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ON THE LOCAL AND GLOBAL EXTERIOR SQUARE L-FUNCTIONS OF GL_n

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Abstract. We show that the local exterior square L-functions of GL_n constructed via the theory of integral representations by Jacquet and Shalika coincide with those constructed by the Langlands–Shahidi method for square integrable representations (and for all irreducible representations when n is even). We also deduce several local and global consequences.

1. Introduction

Let F be a number field, v a place of F and F_v its completion. To any irreducible admissible representation π_v of $\mathrm{GL}_n(F_v)$ the local Langlands correspondence attaches an n-dimensional representation $\rho_{F_v}(\pi_v)$ of the Weil–Deligne group, when F_v is a p-adic field, and of the Weil group, when F_v is archimedean. The exterior square L-function of π_v is defined via this correspondence as a Galois L-function —

$$L(s, \pi_v, \wedge^2) := L(s, \wedge^2(\rho_{F_v}(\pi_v))),$$

and there have been different approaches to establishing its analytic properties. In [11], Jacquet and Shalika suggested that $L_{JS}(s, \pi_v, \wedge^2)$, defined as the "greatest common divisor" of certain local integrals denoted $J(s, W_v, \phi_v)$, when n is even, and $J(s, W_v)$, when n is odd, should yield the local exterior square L-function, when v is p-adic. At the archimedean places, $L_{JS}(s, \pi_v, \wedge^2)$ is defined via the local Langlands correspondence, as before.

On the other hand, the Langlands–Shahidi method also provides a potential construction for this L-function, which we denote by $L_{Sh}(s, \pi_v, \wedge^2)$, for any place v. The corresponding global L-functions are defined by

$$L_{JS}(s, \pi, \wedge^2) = \prod_v L_{JS}(s, \pi_v, \wedge^2) \quad \text{and} \quad L_{Sh}(s, \pi, \wedge^2) = \prod_v L_{Sh}(s, \pi_v, \wedge^2).$$

The main result of this paper is the following theorem.

Theorem 1.1. *If π_v is an irreducible, smooth, square integrable representation of $\mathrm{GL}_n(F_v)$, we have*

$$L_{JS}(s, \pi_v, \wedge^2) = L_{Sh}(s, \pi_v, \wedge^2). \tag{1.1}$$

When n is even, it is possible to express the exterior square L-function of an arbitrary irreducible generic representation in terms of L-functions of the inducing quasi-square integrable data (see [5]). This allows us to prove the following theorem.

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Theorem 1.2. *If π_v is an irreducible generic representation of $\mathrm{GL}_{2n}(F_v)$, we have*

$$L_{JS}(s, \pi_v, \wedge^2) = L_{Sh}(s, \pi_v, \wedge^2). \tag{1.2}$$

As an immediate corollary, we obtain the following global result.

Theorem 1.3. *Let $\pi = \otimes'_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, then*

$$L_{JS}(s, \pi, \wedge^2) = L_{Sh}(s, \pi, \wedge^2). \tag{1.3}$$

The analogous global theorem in the odd case is less satisfactory, but probably suffices for many applications.

The results of Henniart in [9] (see Theorem 4.3 in Section 4.2) show that the equality of the local and global Langlands–Shahidi L-functions with the corresponding local and global Galois L-functions. This enables us to deduce the following corollary.

Corollary 1.4. *Under the hypotheses of Theorems 1.1–1.3, we have*

$$L_{JS}(s, \pi_v, \wedge^2) = L(s, \pi_v, \wedge^2) \quad \text{and} \quad L_{JS}(s, \pi, \wedge^2) = L(s, \pi, \wedge^2).$$

Several other pleasant consequences, both local and global, follow from the equalities of L-functions established above. The analytic properties of $L_{Sh}(s, \pi, \wedge^2)$ – entireness, the functional equation and boundedness in vertical strips — have been established by Shahidi, Kim and Gelbart–Shahidi in a series of papers ([6, 14, 15, 17]) in many cases. Theorem 1.3 of this paper, allows us to show that $L_{JS}(s, \pi, \wedge^2)$ has the same properties. Our proof of the holomorphy results is different from the one due to Belt (see Theorem 5.2 of [3]), which excludes the ramified places. The functional equation and boundedness in vertical strips are new results for $L_{JS}(s, \pi, \wedge^2)$. On the other hand, Belt's global theorem, for the case when n is even and π self-dual, allows us to deduce the corresponding analytic behaviour of $L_{Sh}(s, \pi, \wedge^2)$. The analytic properties of the global exterior square L-functions are discussed in Corollaries 7.3–7.5 in Section 7 of this paper.

There are also consequences for the local L-functions. We are able to give a characterization of self-dual square integrable representations in terms of the existence of a pole for the local symmetric square L-function in Corollary 7.1. A comparison of the two global L-functions also enables us to push the theory of the local exterior square L-functions via integral representations a little further. It is now possible to define a local ϵ-factor and establish a local functional equation for square integrable representations, in fact, for any generic representation occurring as a local constituent of a cuspidal automorphic representation in the even case (see Theorem 8.1). This is a somewhat indirect method of obtaining the functional equation — indeed the local functional equation has no direct analogue in the Langlands–Shahidi constructions. We hope to find a proof within the local theory of integral representations in the future, which will work for all irreducible representations.

Theorem 1.1 is actually proved by global methods by using techniques similar to those used (more recently) in [1, 9], but we need to work somewhat harder since the local theory is not as complete in our case. In particular, we note that although the integral representation yields the L-function at the unramified places for a suitable choice of test vector, it is not known whether this choice yields the generator of the relevant fractional ideal (that is, whether it is the "greatest common divisor" of the local integrals). Indeed, this last fact only follows after we have proved Theorem 1.2 in the even case, and remains an open problem in the odd case.

The crucial new inputs are the recent holomorphy and non-vanishing results of the first author (see Theorem M of [13]) at the finite places, and a non-vanishing result of Dustin Belt (Theorem 2.2 of [3]) at the archimedean places. The idea is to embed the square integrable representation as the local component of a cuspidal automorphic representation. One then takes the quotient of the global integral of Jacquet and Shalika by $L_{Sh}(s, \pi, \wedge^2)$ to obtain a quotient of finitely many local factors. For suitable choices of local data, we can show that the quotient of the non-archimedean factors must be entire and non-vanishing. This last argument is dependent on some slightly finer local analysis, involving the local ϵ-factors, which must be suitably defined in our context. Once this is done, arguments involving the locations of the possible poles allow us to conclude that the relevant quotient is identically 1, yielding the equality of the two L-functions.

2. Notation and preliminaries

Throughout this paper F will be a number field, v a place of F and F_v its completion at the place v. Let $\mathbb{A}_F$ denote the ring of adèles over F. Let $|x_v|_v$ denote the absolute value of an element x_v of F_v and q_v be the cardinality of the residue field of F_v. If x is in $\mathbb{A}_F$ and the x_v are its local components, $|x|$ denotes the product $\prod_v |x_v|_v$ of the local absolute values. We denote by $\mathcal{S}(F_v^n)$ (resp. $\mathcal{S}(\mathbb{A}_F^n)$) the space of Schwartz–Bruhat functions on F_v^n (resp. $\mathbb{A}_F^n$). For $\phi_v \in \mathcal{S}(F_v^n)$ (resp. $\phi \in \mathcal{S}(\mathbb{A}_F^n)$), we denote by $\hat{\phi}_v$ (resp. $\hat{\phi}$) the Fourier transform of ϕ_v (resp. ϕ).

We let G stand for the group GL_n. The F points of G will be denoted $\mathrm{G}(F)$, its F_v points by G_v and its $\mathbb{A}_F$ points by G. We follow this convention, whenever convenient, for all the algebraic groups defined over F that arise in this paper. We let N be the standard maximal unipotent subgroup, that is, the subgroup of G consisting of upper triangular matrices with 1 in each diagonal entry, and we let Z be the centre of G. We will often need to consider the groups GL_r, when $r = 2n$ and $r = 2n + 1$. In this case, we denote the standard maximal unipotent subgroup by N_r. Let M be the space of all $n \times n$ matrices and V the subspace of all upper triangular $n \times n$ matrices. We define the elements

$$w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \quad \text{and} \quad w_{n,n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

in G_v and $GL_{2n}(F_v)$, respectively.

