \leq \|Dv\|_{L^\infty(V)}.$$

Jensen proved in [18] that $u \in W^{1,\infty}(\Omega)$ is an absolutely minimizing Lipschitz extension with a given Lipschitz continuous boundary data g iff u is a viscosity solution of the infinity Laplacian equation:

$$\begin{cases} \Delta_\infty u := \sum_{1 \leq i,j \leq n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Moreover, (1.1) has a unique viscosity solution with any given continuous boundary data. The reader can refer to Armstrong and Smart [2] for a nice new proof of Jensen's uniqueness theorem. After Jensen's celebrated work, there has been an explosion of interest in the infinity Laplacian equation and its generalizations. Two natural extensions include: (i) absolute minimal Lipschitz extensions with respect to more general metrics on $\mathbb{R}^n$ (see, e.g., [7]); and (ii) absolute minimizers of quasiconvex functions of the gradient (see, e.g., [1, 4–6, 9, 10]). We would like to mention beautiful connections between the infinity harmonic functions and the differential game theory first discovered by Peres et al. [19] and later by Barron et al. [8] for Aronsson's equations.

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Viscosity solutions of the infinity Laplacian equation (1.1) are also called *infinity harmonic functions*. One of the most important problems concerning infinity harmonic function is its C^1-regularity. When $n = 2$, this has been proved by Savin [20], and the $C^{1,\alpha}$-regularity was subsequently obtained by Evans and Savin [15]. Very recently, Evans and Smart [16, 17] made a breakthrough in dimensions $n \geq 3$ by showing that any infinity harmonic function is differentiable everywhere. While the continuity of gradient of u remains an open question.

In this short article, we will study the boundary regularity of infinity harmonic functions. We are able to prove

Theorem 1.1. *Suppose that* $\Omega \subset \mathbb{R}^2$ *is a bounded domain with* $\partial\Omega \in C^2$. *Assume that* $g \in C^2(\mathbb{R}^2)$ *and* $u \in C(\overline{\Omega})$ *is the viscosity solution of the infinity Laplacian equation (1.1). Then* $u \in C^1(\overline{\Omega})$. *Moreover, for any* $\delta > 0$, *there exists* $\epsilon_\delta > 0$ *depending only on* $||g||_{C^2(\mathbb{R}^2)}$ *and* $||\partial\Omega||_{C^2}$ *such that for* $x, y \in \overline{\Omega}$,

$$|x - y| \leq \epsilon_\delta \Rightarrow |Du(x) - Du(y)| \leq \delta. \tag{1.2}$$

Here $||\partial\Omega||_{C^2}$ is understood as follows: We say that $||\partial\Omega||_{C^2} \leq C < +\infty$, if there exist $0 < r_C < R_C < +\infty$ such that $\Omega \subset B_{R_C}(O)$ and for any $x = (x_1, x_2) \in \partial\Omega$, after suitable rotation, there exists $f^{(x)}(t) \in C^2(\mathbb{R})$ such that $||f^{(x)}||_{C^2(\mathbb{R})} \leq C$, $f^{(x)}(0) = \frac{d}{dt}f^{(x)}(0) = 0$ and for all $r \in (0, r_C)$

$$B_r(x) \cap \Omega = \{x\} + (B_r(O) \cap \{y = (y_1, y_2)|\ y_2 > f^{(x)}(y_1)\})$$

and

$$B_r(x) \cap \partial\Omega = \{x\} + (B_r(O) \cap \{y = (y_1, y_2)|\ y_2 = f^{(x)}(y_1)\}).$$

The C^2 assumption can actually be relaxed to $C^{1,1}$ and the above definition of norm is equivalent to saying that Ω has a uniform interior and exterior ball condition.

Sketch of the ideas of proof of Theorem 1.1: The C^2-regularities of both $\partial\Omega$ and g assure the existence of classical solutions of the eikonal equation: $|Du| = \textit{constant}$ near $\partial\Omega$, which serve as barrier functions. Using interior estimate established in [20] and routine scaling arguments, to prove Theorem 1.1, it suffices to show that u locally lies between two barrier functions that are C^1-close. One side bound comes easily from the method of characteristics. The proof for the other side bound is more tricky and we utilize some ideas of [20], but is simpler than [20]. The C^2-regularity assumption is necessary to implement the method of characteristics. It remains an interesting question whether Theorem 1.1 holds when g and $\partial\Omega$ are assumed to be C^1, a more natural assumption. It is also an interesting question to ask whether the $C^{1,\alpha}$-interior regularity by Evans and Savin [15] holds up to the boundary for infinity harmonic functions.

Using the tool of *comparison with cones* by Crandall et al. [13], we also establish the differentiability of infinity harmonic functions on the boundary in all dimensions.