Let ψ_v be a non-trivial additive character of F_v. We may view ψ_v as a character of N_v by setting $\psi_v(n) = \psi_v\Big(\sum_{i=1}^{n-1} n_{i,i+1}\Big)$ for $n \in N_v$. Similarly, a global additive character ψ of $\mathbb{A}_F$ can be viewed as a character of N. For a representation π_v of G_v on a vector space U, let $U^*_{\psi_v}$ be the space of all linear forms on U satisfying $\lambda(\pi_v(n)v) = \psi_v(n)\lambda(v)$, for all $v \in U$ and $n \in N_v$. If $\dim U^*_{\psi_v} = 1$, we denote by

$\mathcal{W}(\pi_v, \psi_v)$ the Whittaker model of π_v, defined by

$$\mathcal{W}(\pi_v, \psi_v) = \{W_v : G_v \to \mathbb{C} \mid W_v(g) = \lambda(\pi_v(g)v)\},$$

where $v \in U$ and $\lambda \in U_\psi^*$. Note that G_v acts on $\mathcal{W}(\pi_v, \psi_v)$ by right translation and we have $W_v(ng) = \psi_v(n)W_v(g)$ for $n \in N_v$ and $g \in G_v$. We say that a representation π_v is generic if it is irreducible and $\dim U_{\psi_v}^* = 1$.

If (π_v, U) is a representation of G_v, $(\tilde{\pi}_v, \tilde{U})$ will denote its contragredient representation. If π_v has a Whittaker model and $W_v \in \mathcal{W}(\pi_v, \psi_v)$, we define the function $\tilde{W}_v$ on G_v by $\tilde{W}_v(g) = W_v(w_n^\iota g)$, where ${}^\iota g = {}^t g^{-1}$. The Whittaker model of $\tilde{\pi}_v$, $\mathcal{W}(\tilde{\pi}_v, \bar{\psi}_v)$, consists precisely of the set of functions $\{\tilde{W} \mid W \in \mathcal{W}(\pi_v, \psi_v)\}$.

Let ω_{π_v} be the central character of π_v, if it exists. We say that an irreducible smooth representation (π_v, U) of G_v is square integrable if its central character is unitary and

$$\int_{Z_v \backslash G_v} |f(\pi(g)u)|^2 dg < \infty,$$

for all $u \in U$ and $f \in \tilde{U}$. A smooth irreducible representation π of G_v is called quasi-square integrable if it becomes square integrable after twisting by a suitable quasi-character of G_v.

3. The integral representation of Jacquet and Shalika

3.1. The local theory. Let π_v be an irreducible generic representation of $\mathrm{GL}_r(F_v)$. In [11] Jacquet and Shalika give an integral representation for the exterior square L-function $L(s, \pi_v, \wedge^2)$, using certain families of integrals. If r is even, say $r = 2n$, we let

$$J(s, W_v, \phi_v) = \int_{N_v \backslash G_v} \int_{V_v \backslash M_v} W_v\left(\sigma \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right) \psi_v(-\operatorname{Tr} X) dX \phi_v(e_n g) |\det g|_v^s dg, \tag{3.1}$$

for each $W_v \in \mathcal{W}(\pi_v, \psi_v)$ and ϕ_v in $\mathcal{S}(F_v^n)$, where s is in $\mathbb{C}$, and σ is the permutation given by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n & | & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & | & 2 & 4 & \cdots & 2n \end{pmatrix}.$$

If r is odd, say $r = 2n+1$, we consider

$$J(s, W_v) = \int_{N_v \backslash G_v} \int_{V_v \backslash M_v} W_v\left(\sigma \begin{pmatrix} I_n & X & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \psi_v(-\operatorname{Tr} X) dX |\det g|_v^{s-1} dg, \tag{3.2}$$

for each W_v in $\mathcal{W}(\pi_v, \psi_v)$, where σ is the permutation given by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n & | & n+1 & n+2 & \cdots & 2n & 2n+1 \\ 1 & 3 & \cdots & 2n-1 & | & 2 & 4 & \cdots & 2n & 2n+1 \end{pmatrix}.$$

Combining Proposition 1 of Section 7 and Proposition 3 of Section 9 of [11], we can state the following result.

Proposition 3.1. *Let π_v be an irreducible unitary generic representation of* $\mathrm{GL}_r(F_v)$*. There exists $\eta > 0$ such that the integrals $J(s, W_v, \phi_v)$ (resp. $J(s, W_v)$) converge absolutely for $Re(s) > 1 - \eta$.*

We can use Proposition 2 of Section 7 and Proposition 4 of Section 9 of [11] to obtain the following proposition for unramified representations.

Proposition 3.2. *Suppose that F_v is a p-adic field and that π_v is an unramified representation of* $\mathrm{GL}_r(F_v)$*. If r is even (resp. odd) we can choose $W_v^0 \in \mathcal{W}(\pi_v, \psi_v)$ and $\phi_v^0 \in \mathcal{S}(F_v^n)$ (resp. $W_v^0 \in \mathcal{W}(\pi_v, \psi_v)$) such that*

$$J(s, W_v^0, \phi_v^0) = L(s, \pi_v, \wedge^2) \quad (\textit{resp.} \quad J(s, W_v^0) = L(s, \pi_v, \wedge^2)).$$

In both the p-adic and archimedean cases it is not hard to see that the integrals above can be meromorphically continued to the whole of $\mathbb{C}$.

Let F_v be a p-adic field. It is easy to see from the proof of the above theorem that the integrals $J(s, W_v, \phi_v)$ (resp. $J(s, W_v)$) are rational functions in q_v^{-s} (see Proposition 2.3 of [13]). For g in G_v, define elements g_1 in $\mathrm{GL}_{2n}(F_v)$ and g_2 in $\mathrm{GL}_{2n+1}(F_v)$ as follows:

$$g_1 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

$$J(s, \pi_v(g_1)W_v, R(g)\phi_v) = |\mathrm{det} g|_v^{-s} J(s, W_v, \phi_v),$$

where R denotes the right translation action of G_v on $\mathcal{S}(F_v^n)$, and

$$J(s, \pi_v(g_2)W_v) = |\mathrm{det} g|_v^{-s} J(s, W_v).$$

This shows that the $\mathbb{C}$-vector space $\mathcal{I}(\pi_v)$ generated by the integrals $J(s, W_v, \phi_v)$ (resp. $J(s, W_v)$) in $\mathbb{C}(q_v^{-s})$ is a fractional ideal of $\mathbb{C}[q_v^{-s}, q_v^s]$. Since $\mathbb{C}[q_v^{-s}, q_v^s]$ is a principal ideal domain, the fractional ideal $\mathcal{I}(\pi_v)$ is a principal fractional ideal. We now invoke the following theorem of Belt (see Theorem 2.2 of [3]).

Theorem 3.3. *Let v be any place of F. For each $s_0 \in \mathbb{C}$, there exist W_v in $\mathcal{W}(\pi_v, \psi_v)$ and ϕ_v in $\mathcal{S}(F_v^n)$ (resp. W_v in $\mathcal{W}(\pi_v, \psi_v)$) such that $J(s_0, W_v, \phi_v) \neq 0$ (resp. $J(s_0, W_v) \neq 0$).*

This theorem extends to arbitrary s_0 an earlier result of Jacquet and Shalika for $s_0 = 1$ (see [11]). Using the above theorem we see that 1 lies in $\mathcal{I}(\pi_v)$. As a result we can make the following definition.

Definition 3.4. *We define the exterior square L-function $L_{JS}(s, \pi_v, \wedge^2)$ as the generator of $\mathcal{I}(\pi_v)$ of the form $L_{JS}(s, \pi_v, \wedge^2) = \frac{1}{P(q_v^{-s})}$, where $P(q_v^{-s})$ is a polynomial in $\mathbb{C}[q_v^{-s}]$ and $P(0) = 1$.*

Remark 3.5. As remarked before, although Proposition 3.2 allows us to choose local data so that the integral representation gives the L-function $L(s, \pi_v, \wedge^2)$ when π_v is unramified, it is by no means clear that this choice yields the L-function $L_{JS}(s, \pi_v, \wedge^2)$ defined above. Indeed, it is only after we prove Theorem 1.2 that we will be able to establish this fact, and even then, only in the even case. In the odd case, we are unable to establish this in this paper.

Remark 3.6. We will also use Belt's result in the archimedean case, but not to define the L-function. We emphasize that in this paper, the archimedean L-function $L_{JS}(s, \pi_v, \wedge^2)$ is defined via the local Langlands correspondence.

If F_v is a p-adic field and π_v is square integrable Theorem M of [13] goes further.

Theorem 3.7. *Let π_v be an irreducible smooth square integrable representation of $\mathrm{GL}_r(F_v)$, where F_v is a p-adic field. Then the L-function $L_{JS}(s, \pi_v, \wedge^2)$ is regular in the region $Re(s) > 0$, if r is even, and in the region $Re(s) \geq 0$, if r is odd.*

Remark 3.8. In Theorem N of [13], the first author also proved the non-vanishing of the local integrals $J(s, W_v, \phi_v)$ and $J(s, W_v)$ in $\mathrm{Re}(s) > 0$ for square integrable representations over a p-adic field.