Theorem 1.2. *For* $n \geq 2$, *let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* $\partial\Omega \in C^1$ *and* $g \in C^1(\mathbb{R}^n)$. *Assume that* u *is the viscosity solution of the infinity Laplacian equation (1.1). Then* u *is differentiable on the boundary, i.e., for any* $x_0 \in \partial\Omega$, *there exists* $Du(x_0) \in \mathbb{R}^n$ *such that*

$$u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + o(|x - x_0|) \quad \textit{for all } x \in \overline{\Omega}.$$

Remark 1.1. The interior differentiability of infinity harmonic functions in all dimensions has been proved by Evans and Smart [16]. It is not clear to us whether the C^1 assumption of g and $\partial\Omega$ in Theorem 1.2 can be relaxed to be everywhere differentiable. We need the continuity of the gradient of g and $\partial\Omega$ to derive (2.1) in the next section.

2. Boundary differentiability and proof of Theorem 1.2

In this section, we will assume that $\partial\Omega \in C^1$ and $g \in C^1(\mathbb{R}^n)$ and $u \in C(\overline{\Omega})$ is a viscosity solution of (1.1). We will prove the boundary differentiability Theorem 1.2.

For $x \in \overline{\Omega}$ and $r > 0$, we define

$$S_r^+(x) = \max_{y\in\partial(B_r(x)\cap\Omega)\setminus\{x\}} \frac{u(y)-u(x)}{|y-x|}$$

and

$$S_r^-(x) = \max_{y\in\partial(B_r(x)\cap\Omega)\setminus\{x\}} \frac{u(x)-u(y)}{|y-x|}.$$

By the comparison principle with cones as in [12,13], it is readily seen that both S_r^+ and S_r^- are monotone increasing functions of $r > 0$. Hence, for any $x \in \overline{\Omega}$, we have that

$$S^+(x) = \lim_{r\to 0} S_r^+(x) \quad \text{and} \quad S^-(x) = \lim_{r\to 0} S_r^-(x)$$

exist. Let

$$S(x) = \max\left\{S^+(x), S^-(x)\right\}.$$

Then it is standard that the following properties of $S(x)$ hold, whose proof is left to the readers. Note that by Evan and Smart [16,17], $Du(x)$ exists for all $x \in \Omega$.

Lemma 2.1. *(i) For $x \in \Omega$,*

$$S^+(x) = S^-(x) = S(x) = |Du(x)|.$$

(ii) For $x \in \partial\Omega$,

$$\min\{S^+(x), S^-(x)\} \geq |D_T g(x)|,$$

where $D_T g$ denotes the tangential gradient of g on $\partial\Omega$.
(iii) $S(x)$ is upper-semicontinuous, i.e.,

$$\limsup_{y\to x} S(y) \leq S(x) \quad \forall x \in \overline{\Omega}. \tag{2.1}$$

We first prove Aronsson's tightness property for infinity harmonic functions in $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : \ x_n \geq 0\}$, such a property was first proved by Crandall and Evans [13] for infinity harmonic functions in $\mathbb{R}^n$.

Lemma 2.2. *Suppose $w = w(x', x_n) \in W^{1,\infty}(\mathbb{R}^n_+)$ and*

$$|Dw(x)| \leq 1 \quad a.e.\ x \in \mathbb{R}^n_+.$$

Let $e = (e', e_n) \in \mathbb{R}^n$ be a unit vector with $e_n \geq 0$. Assume that $w(x', 0) = e' \cdot x'$ for all $x' \in \mathbb{R}^{n-1}$ and for $t > 0$ $w(te) = t$. Then $w(x) = e \cdot x$ for $x \in \mathbb{R}^n_+$.

Proof. For $t > 0$ and $x = (x', x_n) \in \mathbb{R}^n_+$, we have that

$$w(te) - w(x) \leq |te - x|$$

so that

$$w(x) \geq t - |te - x| = \frac{2e \cdot x - t^{-1}|x|^2}{1 + |e - t^{-1}x|}.$$

This, after taking $t \to +\infty$, implies

$$w(x) \geq e \cdot x, \quad \forall x \in \mathbb{R}^n_+.$$

It remains to show

$$w(x) \leq e \cdot x, \quad \forall x \in \mathbb{R}^n_+. \tag{2.2}$$

Case 1: $e_n = 0$. Then we have $-te \in \mathbb{R}^n_+$ and

$$w(x) \leq w(-te) + |x + te| = -t + |x + te|.$$

Hence

$$-w(x) \geq t - |x + te| = \frac{-2e \cdot x - t^{-1}|x|^2}{1 + |e + t^{-1}x|},$$

so that (2.2) follows by taking $t \to +\infty$.

Case 2: $e_n > 0$. Then we have that for any $x \in \mathbb{R}^n_+$,

$$w(x) \leq w\left(x' - \frac{x_n}{e_n}e', 0\right) + \left|\left(\frac{x_n}{e_n}e', x_n\right)\right| = e' \cdot x' - \frac{x_n}{e_n}|e'|^2 + \frac{x_n}{e_n} = e \cdot x.$$

This completes the proof. □

Proof of Theorem 1.2. Since $\partial\Omega \in C^1$, by suitable rotations and translations we may assume that $x_0 = 0 \in \partial\Omega$ and for some $r > 0$

$$\Omega \cap B_r(0) = \{(x', x_n) \in B_r(0) \mid x_n > f(x')\},$$

where $f \in C^1(\mathbb{R}^{n-1})$, $f(0) = 0$ and $Df(0) = 0$. Without loss of generality, we may assume that

$$S^+(0) \geq S^-(0)$$

so that $S(0) = \max\{S^+(0), S^-(0)\} = S^+(0)$. Our goal is to show that

$$Du(0) = p_0 := \left(D_T g(0), \sqrt{S^2(0) - |D_T g(0)|^2}\right). \tag{2.3}$$

Here $D_T g(0) = \left(\frac{\partial g}{\partial x_1}, \frac{g}{\partial x_2}, \ldots, \frac{\partial g}{\partial x_{n-1}}\right)(0)$ is the tangential gradient of g at $0 \in \partial\Omega$. If $S(0) = 0$, this follows immediately from Lemma 2.1. So we may assume after scalings that $S(0) = 1$. For $\lim_{m\to+\infty} \lambda_m = 0$, set $\Omega_m = \lambda_m^{-1}\Omega$ and define

$$u_m(x) = \frac{u(\lambda_m x) - g(0)}{\lambda_m}, \quad x \in \Omega_m.$$

Since $u_m(0) = 0$ and u is an absolute minimal Lipschitz extension, we have

$$\|Du_m\|_{L^\infty(\Omega_m)} = \|Du\|_{L^\infty(\Omega)} \leq \|Dg\|_{L^\infty(\Omega)}$$

so that[1]

$$\|u_m\|_{L^\infty(\Omega_m \cap B_R)} + \|Du_m\|_{L^\infty(\Omega_m)} \leq (1 + R)\|Dg\|_{L^\infty(\Omega)}, \quad \forall R > 0.$$

[1]If $Dg = 0$ (i.e., g is constant), then u is constant so that $u_m \equiv 0$. Hence we may assume $Dg \not\equiv 0$.

Since $\lim_{m\to\infty} \Omega_m = \mathbb{R}^n_+$, we may assume that $u_m \to w$ locally uniformly in $\mathbb{R}^n_+$. It is clear that

- $w \in W^{1,\infty}(\mathbb{R}^n_+)$ is an infinity harmonic function in $\mathbb{R}^{n-1} \times (0, +\infty)$,
- $w(x', 0) = D_T g(0) \cdot x'$ for $x' \in \mathbb{R}^{n-1}$,
-

$$|Dw|(x) \le S(0) = 1 \text{ a.e. } x \in \mathbb{R}^n_+. \tag{2.4}$$

We need to verify that

$$w(x) = p_0 \cdot x, \quad \forall x = (x', x_n) \in \mathbb{R}^n_+, \tag{2.5}$$

with p_0 given by (2.3).

Since $g \in C^1$, by the definition of S^+ there exists $r_0 > 0$ such that for any $0 < r \le r_0$ there exists $x_r \in \partial B_r \cap \overline{\Omega}$ such that

$$\lim_{r\to 0} \frac{u(x_r) - g(0)}{r} = S^+(0) = 1.$$

Note that if $|D_T g(0)| < 1$, we may in fact choose $x_r \in \partial B_r \cap \overline{\Omega}$ satisfying

$$\frac{u(x_r) - g(0)}{r} = S_r^+(0).$$

We now claim that for each $k \in \mathbb{N}$, there exists a unit vector $e_k = (e_k', (e_k)_n)$ with $(e_k)_n \ge 0$ such that

$$w(te_k) = t \quad \text{for } t \in [0, k]. \tag{2.6}$$

In fact, taking possible subsequences, we may assume that (for $r = k\lambda_m$)

$$\lim_{m\to+\infty} \frac{x_{k\lambda_m}}{k\lambda_m} = e_k.$$

Then $ke_k = \frac{x_{k\lambda_m}}{\lambda_m} + o(1)$ for $\lim_{m\to+\infty} o(1) = 0$. Hence

$$w(ke_k) = \lim_{m\to+\infty} \frac{u(x_{k\lambda_m}) - g(0)}{\lambda_m} = k.$$

This and (2.4) yield (2.6). After taking a subsequence if necessary, we assume that

$$\lim_{k\to+\infty} e_k = e$$

for a unit vector $e = (e', e_n)$ with $e_n \ge 0$. By (2.6), it is clear that

$$w(te) = t, \quad \forall t > 0.$$

Hence Lemma 2.2 implies $w(x) = e \cdot x$. Since $w(x', 0) = D_T g(0) \cdot x'$, we have $e' = D_T g(0)$. Combining with $e_n \ge 0$ and $|e| = 1$, we conclude that $e_n = \sqrt{1 - |D_T g(0)|^2}$ and hence (2.3) holds. This completes the proof. $\square$