If F_v is an archimedean local field, we can extract the following proposition from [11], or from the more explicit calculation in [3] (see Proposition 3.4 and the proof of Theorem 2.2 in Section 3.4).

Proposition 3.9. *Let a and b be real numbers. There is a finite set of points $P(a, b)$ in the strip $a \leq Re(s) \leq b$ (independent of the choice of W_v and ϕ_v) such that the set of poles of the integrals $J(s, W_v, \phi_v)$ (resp. $J(s, W_v)$) is contained in $P(a, b)$.*

Proof. It is easy to see from the proof of Proposition 1 of Section 7 and Proposition 3 of Section 9 of [11] that the integral $J(s, W_v, \phi_v)$ (resp. $J(s, W_v)$) is a finite sum of products of entire functions and Tate integrals. The exponents of the quasi-characters occurring in the Tate integrals are finite in number and are independent of the choice of W_v and ϕ_v by Proposition 6 of [11]. Since these exponents determine the poles of Tate integrals, and since the latter have at most a finite number of poles in any vertical strip (page 155 of [11]), the existence of the finite set $P(a, b)$ follows. $\square$

3.2. The global theory. As before, let F be a number field. Let Φ be a function in $\mathcal{S}(\mathbb{A}_F^n)$, the space of Schwartz–Bruhat functions on $\mathbb{A}_F^n$. We denote by $\mathrm{P}_{n-1,n}$ the parabolic subgroup of type $(n-1, 1)$ in G. Let π be a unitary cuspidal automorphic representation of GL_r. We denote by ω_π the central character of π. For a non-trivial additive character ψ of $\mathbb{A}_F/F$ and a form φ in the space of π, we consider, when $r = 2n$, the integral

$$I(s, \varphi, \Phi) = \int_{\mathrm{G}(F)\backslash G/Z)} \int_{M(F)\backslash M} \varphi\left(\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right) \psi(\mathrm{Tr}\, X) dX\, E(g, \Phi, s) dg, \tag{3.3}$$

where $E(g, \Phi, s)$ is the Eisenstein series

$$E(g, \Phi, s) = \sum_{\gamma \in \mathrm{P}_{n-1,n}(F)\backslash \mathrm{G}(F)} f(\gamma g, s),$$

with
$$f(g,s) = |\det g|^s \int_{\mathbb{A}_F^\times} \Phi(e_n a g)|a|^{ns}\omega_\pi(a)da.$$

If $r = 2n+1$, consider

$$I(s,\varphi) = \int_{G(F)\backslash G}\int_{F^n\backslash \mathbb{A}_F^n}\int_{M(F)\backslash M} \varphi\left(\begin{pmatrix} I_n & X & Y \\ 0 & I_n & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1\end{pmatrix}\right) \psi(\operatorname{Tr} X)dX dY |\det g|^{s-1}dg. \tag{3.4}$$

Jacquet and Shalika have shown that the integral $I(s,\varphi)$ converges absolutely for all s (see Proposition 1 of Section 9 of [11]). They have also shown that the integral $I(s,\varphi,\Phi)$ converges for all s except at the singularities of the Eisenstein series (see Section 5 of [11]). Lemma 4.2 of [10] shows that the following theorem holds.

Theorem 3.10. *The Eisenstein series $E(g,\Phi,s)$ is absolutely convergent for $Re(s) > 1$. It has a meromorphic continuation to the entire complex plane and satisfies the functional equation*
$$E(g,\Phi,s) = E({}^\iota g,\hat{\Phi},1-s).$$

From the above theorem and the proof of Proposition 1 of Section 9 of [11] (see page 220), we get the following theorem.

Theorem 3.11. *The integrals* (3.3) *and* (3.4) *satisfy the functional equations*
$$I(s,\varphi,\Phi) = I(1-s,\rho(w_{n,n})\tilde{\varphi},\hat{\Phi})$$
and
$$I(s,\varphi) = I(1-s,\varphi'),$$
where $\tilde{\varphi}(g) = \varphi({}^\iota g)$, φ' is a suitable translate of $\tilde{\varphi}$, and ρ denotes the right translation action.

We now define global analogues of the local integrals appearing in Section 3.1. Let
$$W_\varphi(g) = \int_{N_r(F)\backslash N_r} \varphi(ug)\psi(u)du$$
be the Whittaker function associated to φ, where, as before, we view ψ as a character of N_r by setting $\psi(u) = \prod_{j=1}^{r-1}\psi(u_{j,j+1})$. If $r = 2n$, consider
$$J(s,W_\varphi,\Phi) = \int_{N\backslash G}\int_{V'\backslash M} W_\varphi\left(\sigma\begin{pmatrix} I_n & X \\ 0 & I_n\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g\end{pmatrix}\right) \psi(\operatorname{Tr} X)dX\,\Phi(e_n g)|\det g|^s dg,$$
where V$'$ is the subspace of strictly upper triangular matrices in M and σ is the permutation given by
$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n & | & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & | & 2 & 4 & \cdots & 2n\end{pmatrix}.$$

If $r = 2n+1$, consider

$$J(s, W_\varphi) = \int_{N\backslash G}\int_{V'\backslash M} W_\varphi\left(\sigma\begin{pmatrix} I_n & X & 0\\ 0 & I_n & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} g & 0 & 0\\ 0 & g & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$
$$\psi(\mathrm{Tr}\, X)dX|\det g|^{s-1}dg,$$

where σ is the permutation given by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n & | & n+1 & n+2 & \cdots & 2n & 2n+1\\ 1 & 3 & \cdots & 2n-1 & | & 2 & 4 & \cdots & 2n & 2n+1\end{pmatrix}.$$

We can combine Proposition 5 of Section 6 and Proposition 2 of Section 9 of [11] to get the following proposition.

Proposition 3.12. *For* $\mathrm{Re}(s)$ *sufficiently large we obtain the following equalities.*

(1) *The integral* $J(s, W_\varphi, \Phi)$ *converges absolutely and*
$$I(s, \varphi, \Phi) = J(s, W_\varphi, \Phi).$$
(2) *The integral* $J(s, W_\varphi)$ *converges absolutely and*
$$I(s, \varphi) = J(s, W_\varphi).$$

The global integrals are easily related to the local integrals for decomposable vectors. If $W_\varphi = \prod_v W_v$ and $\Phi = \prod_v \Phi_v$, where v runs over all the absolute values of F, then for $\mathrm{Re}(s)$ sufficiently large, we have

$$J(s, W_\varphi, \Phi) = \prod_v J(s, W_v, \Phi_v) \quad \text{and} \quad J(s, W_\varphi) = \prod_v J(s, W_v).$$

4. The Langlands–Shahidi method

4.1. The local theory. Let $L_{Sh}(s, \pi_v, \wedge^2)$ be the exterior square L-function defined by the Langlands–Shahidi method. For a tempered representation π_v of $\mathrm{GL}_r(F_v)$ over a p-adic field F_v, the L-function $L_{Sh}(s, \pi_v, \wedge^2)$ is defined as the inverse of a certain unique polynomial $P(q_v^{-s})$ in q_v^{-s} satisfying $P(0) = 1$, and such that $P(q_v^{-s})$ is the numerator of a certain gamma factor $\gamma(s, \pi_v, r_i, \psi_v)$ defined in [17]. We refer to [17] for the precise definition of $L_{Sh}(s, \pi_v, \wedge^2)$.

Proposition 4.1. [17, Proposition 7.2] *If* π_v *is a tempered representation of* $\mathrm{GL}_r(F_v)$, *where* F_v *is a* p*-adic field, the local* L*-function* $L_{Sh}(s, \pi_v, \wedge^2)$ *is holomorphic in the region* $Re(s) > 0$.

If F_v is an archimedean local field the L-functions are known to have the following form.

Proposition 4.2. *The* L*-function* $L_{Sh}(s, \pi_v, \wedge^2)$ *is a product of gamma functions of the form* $c\pi^{-s/2}\Gamma(\frac{s+b}{2})$ *for constants* $c \neq 0$ *and* b *in* $\mathbb{C}$.

When v is a finite place of F, Henniart has shown that the Langlands–Shahidi and Galois L-functions are equal (this was already known for the finite unramified places by [15]). Combining this with a result of Shahidi for the archimedean places in [16], we can state the following theorem.