3. C^1-boundary regularity and proof of Theorem 1.1

In this section, we will assume that $n = 2$, $\partial\Omega \in C^2$, $g \in C^2(\mathbb{R}^2)$, and $u \in C(\overline{\Omega})$ is a viscosity solution of (1.1). We will prove the C^1-boundary regularity Theorem 1.1.

Write $e = (e_1, e_2)$. Assume that $|e| = 1$ and $e_2 = \tau > 0$. For $\mu, \nu > 0$, let $B_{\mu,\nu}$ denote the parallelogram

$$B_{\mu,\nu} = \left\{ te + (s, 0) \Big| \; t \in \left[-\frac{1}{4}, \mu\right], s \in [-\nu, \nu] \right\}.$$

We assume that

$$\Omega = B_{1,1} \cap \{(x_1, x_2) |\; x_2 > f(x_1)\}, \quad \Gamma = \partial\Omega \cap \{(x_1, x_2) \in B_{1,1} \mid x_2 = f(x_1)\}$$

for a function $f \in C^2(\mathbb{R})$ and $f(0) = f'(0) = 0$. Let $O = (0, 0) \in \Gamma$. See figure 1 below.

Lemma 3.1. *Assume $|f'| \leq \epsilon$ and $e_2 = \tau > 0$. Suppose that $u \in C(\overline{\Omega})$ is infinity harmonic function in Ω satisfying that*
(i)

$$u = g \quad \text{on } \Gamma;$$

(ii)

$$|u(x) - e \cdot x| \leq \epsilon \quad \text{in } \overline{\Omega}.$$

Assume that $w \in C^1(\Omega) \cap C(\overline{\Omega})$ is a solution of

$$\begin{cases} |Dw| = 1 - \delta & \text{in } \Omega, \\ w = g & \text{on } \Gamma. \end{cases}$$

For any fixed $\delta, \tau > 0$, if ϵ is sufficiently small then we have that

$$u(x) \geq w(x) \quad \text{for } x \in \overline{\Omega} \cap B_{1,\frac{1}{4}}.$$

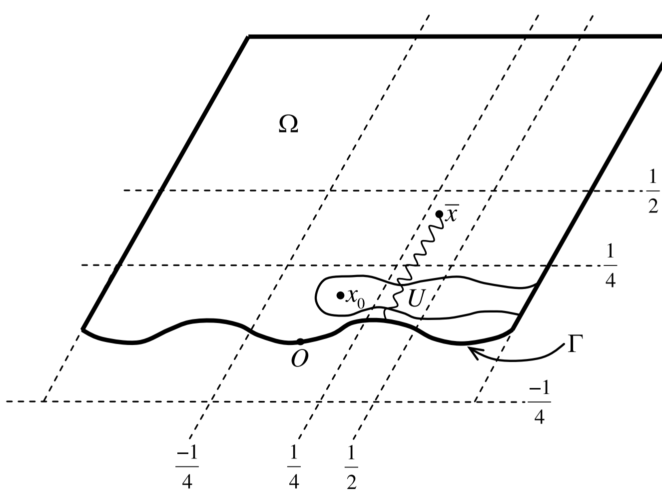

FIGURE 1. Proof of Lemma 3.1.

Proof. We argue by contradiction. Suppose that there exists $x_0 \in \Omega \cap B_{1,\frac{1}{4}}$ such that $u(x_0) < w(x_0)$. Note that when ϵ is small, within $B_{1,1}$, each line $x + te$ intersects the curve $\{x_2 = f(x_1)\}$ exactly once. Denote U as the connected component of $\{u < w\}$ containing x_0. Since $|w(te + x) - g(x)| \leq (1-\delta)t$ for $x \in \Gamma$ and $x + te \in \Omega$, it is clear that if ϵ is sufficiently small then

$$U \subset \Omega \cap B_{\frac{1}{4},1}.$$

See figure 1 above. Also, U should stretch all the way to $\partial\Omega \backslash \Gamma$ although $\partial U \cap \Gamma$ might not be empty. Without loss of generality, we assume

$$\partial U \cap \left\{ te + (1,0) \middle| t \in \left[-\frac{1}{4}, \frac{1}{4}\right] \right\} \neq \emptyset.$$

Let K be the line segment $\left\{(\frac{3}{8}, 0) + \lambda e : \ \lambda \in [\frac{1}{4}, \frac{1}{2}]\right\}$. According to (ii) , if ϵ is small enough, then there must exist $\bar{x} \in K$ such that

$$|Du(\bar{x})| > 1 - 10\epsilon.$$

Let $\xi(t) : (-T, 0] \to \Omega$ be a backward generalized gradient flow from $\bar{x}$, i.e., $\xi(0) = \bar{x}$, $\xi(-T) \in \partial\Omega$,

$$|Du(\xi(t))| \geq |Du(\bar{x})| \geq 1 - 10\epsilon, \quad -T \leq t \leq 0$$

and

$$u(\bar{x}) - u(\xi(t)) \geq \int_t^0 |\dot{\xi}(s)| \, ds \geq (1 - 10\epsilon)|\bar{x} - \xi(t)|, \quad -T \leq t \leq 0.$$

See [11] for the construction of ξ. Let S denote the strip bounded by two lines $L_1 = \frac{1}{4} + \lambda e$ and $L_2 = \frac{1}{2} + \lambda e$. According to (ii), when ϵ is small enough, the whole curve ξ must lie within the strip S and $\xi(-T) \in \Gamma$. Hence, there exists $t_0 \in (-T, 0)$ such that $\xi(t_0) \in S \cap U$. This leads a contradiction if we are able to establish the following claim.

Claim. If ϵ is sufficiently small, then

$$\sup_{x \in U \cap S} |Du(x)| \leq 1 - 12\epsilon.$$

In fact, we again argue by contradiction. Assume that there is a $\tilde{x} \in U \cap S$ such that

$$|Du(\tilde{x})| > 1 - 12\epsilon.$$

Let $\tilde{\xi}(t) : (-\tilde{T}, 0] \to U$ be a backward gradient flow from $\tilde{x}$ such that $\xi(-\tilde{T}) \in \partial U$. Since

$$u(\tilde{x}) - u(\tilde{\xi}(-\tilde{T})) \geq (1 - 12\epsilon) \int_{-\tilde{T}}^0 |\dot{\tilde{\xi}}(s)| \, ds,$$

we have that $u(\tilde{\xi}(-\tilde{T})) < w(\tilde{\xi}(-\tilde{T}))$ provided that $12\epsilon < \delta$. Hence $\tilde{\xi}(-\tilde{T}) \in \Big\{te + (1,0) | t \in [-\frac{1}{4}, \frac{1}{4}]\Big\}$. Then by (ii),

$$e \cdot (\tilde{x} - \tilde{\xi}(-\tilde{T})) \geq (1 - 12\epsilon)|\tilde{x} - \tilde{\xi}(-\tilde{T})| - 2\epsilon.$$

This is impossible provided that ϵ is small enough. □

Let f be the same function as in the statement of Lemma 3.1. Denote

$$\Sigma_t = B_t(O) \cap \{(x_1, x_2)|\ x_2 > f(x_1)\}.$$

and

$$\Gamma_t = \overline{B_t(O)} \cap \{(x_1, x_2)|\ x_2 = f(x_1)\}.$$

See figure 2 below.

Lemma 3.2. *Assume* $|f'| \leq \epsilon$, $|f''| \leq 1$ *and* $|g|_{C^2(\mathbb{R}^2)} \leq 1$. *Suppose that* u *is infinity harmonic in* Σ_1 *and* $u = g$ *on* Γ_1. *Assume that*

$$\max_{x\in\overline{\Sigma_1}} |u - e\cdot x| \leq \epsilon \text{ and } \max_{x\in\Gamma_1} |(Dg - e)_T| \leq \epsilon. \tag{3.1}$$

Here $(Dg - e)_T$ *denotes the tangential component of* $(Dg - e)$ *along the boundary* Γ_1. *Then for any* $\tau > 0$, *there exists* $\epsilon_{e,\tau} > 0$ *depending only on* e *and* τ *such that when* $\epsilon \leq \epsilon_{e,\tau}$,

$$|Du(x) - e| \leq \tau \quad \textit{for all } x \in \overline{\Sigma_{\frac{1}{2}}}. \tag{3.2}$$

Proof: When $\epsilon > 0$ is sufficiently small, $\partial B_t(O) \cap \{(x_1, x_2)|\ x_2 = f(x_1)\}$ contains exactly two points, for $t \in (0, 1]$. Due to (3.1) and $|f'| \leq \epsilon$, by comparison with cones (first on the boundary and then in the interior), it is easy to prove that

$$\sup_{\overline{\Sigma_{\frac{3}{4}}}} |Du(x)| \leq |e| + C\epsilon. \tag{3.3}$$

If $|e| = 0$, then (3.2) follows from (3.3) immediately. Now we assume $|e| = \mu > 0$.