Theorem 4.3. *Let π_v be a smooth irreducible representation of $\mathrm{GL}_r(F_v)$. Then*

$$L_{Sh}(s, \pi_v, \wedge^2) = L(s, \pi_v, \wedge^2).$$

4.2. The global theory. Let $\pi = \otimes'_v \pi_v$ be a unitary cuspidal representation of GL_r. We define the completed (global) L-function as

$$L_{Sh}(s, \pi, \wedge^2) = \prod_v L_{Sh}(s, \pi_v, \wedge^2),$$

where v runs over all the absolute values of F. Combining Propositions 3.1 and 3.4 of [14] we obtain:

Proposition 4.4. *The L-function $L_{Sh}(s, \pi, \wedge^2)$ is holomorphic for $Re(s) > 1$.*

Combining Theorem 3.5 and Proposition 3.6 of [14]) gives

Theorem 4.5. *If n is even and π is non-self-dual, or if n is odd, the L-function $L_{Sh}(s, \pi, \wedge^2)$ is entire.*

In Theorem 7.7 of [17] Shahidi shows that the L-function $L_{Sh}(s, \pi, \wedge^2)$ satisfies a functional equation.

Theorem 4.6. *The L-function $L_{Sh}(s, \pi, \wedge^2)$ admits a meromorphic continuation to the entire complex plane and satisfies a functional equation*

$$L_{Sh}(s, \pi, \wedge^2) = \epsilon_{Sh}(s, \pi, \wedge^2) L_{Sh}(1 - s, \tilde{\pi}, \wedge^2), \tag{4.1}$$

where the function $\epsilon_{Sh}(s, \pi, \wedge^2)$ is entire and non-vanishing, and $\tilde{\pi}$ denotes the representation contragredient to π.

5. A lemma about ϵ-factors

In this section we prove a lemma (Lemma 5.2) about proportionality factors that appear when relating the local exterior square L-function of the contragredient representation to the integral representation involving Whittaker functions "dual" to the spherical vector. As we will see in Section 8, these proportionality factors are the ϵ-factors that appear in the local functional equation.

Let $\pi = \otimes'_v \pi_v$ be a unitary cuspidal automorphic representation of GL_r. Let S_∞ denote the set of archimedean places of F and S_r the set of finite places v for which π_v is not unramified. We set $S = S_\infty \cup S_r$. With the notation of Proposition 3.2 and using Theorem 4.3, we have the following proposition which yields the equality of the L-functions for $v \notin S$.

Proposition 5.1. *Let π_v be an unramified representation of a p-adic field F_v. Then*

$$J(s, W_v^0, \phi_v) \text{ (resp. } J(s, W_v^0)) = L_{Sh}(s, \pi_v, \wedge^2) = L(s, \pi_v, \wedge^2). \tag{5.1}$$

Let $v_0 \in S_r$. Since the local L-function is defined as a generator of $\mathcal{I}(\pi_{v_0})$, there exist Whittaker functions W_{i,v_0} and Schwartz–Bruhat functions ϕ_{i,v_0} such that

(5.2)
$$L_{JS}(s,\pi_{v_0},\wedge^2)=\sum_{i=1}^{n_{v_0}} J(s,W_{i,v_0},\phi_{i,v_0}) \text{ (resp. } L_{JS}(s,\pi_{v_0},\wedge^2)=\sum_{i=1}^{n_{v_0}} J(s,W_{i,v_0})).$$

Applying the same reasoning to the contragredient representation $\tilde{\pi}_{v_0}$, we get

$$\sum_{i=1}^{n_{v_0}} J(s,\rho(w_{n,n})\tilde{W}_{i,v_0},\hat{\phi}_{i,v_0}) = M_1(q_{v_0}^{-s})L_{JS}(s,\tilde{\pi}_{v_0},\wedge^2) \tag{5.3}$$

(resp. $\sum_{i=1}^{n_{v_0}} J(s,\tilde{W}_{i,v_0}) = M_1(q_{v_0}^{-s})L_{JS}(s,\tilde{\pi}_{v_0},\wedge^2)$), where $M_1(X)$ is a polynomial in $\mathbb{C}[X,X^{-1}]$. Similarly, there exist W'_{i,v_0} and ϕ'_{i,v_0} such that

$$L_{JS}(s,\tilde{\pi}_{v_0},\wedge^2)=\sum_{i=1}^{m_{v_0}} J(s,\rho(w_{n,n})\tilde{W}'_{i,v_0},\hat{\phi}'_{i,v_0}) \tag{5.4}$$

and

$$\sum_{i=1}^{m_{v_0}} J(s,W'_{i,v_0},\phi'_{i,v_0}) = M_2(q_{v_0}^{-s})L_{JS}(s,\pi_{v_0},\wedge^2) \tag{5.5}$$

(resp. $\sum_{i=1}^{m_{v_0}} J(s,W'_{i,v_0}) = M_2(q_{v_0}^{-s})L_{JS}(s,\pi_{v_0},\wedge^2)$), where $M_2(X)$ is a polynomial in $\mathbb{C}[X,X^{-1}]$. In what follows, by a monomial in $\mathbb{C}[X,X^{-1}]$ we will mean a polynomial of the form cX^m, with $m\in\mathbb{Z}$.

Lemma 5.2. *The polynomials M_1 and M_2 are monomials in $q_{v_0}^{-s}$.*

Proof. We will give the proof only for the case $r=2n$, the odd case being entirely analogous (and simpler). For $v\in S_r \smallsetminus \{v_0\}$, we make a specific choice of W_v and ϕ_v such that the integrals $J(s,W_v,\phi_v)$ are not identically zero. For $v\in S_\infty$ we may take W_v and ϕ_v arbitrary, again such that $J(s,W_v,\phi_v)$ is not identically zero. Let

$$W_i = W_{i,v_0}\cdot \prod_{v\in S\smallsetminus\{v_0\}} W_v \cdot \prod_{v\notin S} W_v^0 \quad\text{and}\quad \Phi_i=\phi_{i,v_0}\cdot \prod_{v\in S\smallsetminus\{v_0\}}\phi_v\cdot\prod_{v\notin S}\phi_v^0.$$

Let

$$F_1(s,\pi)=\sum_{i=1}^{n_{v_0}} J(s,W_i,\Phi_i) \quad\text{and}\quad F_2(s,\tilde{\pi})=\sum_{i=1}^{n_{v_0}} J(s,\rho(w_{n,n})\tilde{W}_i,\hat{\Phi}_i).$$

Using (5.2) and (5.3) respectively, we obtain

$$F_1(s,\pi)=L_{JS}(s,\pi_{v_0},\wedge^2)\cdot\prod_{v\in S\smallsetminus\{v_0\}} J(s,W_v,\phi_v)\cdot\prod_{v\notin S} J(s,W_v^0,\phi_v^0) \tag{5.6}$$

and

(5.7)
$$F_2(s,\tilde{\pi})=M_1(q_{v_0}^{-s})L_{JS}(s,\tilde{\pi}_{v_0},\wedge^2)\cdot\prod_{v\in S\smallsetminus\{v_0\}} J(s,\rho(w_{n,v})\tilde{W}_v,\hat{\phi}_v)\cdot\prod_{v\notin S} J(s,\tilde{W}_v^0,\hat{\phi}_v^0).$$

From Theorem 3.11, Proposition 3.12 and Theorem 4.6, we have

$$\frac{F_1(s,\pi)}{L_{Sh}(s,\pi,\wedge^2)} = \frac{F_2(1-s,\tilde{\pi})}{\epsilon_{Sh}(s,\pi,\wedge^2)L_{Sh}(1-s,\tilde{\pi},\wedge^2)}. \tag{5.8}$$

This gives, using equations (5.1), (5.6) and (5.7),

$$\begin{aligned} &\frac{L_{JS}(s,\pi_{v_0},\wedge^2)}{L_{Sh}(s,\pi_{v_0},\wedge^2)} \cdot \prod_{v\in S\smallsetminus\{v_0\}} \frac{J(s,W_v,\pi_v)}{L_{Sh}(s,\pi_v,\wedge^2)} \\ &\quad = \frac{M_1(q_{v_0}^{-s})}{\epsilon_{Sh}(s,\pi,\wedge^2)} \cdot \frac{L_{JS}(1-s,\tilde{\pi}_{v_0},\wedge^2)}{L_{Sh}(1-s,\tilde{\pi}_{v_0},\wedge^2)} \prod_{v\in S\smallsetminus\{v_0\}} \frac{J(1-s,\rho(w_{n,n})\tilde{W}_v,\hat{\pi}_v)}{L_{Sh}(1-s,\tilde{\pi}_v,\wedge^2)}. \end{aligned} \tag{5.9}$$