Claim: Given $\delta > 0$, when $\epsilon(\leq \min\{\frac{\delta}{2}, \frac{\mu}{2}\})$ is small enough, there exists a positive constant $\hat{r} \in (0, \frac{1}{6})$ depending only on e and δ such that for any point $x \in \Gamma_{\frac{2}{3}}$, we can find two barrier functions $w_x^{\pm}(y) \in C^1(B_{\hat{r}}(x))$ satisfying

$$w_x^-(y) \leq u(y) \leq w_x^+(y) \quad \text{in } \overline{B_{\hat{r}}(x) \cap \Sigma_1} \tag{3.4}$$

and

$$\max\{|Dw_x^+(y) - e|, |Dw_x^-(y) - e|\} \leq 2\delta \quad \text{in } \overline{B_{\hat{r}}(x)}. \tag{3.5}$$

For simplicity, we will only prove this claim for $x = O = (0, 0)$ (the proof for other points can be done similarly). Since $f'(0) = 0$, $D_T g(O) = g_{x_1}(0)$. Denote $g_{x_1}(0) = s$ and $e = (e_1, e_2)$. Then by (3.1), $|s - e_1| \leq \epsilon$.

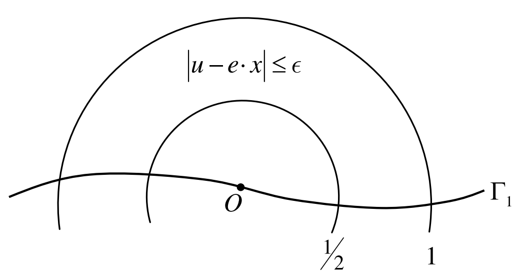

FIGURE 2. Uniform control.

Case 1: $e_2 = 0$. Then $|e_1| = \mu$. Choose ϵ small enough such that by (3.3),

$$\sup_{\overline{\Sigma_{\frac{3}{4}}}} |Du(x)| \le \sqrt{s^2+\delta^2}. \tag{3.6}$$

Using the method of characteristics (see [14] Chapter 3 for instance), there exist a simply connected open set V containing O such that $V^+ := V \cap \{x_2 > f(x_1)\} \subset \Sigma_{\frac{3}{4}}$ and two barrier functions $w^\pm \in C^2(V)$ that are classical solutions of the eikonal equation:

$$\begin{cases} |Dw^\pm| = \sqrt{s^2+\delta^2} & \text{in } V, \\ \quad w^\pm = g & \text{on } V \cap \Gamma_1 \end{cases}$$

subject to the condition: $Dw^\pm(O) = (g_{x_1}(O), \pm\delta) = (s, \pm\delta)$. Since $|s - e_1| \le \epsilon$, $|s| \le \mu + \delta$. We may choose $r_2 > 0$ depending only on μ and δ such that $\overline{B_{r_2}(O)} \subset V$. From the constructions of $w^\pm$, we have that

$$w^-(x) \le u(x) \le w^+(x) \quad \text{for } x \in B_{r_2}(O) \cap \overline{\Sigma_1}. \tag{3.7}$$

We will indicate the proof of the second inequality in (3.7) (the first inequality in (3.7) can be proved similarly). According to the method of characteristics, for any $x \in B_{r_2}(O) \cap \Sigma_1$, there exists a unique $y_x \in V \cap \Gamma_{\frac{3}{4}}$ and $t_x > 0$ such that

$$\xi(t_x) = x, \quad \xi(0) = y_x$$

and the characteristics $\xi : (0, t_x] \to V^+$ satisfies that

$$\dot{\xi}(t) = \frac{Dw^+(\xi(t))}{\sqrt{s^2+\delta^2}}.$$

Hence, by (3.6), we have

$$\frac{d}{dt}\Big(u(\xi(t)) - w^+(\xi(t))\Big) = \frac{Du(\xi(t)) \cdot Dw^+(\xi(t))}{\sqrt{s^2+\delta^2}} - \sqrt{s^2+\delta^2} \le 0, \ 0 \le t \le t_x.$$

This implies $u(x) \le w^+(x)$. We would like to point out that ξ is actually a straight line and

$$Dw^+(\xi(t)) \equiv D_T g(y_x)\tau(y_x) + n(y_x)\sqrt{s^2+\delta^2 - D_T^2 g(y_x)}.$$

Here $\tau(y_x) = \frac{(1, f'(y_{x1}))}{\sqrt{1+(f'(y_{x1}))^2}}$ is the unit tangential direction of Γ_1 at $y_x = (y_{x1}, y_{x2})$, $n(y_x) = \frac{(-f'(y_{x1}), 1)}{\sqrt{1+(f'(y_{x1}))^2}}$ is the inward normal vector of Γ_1 at y_x, and $D_T g(y_x) = Dg(y_x) \cdot \tau(y_x)$.