By applying the same reasoning as above to $\tilde{\pi}$, we get

$$\begin{aligned} &\frac{L_{JS}(s,\tilde{\pi}_{v_0},\wedge^2)}{L_{Sh}(s,\tilde{\pi}_{v_0},\wedge^2)} \cdot \prod_{v\in S\smallsetminus\{v_0\}} \frac{J(s,\rho(w_{n,n})\tilde{W}_v,\tilde{\pi}_v)}{L_{Sh}(s,\tilde{\pi}_v,\wedge^2)} \\ &\quad = \omega_\pi(-1)\frac{M_2(q_{v_0}^{-s})}{\epsilon_{Sh}(s,\tilde{\pi},\wedge^2)} \frac{L_{JS}(1-s,\pi_{v_0},\wedge^2)}{L_{Sh}(1-s,\pi_{v_0},\wedge^2)} \cdot \prod_{v\in S\smallsetminus\{v_0\}} \frac{J(1-s,W_v,\pi_v)}{L_{Sh}(1-s,\pi_v,\wedge^2)}, \end{aligned} \tag{5.10}$$

where the factor $\omega_\pi(-1)$ arises because $\hat{\hat{\phi}}(x) = \phi(-x)$. Combining (5.9) and (5.10), we obtain

$$M_1(q_{v_0}^{-s})M_2(q_{v_0}^{-(1-s)}) = \omega_\pi(-1)\epsilon_{Sh}(s,\pi,\wedge^2)\epsilon_{Sh}(1-s,\tilde{\pi},\wedge^2) = \pm 1.$$

Hence, the polynomials M_1 and M_2 are non-vanishing, and it follows that they must be monomials in $q_{v_0}^{-s}$. □

6. The proof of the main theorem

We are now ready to establish Theorem 1.1. We will concentrate on the case $r = 2n$ and omit the case $r = 2n+1$, since the proof follows along similar lines and is, once again, somewhat easier.

Proof. Assume that $r = 2n$. We start with a proposition that allows us to embed a square integral representation as the local component of a (global) cuspidal automorphic representation. We use a weaker form of Lemma 6.5 of Chapter 1 of [2].

Proposition 6.1. *Let v_0 be a place of F. If π_{v_0} is a square integrable representation of $\mathrm{GL}_r(F_{v_0})$, there exists a cuspidal automorphic representation $\Pi = \otimes_v' \Pi_v$ of GL_r such that $\Pi_{v_0} \simeq \pi_{v_0}$.*

We will also need Lemma 5 of [12].

Lemma 6.2. *Let K be a p-adic field. There exists a number field F and a place v_0 of F such that $F_{v_0} = K$, where v_0 is the unique place of F lying over the rational prime p.*

We can and will assume that Π is unitary in the above proposition. Let τ be a square integrable representation of $\mathrm{GL}_n(K)$. We choose F as in the lemma above so that $F_{v_0} = K$. Hence, we may view τ as a (square integrable) representation π_{v_0} of $\mathrm{GL}_{2n}(F_{v_0})$. Using the proposition above, we can find a cuspidal automorphic Π of GL_{2n} with $\Pi_{v_0} = \pi_{v_0}$. We now consider the quotients

$$G_1(s,\Pi) = \frac{F_1(s,\Pi)}{L_{Sh}(s,\Pi,\wedge^2)} \quad \text{and} \quad G_2(s,\tilde{\Pi}) = \frac{F_2(s,\tilde{\Pi})}{M_1(q_{v_0}^{-(1-s)})L_{Sh}(s,\tilde{\Pi},\wedge^2)},$$

where F_1, F_2 and M_1 are as in equations (5.6) and (5.7). From equation (5.8), we have

$$G_1(s,\Pi) = \eta(s,\Pi)G_2(1-s,\tilde{\Pi}), \tag{6.1}$$

where $\eta(s,\Pi) = M_1(q_{v_0}^{-s})/\epsilon_{Sh}(s,\Pi,\wedge^2)$ is an entire function without zeros. On the other hand, at the places where Π is unramified the local integrals in the numerator and the L-functions in the denominator cancel each other out. Hence,

$$G_1(s,\Pi) = \frac{L_{JS}(s,\pi_{v_0},\wedge^2)}{L_{Sh}(s,\pi_{v_0},\wedge^2)} \cdot \prod_{v\in S_r\smallsetminus\{v_0\}} \frac{J(s,W_v,\Phi_v)}{L_{Sh}(s,\Pi_v,\wedge^2)} \cdot \prod_{v\in S_\infty} \frac{J(s,W_v,\Phi_v)}{L_{Sh}(s,\Pi_v,\wedge^2)}.$$

We write this as

$$G_1(s,\Pi) = P(s,\Pi)Q_1(s,\Pi)R_1(s,\Pi),$$

where

$$P(s,\Pi) = \frac{L_{JS}(s,\pi_{v_0},\wedge^2)}{L_{Sh}(s,\pi_{v_0},\wedge^2)}, \quad R_1(s,\Pi) = \prod_{v\in S_\infty} \frac{J(s,W_v,\Phi_v)}{L_{Sh}(s,\Pi_v,\wedge^2)}$$

and

$$Q_1(s,\Pi) = \prod_{v\in S_r\smallsetminus\{v_0\}} \frac{J(s,W_v,\Phi_v)}{L_{Sh}(s,\Pi_v,\wedge^2)} = \prod_{v\in S_r\smallsetminus\{v_0\}} c_1 q_v^{m_1 s} \frac{\prod_{i=1}^{k_v}(1-\alpha_i(v)q_v^{-s})}{\prod_{j=1}^{l_v}(1-\beta_j(v)q_v^{-s})},$$

for some $m_1 \in \mathbb{Z}$ and $c_1 \in \mathbb{C}$, and for integers k_v and l_v and complex numbers $\alpha_i(v)$ and $\beta_j(v)$. Note that by our assumption on F, $(p,q_v) = 1$. Similarly, we have

$$G_2(s,\Pi) = P(s,\tilde{\Pi})Q_2(s,\tilde{\Pi})R_2(s,\tilde{\Pi}),$$

where

$$Q_2(s,\tilde{\Pi}) = \prod_{v\in S_r\smallsetminus\{v_0\}} \frac{J(s,\rho(w_{n,n})\tilde{W}_v,\hat{\Phi}_v)}{L_{Sh}(s,\tilde{\Pi}_v,\wedge^2)} = \prod_{v\in S_r\smallsetminus\{v_0\}} c_2 q_v^{m_2} \frac{\prod_{i=1}^{k'_v}(1-\alpha'_i(v)q_v^{-s})}{\prod_{j=1}^{l'_v}(1-\beta'_j(v)q_v^{-s})},$$

for some $m_2 \in \mathbb{Z}$ and $c_2 \in \mathbb{C}$, and some integers k'_v and l'_v and complex numbers $\alpha'_i(v)$ and $\beta'_j(v)$, and

$$R_2(s,\tilde{\Pi}) = \prod_{v\in S_\infty} \frac{J(s,\rho(w_{n,n})\tilde{W}_v,\hat{\Phi}_v)}{L_{Sh}(s,\tilde{\Pi}_v,\wedge^2)}.$$

By Theorem 3.7 and Proposition 4.1, the functions $P(s,\Pi)$ and $P(s,\tilde{\Pi})$ are regular and non-vanishing in the region $\mathrm{Re}(s) > 0$. By Propositions 3.9 and 4.2, the function $R_1(s,\Pi)$ and $R_2(s,\tilde{\Pi})$ have only finitely many poles in any vertical strip $a \leq \mathrm{Re}(s) \leq b$.

To prove Theorem 1.1, it is enough to prove that the function $P(s,\Pi)$ is entire and nowhere vanishing. It will then follow that $P(s,\Pi)$ must be a monomial in $q_{v_0}^{-s}$. Since both the L-functions $L_{JS}(s,\pi_v,\wedge^2)$ and $L_{Sh}(s,\pi_v,\wedge^2)$ are normalized to have numerator 1, it is immediate that $P(s,\Pi)$ must be identically 1.

Suppose that $P(s,\Pi)$ has a zero at s_0. This means that the function $P(s,\Pi)$ also has zeros at $s_0+2\pi ik/\log q_{v_0}$, for all $k\in\mathbb{Z}$. We claim that all but finitely many of these zeros must also be zeros of $G_1(s,\Pi)$. This fails to happen only if all but finitely many zeros are cancelled by the poles of $Q_1(s,\Pi)R_1(s,\Pi)$. By Proposition 3.9, $R_1(s,\Pi)$ can contribute only finitely many poles on any line with real-part constant, and this set of poles is independent of the choice of W_v and ϕ_v at the archimedean places in the function $F_1(s,\Pi)$. Hence, $Q_1(s,\Pi)$ must have infinitely many poles of this form. The poles of $Q_1(s,\Pi)$ are of the form $s_j+2\pi il/\log q_v$, for all $l\in\mathbb{Z}$, with $v\in S_r\setminus\{v_0\}$. It follows that there is at least one v such that there exist two integers $l_1\neq l_2$ such that

$$s_0+2\pi il_1/\log q_v=s_1+2\pi ik_1/\log q_{v_0}\quad\text{and}\quad s_0+2\pi il_2/\log q_v=s_1+2\pi ik_2/\log q_{v_0}$$

for some k_1 and k_2 in $\mathbb{Z}$ (in fact, there are infinitely many distinct integers with this property). It follows that $\log q_v/\log q_{v_0}$ is rational, which is absurd since $(q_v,q_{v_0})=1$ by choice. Thus, for all but finitely many k, the points $s_0+2\pi ik/\log q_{v_0}$ are zeros of $G_1(s,\Pi)$.

Since $P(s,\Pi)$ is non-vanishing in the region $\mathrm{Re}(s)>0$, we must have $\mathrm{Re}(s_0)\le 0$. From (6.1), we see that all but finitely many of the points $1-s_0+2\pi ik/\log q_{v_0}$ are zeros of the function $G_2(s,\tilde\Pi)$. Since $P(s,\tilde\Pi)$ is non-vanishing in the region $\mathrm{Re}(s)>0$, these zeros have to be the zeros of $Q_2(s,\tilde\Pi)R_2(s,\tilde\Pi)$. Arguing as above, these cannot be zeros of $Q_2(s,\tilde\Pi)$ for infinitely many k. By Proposition 4.2, the poles of $\prod_{v\in S_\infty}L_{Sh}(s,\tilde\Pi_v,\wedge^2)$ lie along horizontal lines. Hence, this product can contribute only finitely many poles on any line with real-part constant. Thus, except for finitely many k, these zeros must be zeros of $\prod_{v\in S_\infty}J(s,\rho(w_{n,n})\tilde W_v,\hat\Phi_v)$, for every $\rho(w_{n,n})\tilde W_v$ and $\hat\Phi_v$ (with $v|\infty$) such that $J(s,\rho(w_{n,n})\tilde W_v,\hat\Phi_v)$ is not identically zero. If $\beta=1-s_0+2\pi il/\log q_{v_0}$ is one of these zeros of $G_2(s,\tilde\Pi)$, this contradicts Theorem 3.3 which asserts that there are W_v and Φ_v such that $\prod_{v\in S_\infty}J(\beta,\rho(w_{n,n})\tilde W_v,\hat\Phi_v)\neq 0$. Hence $P(s,\Pi)$ is non-vanishing.

We now show that $P(s,\Pi)$ must be entire. We rewrite functional equation (6.1) as

$$P(s,\Pi)Q_1(s,\Pi)R_1(s,\Pi)=\eta(s,\Pi)P(1-s,\tilde\Pi)Q_2(1-s,\tilde\Pi)R_2(1-s,\tilde\Pi).\tag{6.2}$$

Proposition 4.2 and the form of the local Langlands–Shahidi L-factor at the finite places show us that $\prod_{v\in S\smallsetminus v_0}L_{Sh}(s,\Pi_v,\wedge^2)$ is nowhere vanishing. By Proposition 3.1, the function $Q_1(s,\Pi)R_1(s,\Pi)$ is holomorphic in $\mathrm{Re}(s)>1-\eta$, for some $\eta>0$, and the function $Q_2(1-s,\tilde\Pi)\ R_2(1-s,\tilde\Pi)$ is holomorphic in $\mathrm{Re}(s)<\eta$. Hence, the function $G_1(s,\Pi)$ is holomorphic in $\mathrm{Re}(s)>1-\eta$ and in $\mathrm{Re}(s)<\eta$. Suppose that $P(s,\Pi)$ has a pole at s_0. This means that the function $P(s,\Pi)$ also has poles at $s_0+2\pi ik/\log q_{v_0}$, $k\in\mathbb{Z}$. The function $P(s,\Pi)$ is holomorphic in the region $\mathrm{Re}(s)>0$, hence, we obtain $\mathrm{Re}(s_0)\le 0$. Since $G_1(s,\Pi)$ is holomorphic in $\mathrm{Re}(s)<\eta$, $\eta>0$, these poles must be cancelled by the zeros of $Q_1(s,\Pi)R_1(s,\Pi)$. Arguing as in the the non-vanishing case, these cannot be zeros of $Q_1(s,\Pi)$ for infinitely many k. By Proposition 4.2, the poles of $\prod_{v\in S_\infty} L_{Sh}(s,\Pi_v,\wedge^2)$ lie along horizontal lines. Hence, this product can contribute only finitely many poles on any line with real-part constant. Thus, except for finitely many k, these poles must be zeros of $\prod_{v\in S_\infty} J(s,W_v,\Phi_v)$, for every W_v and Φ_v (with $v|\infty$) such that $J(s,W_v,\Phi_v)$ is not identically zero. As in the preceeding paragraph, this contradicts Theorem 3.3. Hence, $P(s,\Pi)$ must be entire and this completes the proof of Theorem 1.1. □

7. Extensions and applications of the main theorem

We now use Theorem 1.1 to prove Theorems 1.2 and 1.3 and give a number of other applications.

Recall that the symmetric square L-function of a representation π_v can be defined via the local Langlands correspondence as a Galois L-function. As before, if $\rho_{F_v}(\pi_v)$ corresponds to π_v we define

$$L(s,\pi_v,\mathrm{Sym}^2) = L(s,\mathrm{Sym}^2(\rho_{F_v}(\pi_v))).$$

As a first application of Theorem 1.1, we are able to obtain the following characterization of self-dual-square integrable representations.

Corollary 7.1. *Let π_v be an irreducible smooth square integrable representation of $\mathrm{GL}_{2n}(F_v)$ which has no Shalika functional. Then the symmetric square L-function $L(s,\pi_v,Sym^2)$ has a pole at $s=0$ if and only if π_v is self-dual, that is, if and only if $\pi_v \simeq \tilde{\pi}_v$.*

Proof. We have

$$L(s,\pi_v\times\pi_v) = L(s,\pi_v,\wedge^2)L(s,\pi_v,\mathrm{Sym}^2). \tag{7.1}$$

We know that the L-function $L(s,\pi_v\times\pi_v)$ has a pole at $s=0$ if and only if $\pi_v\simeq\tilde{\pi}_v$. Since $2n$ is even, Corollary 4.4 of [13] shows that the L-function $L_{JS}(s,\pi_v,\wedge^2)$ does not have a pole at $s=0$. Thus, from Theorems 1.1 and 1.4, the L-function $L(s,\pi_v,\wedge^2)$ does not have a pole $s=0$. Hence, the corollary follows from equation (7.1). The converse is trivial. □

To establish the equality of the exterior square L-functions for generic representations in the even case we proceed as follows. We start by proving the equality for quasi-square integrable representations. Let π_v be a quasi-square integrable representation of $\mathrm{GL}_r(F_v)$. Then $\pi_v = \pi_0\otimes\chi$, where π_0 is a square integrable representation and $\chi = \chi_0|\ |^{s_0}, \chi_0$ is an unitary character. Clearly, $\pi_0\otimes\chi_0$ is a square integrable representation. Since $J(s,W\otimes|\ |^{s_0},\phi) = J(s+2s_0,W,\phi)$, we see that

$$L_{JS}(s,\pi_v,\wedge^2) = L_{JS}(s+2s_0,\pi_0\otimes\chi_0,\wedge^2).$$

Using Theorem 1.1, we get

$$L_{JS}(s,\pi_0\otimes\chi_0,\wedge^2) = L_{Sh}(s,\pi_0\otimes\chi_0,\wedge^2). \tag{7.2}$$

Let $\mathcal{A}_r(F_v)$ denote the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_r(F_v)$, and let $\mathcal{G}_r(F_v)$ denote the set of isomorphism classes of Φ-semisimple r-dimensional complex representations of Weil–Deligne group W'_{F_v}. The local Langlands correspondence (proved by Harris and Taylor in [7], see also Henniart [8]) asserts that for each $r \geq 1$, there exists a bijection

$$\rho_{F_v} : \mathcal{A}_r(F_v) \longrightarrow \mathrm{GL}_r(F_v),$$

satisfying certain functorial properties. If σ_v is a smooth irreducible representation of $\mathrm{GL}_r(F_v)$ then (see Theorem 4.3) has shown that

$$L_{Sh}(s, \sigma_v, \wedge^2) = L(s, \sigma_v, \wedge^2) = L(s, \wedge^2 \rho_F(\sigma_v)). \tag{7.3}$$

It is easy to see that $\wedge^2(\rho_F(\sigma_v) \otimes \chi) \simeq \wedge^2(\rho_F(\sigma_v)) \otimes \chi^2$, whence

$$\begin{aligned} L(s, \pi_v, \wedge^2) &= L(s, \wedge^2 \rho_F(\pi_0 \otimes \chi_0| \ |^{s_0})) = L(s, \wedge^2(\rho_F(\pi_0 \otimes \chi_0) \otimes | \ |^{s_0})) \\ &= L(s, \wedge^2(\rho_F(\pi_0 \otimes \chi_0)) \otimes | \ |^{2s_0}) = L(s + 2s_0, \wedge^2 \rho_F(\pi_0 \otimes \chi_0)). \end{aligned} \tag{7.4}$$

From (7.2), (7.3) and (7.4), we have

$$L_{JS}(s, \pi_v, \wedge^2) = L_{Sh}(s, \pi_v, \wedge^2) = L(s, \pi_v, \wedge^2). \tag{7.5}$$

This proves the equality for quasi-square integrable representations, which we record below as a theorem.