Case 2: $e_2 \ne 0$. Without loss of generality, we assume that $e_2 > 0$. For otherwise, we can consider $-u$ and $-e$. Let $0 < \delta < \frac{e_2}{2}$. When ϵ is small enough, by (3.3) we have

$$\sup_{\overline{\Gamma_{\frac{3}{4}}}} |Du(x)| \le \sqrt{s^2 + (e_2+\delta)^2}$$

and

$$\sqrt{s^2 + (e_2-\delta)^2} \le \sqrt{|e|^2 - \delta^2}.$$

Using the method of characteristics, there exist a simply connected open set V containing O such that $V^+ := V \cap \{x_2 > f(x_1)\} \subset \Sigma_{\frac{3}{4}}$ and two barrier functions $w^\pm$ on V which are classical solutions of

$$\begin{cases} |Dw^\pm| = \sqrt{s^2 + (e_2 \pm \delta)^2} & \text{in } V, \\ w^\pm = g & \text{on } V \cap \Gamma_1 \end{cases}$$

subject to the condition: $Dw^\pm(O) = (g_{x_1}(O), e_2 \pm \delta) = (s, e_2 \pm \delta)$. Since $|s| \le |e_1| + \epsilon \le \mu + \delta$, we may Choose $r_2 > 0$ depending only on e and δ such that $\overline{B_{r_2}(O)} \subset V$. From the construction of w^+, we have that

$$u(x) \le w^+(x) \quad \text{for } x \in B_{r_2}(O) \cap \overline{\Sigma_1}.$$

The proof is similar to that of (3.7). Moreover, let $\lambda \in (0,1)$ such that $B_{1,1} \subset B_{\frac{r_2}{\lambda}}(O)$ (see the definition of $B_{1,1}$ at the begin of this section), and consider $u_\lambda(x) = \frac{u(\lambda x) - u(O)}{\lambda}$, $x \in B_{1,1}$. Apply Lemma 3.1 to u_λ, $f_\lambda(t) = \frac{f(\lambda t)}{\lambda}$, $g_\lambda(x) = \frac{g(\lambda x) - g(O)}{\lambda}$, and $w_\lambda(x) = \frac{w^-(\lambda x) - w^-(O)}{\lambda}$, we conclude that when ϵ is small enough, there exists $0 < r_3 = \alpha r_2$ for some $\alpha \in (0,1)$ depending only on e and δ such that

$$u(x) \ge w^-(x) \quad \text{for } x \in B_{r_3}(O) \cap \overline{\Sigma_1}.$$

Hence

$$w^-(x) \le u(x) \le w^+(x) \quad \text{for } x \in B_{r_3}(O) \cap \overline{\Sigma_1}.$$

Note that $|D^\pm w(O) - e| \le \epsilon + \delta$. Also, the module of continuity of $Dw^\pm$ depends only on δ and e. Hence, we may choose $\hat{r} > 0$ depending only on δ and e such that the Claim holds.

Next let $W = \left\{ x \in \Sigma_{\frac{1}{2}} \mid d(x, \Gamma_{\frac{1}{2}}) \le \frac{\hat{r}}{2} \right\}$. When $x \in W$, (3.2) can be derived from our claim and Savin's interior estimate (see [20] Proposition 2) through routine scaling argument. For reader's convenience, we sketch it here. Fix $x_0 \in W$. Choose $y_0 \in \partial\Omega$ such that $|x_0 - y_0| = d(x_0, \partial\Omega) = r_0 < \frac{\hat{r}}{2} \le \frac{1}{12}$. Clearly, $y_0 \in \Gamma_{\frac{2}{3}}$. Denote

$$v(y) = \frac{u(y_0 + r_0(y - y_0)) - u(y_0)}{r_0}, \quad y \in B_1(\bar{x}_0).$$

Then v is an infinity harmonic function in $B_1(\bar{x}_0)$, here $\bar{x}_0 = y_0 + \frac{x_0 - y_0}{r_0}$. By (3.4) and (3.5), we have

$$|v(y) - e \cdot (y - y_0)| \le 4\delta \quad \text{for } y \in B_1(\bar{x}_0).$$

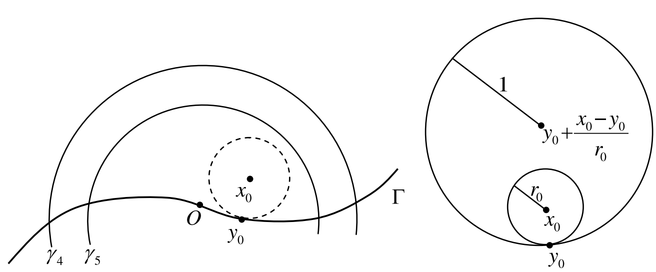

FIGURE 3. Rescaling argument along the boundary.