Theorem 7.2. *Let π_v be a quasi-square integrable representation of* $\mathrm{GL}_r(F_v)$. *Then*

$$L_{JS}(s, \pi_v, \wedge^2) = L_{Sh}(s, \pi_v, \wedge^2) = L(s, \pi_v, \wedge^2).$$

Let π_v be an irreducible generic representation of $\mathrm{GL}_r(F_v)$, where $r = 2n$. We now prove Theorem 1.2.

Proof. It is a theorem of Bernstein and Zelevinsky [4] that π_v is parabolically induced from quasi-square integrable representations. Thus, we can write

$$\pi_v = \mathrm{Ind}(\pi_{1,v} \otimes \pi_{2,v} \otimes \cdots \otimes \pi_{r,v}),$$

where the $\pi_{i,v}$ are quasi-square integrable representations of $\mathrm{GL}_{n_i}(F)$ with $\sum n_i = 2n$. In [5], Cogdell and Piatetski-Shapiro have proved that

$$L_{JS}(s, \pi_v, \wedge^2) = \prod_{i=1}^{r} L_{JS}(s, \pi_{i,v}, \wedge^2) \prod_{\substack{i<j \\ j=2}}^{r} L(s, \pi_{i,v} \times \pi_{j,v}), \tag{7.6}$$

where $L(s, \pi_{i,v} \times \pi_{j,v})$ is the Rankin–Selberg L-function of $\pi_{i,v} \times \pi_{j,v}$. By the local Langlands correspondence, π_v corresponds to

$$\rho_{F_v}(\pi_{1,v}) \oplus \rho_{F_v}(\pi_{2,v}) \oplus \cdots \oplus \rho_{F_v}(\pi_{r,v}).$$

Thus, we have

$$L(s, \pi_v, \wedge^2) = L(s, \wedge^2(\rho_F(\pi_{1,v}) \oplus \rho_F(\pi_{2,v}) \oplus \cdots \oplus \rho_F(\pi_{r,v}))).$$

If $V_1, V_2, \ldots, V_r$ are the spaces on which $\pi_{1,v}, \pi_{2,v}, \ldots, \pi_{r,v}$ act, we can prove that

$$\wedge^2(\oplus_{i=1}^r V_i) \simeq (\oplus_{i=1}^r \wedge^2 V_i) \bigoplus \left(\oplus_{\substack{i<j \\ j=2}}^r (V_i \otimes V_j) \right).$$

It follows that

$$L(s, \pi_v, \wedge^2) = \prod_{i=1}^r L(s, \wedge^2 \rho_F(\pi_{i,v})) \prod_{\substack{i<j \\ j=2}}^r L(s, \rho_F(\pi_{i,v}) \otimes \rho_F(\pi_{j,v})). \tag{7.7}$$

By the local Langlands correspondence, we have

$$L(s, \pi_{i,v} \times \pi_{j,v}) = L(s, \rho_{F_v}(\pi_{i,v}) \otimes \rho_{F_v}(\pi_{j,v})). \tag{7.8}$$

Hence, we have

$$L(s, \pi_v, \wedge^2) = \prod_{i=1}^r L(s, \pi_{i_v}, \wedge^2) \prod_{\substack{i<j \\ j=2}}^r L(s, \pi_{i,v} \times \pi_{j,v}), \tag{7.9}$$

From (7.5), (7.6) and (7.9), we have

$$L_{JS}(s, \pi_v, \wedge^2) = L(s, \pi_v, \wedge^2). \tag{7.10}$$

This completes the proof of Theorem 1.2. □

Theorem 1.3 is an immediate consequence of the Theorems 1.1 and 1.2, combined with the results of Shahidi in [16] for the archimedean places. It allows us to obtain the analytic properties of $L_{JS}(s, \pi, \wedge^2)$, when they are known for $L_{Sh}(s, \pi, \wedge^2)$, and conversely. The analytic properties of $L_{Sh}(s, \pi, \wedge^2)$ due to Kim and Shahidi were recorded in Theorems 4.5 and 4.6 from Section 4. We also use the theorem of Gelbart and Shahidi in [6], which shows that $L_{Sh}(s, \pi, \wedge^2)$ is bounded in vertical strips.

Corollary 7.3. *If π is a unitary cuspidal automorphic representation of GL_{2n}, which is not self-dual, the L-function $L_{JS}(s, \pi, \wedge^2)$ is entire, satisfies the functional equation* (4.1)*, and is bounded in vertical strips.*

In the odd case, we can make only a weaker statement about the integrals since the equality of the local L-functions has not been established even for unramified representations. However, this statement should suffice for many, if not most, applications. As in Section 5, we let S_∞ denote the archimedean places of F and S_r denote the set of places where π_v is not unramified. We denote by S_{ur} the set of finite places where π_v is unramified.

Corollary 7.4. *Let π be a unitary cuspidal automorphic representation of GL_{2n+1} such that the local components π_v at the finite places are either unramified or square integrable. Then*

$$\prod_{v \in S_\infty \cup S_{\text{ur}}} L_{Sh}(s, \pi_v, \wedge^2) \prod_{v \in S_r} L_{JS}(s, \pi_v, \wedge^2)$$

is entire, satisfies the functional equation (4.1)*, and is bounded in vertical strips.*

To prove the corollaries, we simply choose W_v and ϕ_v (resp. W_v) so that $J(s, W_, \phi_v)$ (resp. $J(s, W_v)$ gives $L_{Sh}(s, \pi_v, \wedge^2)$ at each finite place.

The facts that $L_{JS}(s, \pi, \wedge^2)$ has a functional equation and that it is bounded in vertical strips are new results. The corollaries above also strengthen the results of Theorem 5.2 of [3] where all the ramified and archimedean places are excluded for the stated holomorphy result.

For the case when r is even and π is self-dual, we can use Belt's theorem for the partial Jacquet–Shalika L-function to deduce holomorphy results for $L_{Sh}(s, \pi, \wedge^2)$. In conjunction with Section 8 of [11], we can obtain the following corollary.

Corollary 7.5. *Assume that π is a unitary cuspidal automorphic representation of GL_{2n} which is self-dual and that the local components π_v at the archimedean places are tempered. If the central character ω_π is not trivial, then $L_{Sh}(s, \pi, \wedge^2)$ is entire. If ω_π is trivial, $L_{Sh}(s, \pi, \wedge^2)$ is holomorphic at all points except possibly for simple poles at $s = 0$ or $s = 1$. There will be simple poles if and only if π has a non-zero global Shalika period.*

One non-trivial case covered by the corollary above is the following. Let σ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to an arbitrary holomorphic cusp form or to a Maass cusp form on the full modular group, and let π be the symmetric cube lift of σ to $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$. Then the archimedean places of π are known to be tempered.

8. The local functional equation

The main purpose of this section is to obtain a local functional equation for the L-functions $L_{JS}(s, \pi_v, \wedge^2)$ using the same global methods as before. The key point is that we are able to define a local ϵ-factor $\epsilon_{JS}(s, \pi_{v_k}, \psi_{v_k}, \wedge^2)$. Of course, this is conjecturally the same as the ϵ-factor arising in the Langlands–Shahidi method.

Theorem 8.1. *Let F_v be a p-adic field. Assume first that $r = 2n$. If π_v is an irreducible generic representation of $\mathrm{GL}_r(F_v)$, which occurs as the local constituent of a cuspidal automorphic representation π of GL_r, we have*

$$\frac{J(1-s, \rho(w_{n,n})\tilde{W}_v, \hat{\phi}_v)}{L(1-s, \tilde{\pi}_v, \wedge^2)} = \epsilon_{JS}(s, \pi_v, \psi_v, \wedge^2)\frac{J(s, W_v, \phi_v)}{L(s, \pi_v, \wedge^2)},$$

where $\epsilon_{JS}(s, \pi_v, \psi_v, \wedge^2)$ is entire and non-vanishing. If $r = 2n+1$, and π_v is an irreducible square integrable representation of $\mathrm{GL}_r(F_v)$, we have

$$\frac{J(1-s, W_v')}{L(1-s, \tilde{\pi}_v, \wedge^2)} = \epsilon_{JS}(s, \pi_v, \psi_v, \wedge^2)\frac{J(s, W_v)}{L(s, \pi_v, \wedge^2)},$$

where W_v' is some fixed translate of $\tilde{W}_v$, as in Theorem 3.11, and the function $\epsilon_{JS}(s, \pi_v, \psi_v, \wedge^2)$ is entire and non-vanishing.

Note that because of Proposition 6.1, square integrable representations satisfy the hypotheses of the theorem in the even case, and are thus covered in both cases. We will give the proof of this theorem only in the even case since the proof in the odd case is entirely analogous. The factor $\epsilon_{JS}(s, \pi_v, \psi_v, \wedge^2)$ will be defined explicitly in (8.8).

Proof. Let

$$W = \prod_{v \in S} W_v \cdot \prod_{v \notin S} W_v^0 \quad \text{and} \quad \Phi = \prod_{v \in S} \phi_v \cdot \prod_{v \notin S} \phi_v^0,$$

where W_v^0 and ϕ_v^0 are as in equation (5.1) and $S = S_\infty \cup S_r$, as before. From Theorem 3.11, Proposition 3.12, Theorem 4.6 and equation (5.1), we have

$$\prod_{v \in S} \frac{J(s, W_v, \phi_v)}{L_{Sh}(s, \pi_v, \wedge^2)} = \frac{1}{\epsilon(s, \pi, \wedge^2)} \cdot \prod_{v \in S} \frac{J(1-s, \rho(w_{n,n})\tilde{W}_v, \hat{\phi}_v)}{L_{Sh}(1-s, \tilde{\pi}_v, \wedge^2)}. \tag{8.1}$$

We recall the definition of the monomial $M_1(q_{v_1}^{-s})$ made in Section 5 (see (5.3)). Suppose that there are k places in S_r. Fix a place $v_1 \in S_r$. There exist W_{i,v_1} and ϕ_{i,v_1} such that $L_{JS}(s, \pi_{v_1}, \wedge^2) = \sum_{i=1}^{n_{v_1}} J(s, W_{i,v_1}, \phi_{i,v_1})$, and for this choice of data we have

$$\sum_{i=1}^{n_{v_1}} J(1-s, \rho(w_{n,n})\tilde{W}_{i,v_1}, \hat{\phi}_{i,v_1}) = M_1(q_{v_1}^{-s}) L_{JS}(1-s, \tilde{\pi}_{v_1}, \wedge^2). \tag{8.2}$$

Note that since π is a cuspidal automorphic representation, it is globally generic, and hence, every local component π_v is generic. By summing equation (5.2) over i and using Theorem 1.2, we get

$$\prod_{v \in S \setminus \{v_1\}} \frac{J(s, W_v, \phi_v)}{L_{Sh}(s, \pi_v, \wedge^2)} = \frac{M_1(q_{v_1}^{-s})}{\epsilon(s, \pi, \wedge^2)} \times \prod_{v \in S \setminus \{v_1\}} \frac{J(1-s, \rho(w_{n,n})\tilde{W}_v, \hat{\phi}_v)}{L_{Sh}(1-s, \tilde{\pi}_v, \wedge^2)}. \tag{8.3}$$

Fix a place $v_2 \in S \setminus \{v_1\}$. Arguing as above, we have

$$\prod_{v \in S \setminus \{v_1, v_2\}} \frac{J(s, W_v, \phi_v)}{L_{Sh}(s, \pi_v, \wedge^2)} = \frac{M_2(q_{v_2}^{-s}) M_1(q_{v_1}^{-s})}{\epsilon(s, \pi, \wedge^2)} \times \prod_{v \in S \setminus \{v_1, v_2\}} \frac{J(1-s, \rho(w_{n,n})\tilde{W}_v, \hat{\phi}_v)}{L_{Sh}(1-s, \tilde{\pi}_v, \wedge^2)}, \tag{8.4}$$

$M_2(q_{v_2}^{-s})$ is a monomial in $q_{v_2}^{-s}$. Continuing in this way, we get

$$\begin{aligned} \frac{J(s, W_{v_k}, \phi_{v_k})}{L_{Sh}(s, \pi_v, \wedge^2)} \prod_{v \in S_\infty} \frac{J(s, W_v, \phi_v)}{L_{Sh}(s, \pi_v, \wedge^2)} &= \frac{M'(s)}{\epsilon(s, \pi, \wedge^2)} \cdot \frac{J(1-s, \rho(w_{n,n})\tilde{W}_{v_k}, \hat{\phi}_{v_k})}{L_{Sh}(1-s, \tilde{\pi}_{v_k}, \wedge^2)} \\ &\quad \times \prod_{v \in S_\infty} \frac{J(1-s, \rho(w_{n,n})\tilde{W}_v, \hat{\phi}_v)}{L_{Sh}(1-s, \tilde{\pi}_v, \wedge^2)}, \end{aligned} \tag{8.5}$$

where $M'(s) = \prod_{i=1}^{k-1} M_i(q_{v_i}^{-s})$. Again as above, we obtain

$$\prod_{v \in S_\infty} \frac{J(s, W_v, \phi_v)}{L_{Sh}(s, \pi_v, \wedge^2)} = \frac{M(s)}{\epsilon(s, \pi, \wedge^2)} \cdot \prod_{v \in S_\infty} \frac{J(1-s, \rho(w_{n,n})\tilde{W}_v, \hat{\phi}_v)}{L_{Sh}(1-s, \tilde{\pi}_v, \wedge^2)}, \tag{8.6}$$

where $M(s) = \prod_{i=1}^{k} M_i(q_{v_i}^{-s})$. From equations (8.5) and (8.6), we have

$$\frac{J(1-s, \rho(w_{n,n})\tilde{W}_{v_k}, \hat{\phi}_{v_k})}{L_{Sh}(1-s, \tilde{\pi}_{v_k}, \wedge^2)} = M_k(q_{v_k}^{-s}) \frac{J(s, W_{v_k}, \phi_{v_k})}{L_{Sh}(s, \pi_{v_k}, \wedge^2)}. \tag{8.7}$$

If we set

$$\epsilon_{JS}(s, \pi_{v_k}, \psi_{v_k}, \wedge^2) = M_k(q_{v_k}^{-s}), \tag{8.8}$$

we get

$$\frac{J(1-s, \rho(w_{n,n})\tilde{W}_{v_k}, \hat{\phi}_{v_k})}{L_{Sh}(1-s, \tilde{\pi}_{v_k}, \wedge^2)} = \epsilon_{JS}(s, \pi_{v_k}, \psi_{v_k}, \wedge^2) \frac{J(s, W_{v_k}, \phi_{v_k})}{L_{Sh}(s, \pi_{v_k}, \wedge^2)}. \tag{8.9}$$

The ordering of the ramified places as $1, 2, \ldots, k$ was completely arbitrary. Thus equation (8.9) holds when k is replaced by any i, $1 \le i \le k$. Note that we also get the local functional equation at unramified finite places. Let v_{ur} be an unramified finite place. If we choose

$$W = W_{v_{\mathrm{ur}}} \cdot \prod_{v \in S} W_v \cdot \prod_{v \notin S \cup \{v_{\mathrm{ur}}\}} W_v^0 \quad \text{and} \quad \Phi = \phi_{v_{\mathrm{ur}}} \cdot \prod_{v \in S} \phi_v \cdot \prod_{v \notin S \cup \{v_{\mathrm{ur}}\}} \phi_v^0,$$

and argue as above we get the local functional equation at v_{ur}. □

Remark 8.2. It may appear *a priori* that the function $M_1(q_v^{-s})$ depends on the choices of vectors W_{i,v_1} and functions ϕ_{i,v_1} made in (8.2) in order to obtain $L_{JS}(s, \pi_v, \wedge^2)$. However, once we obtain the local functional equation, $M_1(q_v^{-s})$ is defined completely by this equation, and is thus independent of this choice. The ϵ-factor we have defined thus depends only on the representation π_{v_1}.

Remark 8.3. When $F = \mathbb{Q}$, we can use the same arguments to get a local functional equation at the archimedean place.

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