Let $\widetilde{v}(z) = v(\bar{x}_0 + z) + e \cdot y_0 - e \cdot \bar{x}_0$ for $z \in B_1(O)$. Then we have

$$|\widetilde{v}(z) - e \cdot z| \leq 4\delta, \ z \in B_1(O).$$

By Savin's interior estimate ([20] Proposition 2), for any given $\tau > 0$, if δ is chosen to be sufficiently small, we have that

$$|Du(x_0) - e| = |Dv(\bar{x}_0) - e| = |D\widetilde{v}(O) - e| \leq \tau.$$

If $x \in \Sigma_{\frac{1}{2}} \backslash W$, (3.2) follows immediately from Savin's interior estimate ([20] Proposition 2).

Proof of Theorem 1.1. It suffices to prove (1.2). We argue by contradiction. If it were false, then there would exist $\tau > 0$, a sequence of C^2 bounded domains Ω_m, boundary values $g_m \in C^2(\mathbb{R}^2)$, and infinity harmonic functions $u_m \in C(\overline{\Omega}_m)$, and two sequences of points $\{x_m\}$ and $\{y_m\}$ in $\overline{\Omega}_m$ such that

$$||g_m||_{C^2(\mathbb{R}^2)} \leq 1, \quad ||\Omega_m||_{C^2} \leq C, \tag{3.8}$$

$$|x_m - y_m| \leq \frac{1}{m} \quad \text{and} \quad |Du_m(x_m) - Du_m(y_m)| \geq 4\tau. \tag{3.9}$$

Upon taking possible subsequences, we may assume that there exist a bounded $C^{1,1}$ domain Ω (i.e., $\partial\Omega \in C^{1,1}$) and $g \in C^{1,1}(\mathbb{R}^2)$ such that $\Omega_m \to \Omega$ and $g_m \to g$ in C^1 as $m \to +\infty$. Due to Savin's interior estimate [20] or the $C^{1,\alpha}$ regularity in [15], x_m and y_m must converge to a point on $\partial\Omega$. Let us assume that

$$\lim_{m\to+\infty} x_m = \lim_{m\to+\infty} y_m = (0,0) = O \in \partial\Omega.$$

By suitable translations and rotations, we may assume that $O \in \partial\Omega_m$ and there exists some $r > 0$ such that for all $m \geq 1$

$$\Omega_m \cap B_r(O) = \{(y_1, y_2) \in B_r(O) \mid y_2 > f_m(y_1)\},$$

for some $f_m \in C^2(\mathbb{R})$, $f_m(0) = 0$, $f_m'(0) = 0$ and $||f_m||_{C^2(\mathbb{R})} \leq C$. Next, we suppose as $m \to \infty$,

$$u_m \to u \quad \text{uniformly in } C(\overline{\Omega}).$$

Here $u \in C(\overline{\Omega})$ is the infinity harmonic function satisfying $u = g$ on $\partial\Omega$. According to Theorem 1.2, u is differentiable at O. Denote $e = Du(0)$. For τ and e, let $\epsilon = \epsilon_{e,\tau}$ be the same number as in Lemma 3.2. Choose a positive number $\lambda_\epsilon < \min\{r, \epsilon\}$ such that

$$\left|\frac{u(\lambda_\epsilon x) - u(O)}{\lambda_\epsilon} - e \cdot x\right| \leq \frac{\epsilon}{2} \quad \text{for } x \in \lambda_\epsilon^{-1}(B_{\lambda_\epsilon}(O) \cap \Omega)$$

and

$$\left|(Dg - e)_T\right| \leq \frac{\epsilon}{2} \quad \text{for } x \in B_{\lambda_\epsilon}(O) \cap \partial\Omega.$$

Hence when m is large enough,

$$\left|\frac{u_m(\lambda_\epsilon x) - u_m(O)}{\lambda_\epsilon} - e \cdot x\right| \leq \epsilon \quad \text{for } x \in \lambda_\epsilon^{-1}(B_{\lambda_\epsilon}(O) \cap \Omega_m)$$

and

$$\left|(Dg_m - e)_T\right| \leq \epsilon \quad \text{for } x \in B_{\lambda_\epsilon}(O) \cap \partial\Omega_m.$$

Set $v_m(x) = \frac{u_m(\lambda_\epsilon x) - u_m(O)}{\lambda_\epsilon}$. Apply Lemma 3.2 to $\widetilde{u} = v_m$, $\widetilde{f}(t) = f_m(\lambda_\epsilon t)$ and $\widetilde{g} = \frac{g_m(\lambda_\epsilon x) - g_m(O)}{\lambda_\epsilon}$, we have that

$$|Du_m(\lambda_\epsilon x) - e| = |Dv_m(x) - e| \leq \tau \quad \text{in } x \in \lambda_\epsilon^{-1}\Big(B_{\frac{\lambda_\epsilon}{2}}(O) \cap \Omega_m\Big).$$

This contradicts to (3.9) when m is sufficiently large. The proof is no complete. □

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