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ON THE SINGULAR LOCUS OF CERTAIN SUBVARIETIES OF SPRINGER FIBERS

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Abstract. Let $x \in \mathrm{End}(\mathbb{K}^n)$ be an endomorphism such that $x^2 = 0$ (where $\mathbb{K}$ is an algebraically closed field). The corresponding Springer fiber $\mathcal{F}_x$ is the algebraic variety of x-stable complete flags. In the present case, $\mathcal{F}_x$ has a suitable decomposition into a finite number of orbits under the action of the centralizer of x. The closures of these orbits may be singular. In this paper, we give a combinatorial description of the singular locus of the orbit closures. In particular, we deduce a description of the singular locus of the irreducible components of $\mathcal{F}_x$.

1. Introduction

Throughout this paper, we fix an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. We also fix a vector space V of finite dimension n. A chain $(V_0 \subset V_1 \subset \cdots \subset V_n)$ of subspaces of V such that $\dim V_i = i$, for all i, is called a *complete flag*. By $\mathcal{F}$, we denote the set of all the complete flags. It has a natural structure of algebraic projective variety and it admits certain remarkable subvarieties. The Schubert varieties are the most classical example.

1.1. Schubert varieties and their singular loci. Choose a basis $(e_1, \ldots, e_n)$ of V, and let $B \subset GL(V)$ be the subgroup of linear automorphisms whose matrix in the basis is upper triangular. Then, $\mathcal{F}$ consists of a finite number of orbits for the natural action of B, called *Schubert cells*, and their closures in the Zariski topology are called *Schubert varieties*. The Schubert varieties are parameterized by the elements $w \in \mathbf{S}_n$ of the symmetric group: each Schubert cell can be written $X_w^0 := B \cdot F_w$, where $F_w := (\langle e_{w_1}, \ldots, e_{w_i} \rangle_{\mathbb{K}})_{i=0}^n$, and thus each Schubert variety is of the form $X_w := \overline{B \cdot F_w}$. Moreover, $\dim X_w = \ell(w)$ (the Bruhat length of w), and one has $X_{w'} \subset \overline{X_w}$ if and only if $w' \leq w$, where $\leq$ stands for the Bruhat order.

The simplicity of the combinatorics allowed V. Lakshmibai and C.S. Seshadri [7] to determine the singular locus of the Schubert varieties, in the following manner. Given $w \in \mathbf{S}_n$, the corresponding *Bruhat graph* is by definition the graph whose set of vertices is $\{v \in \mathbf{S}_n : v \leq w\}$ and with an edge between v and v' whenever one has $v' = v(i : j)$, where $(i : j) \in \mathbf{S}_n$ is the transposition that switches two distinct integers i and j. Then, the singular locus of the Schubert variety X_w is the union of the Schubert cells X_v^0 corresponding to the singular vertices v of the Bruhat graph of w, i.e., those vertices, which are incident with more than $\ell(w)$ edges. The reason is that, for each edge (v, v') in the Bruhat graph, one gets a projective curve in X_w

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between the points F_v and $F_{v'}$, giving rise to a tangent vector $z_{v,v'} \in T_{F_v}X_w$. The vectors $z_{v,v'}$ corresponding to the various edges (v, v') form a basis of $T_{F_v}X_w$. Thus, $\dim T_{F_v}X_w$ is equal to the number of edges at v in the Bruhat graph.

1.2. Springer fibers. In this paper, we are rather interested in another family of varieties of complete flags. Given a nilpotent endomorphism $x \in \mathrm{End}(V)$, we let $\mathcal{F}_x \subset \mathcal{F}$ be the subset of x-stable complete flags, i.e., flags $(V_0, \ldots, V_n)$ such that $x(V_i) \subset V_{i-1}$ for each i. Then, $\mathcal{F}_x$ is a closed subvariety of $\mathcal{F}$ and it is called a *Springer fiber*. The Springer fiber $\mathcal{F}_x$ is an equidimensional variety (see [10]), it is usually not irreducible, and its irreducible components may be singular. Springer fibers arise in some problems in representation theory (see, for instance, [1,6,12,13]). Their elementary properties are described in [11], and the singularity of their irreducible components has been studied recently in, e.g., [3–5], [9] and references therein.

Let $Z_x := \{g \in GL(V) : gxg^{-1} = x\}$ be the centralizer of x. Thus, Z_x naturally acts on the complete flags and leaves the Springer fiber $\mathcal{F}_x$ stable. However, for x arbitrary, $\mathcal{F}_x$ may contain an infinite number of Z_x-orbits, and the inclusion relations between the Z_x-orbit closures, or of the Z_x-orbits in the components of $\mathcal{F}_x$, seem to be quite complicated.

1.3. Springer fibers in the case $x^2 = 0$. Hereafter, we focus on the situation where x is an endomorphism such that $x^2 = 0$. This situation is more favorable for at least two reasons:

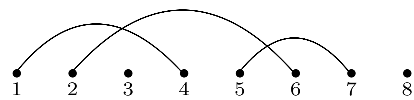
- Here, the Springer fiber $\mathcal{F}_x$ consists of a finite number of Z_x-orbits, and each orbit is driven by a special flag of the form F_w (cf. Section 1.1) corresponding to the choice of a Jordan basis $(e_1, \ldots, e_n)$.
- The orbits are suitably parameterized by a family of graphs called link patterns, and the elementary properties of the orbits (dimension, inclusion relations between closures) are known in terms of combinatorial properties of these graphs.

In Section 2, we recall from [4], [8] the description of the Z_x-orbits of $\mathcal{F}_x$ that we have just summed up. Relying on this, criteria for the smoothness of the components of $\mathcal{F}_x$ have already been established in [4]. Here, we are able to determine the singular locus of each Z_x-orbit closure of the Springer fiber $\mathcal{F}_x$ (in particular, of each irreducible component), by implementing a technique analogous to the one used in the case of the Schubert varieties (cf. Section 1.1). Our main result is stated in Section 3 and the proof is given in Section 4.

2. Description of the Z_x-orbits of $\mathcal{F}_x$

As in Section 1.3, $x \in \mathrm{End}(V)$ is an endomorphism such that $x^2 = 0$. Let $k = \mathrm{rank}\, x$. Thus, x has exactly k Jordan blocks of size 2 and $n - 2k$ Jordan blocks of size 1. In this section, our purpose is to recall the construction of the Z_x-orbits of the Springer fiber $\mathcal{F}_x$. This construction was given in [8] in a slightly different setting and has been adapted to the present setting in [4]. The combinatorial objects which enter the construction are presented in the next definition.

Definition 1. Let $\mathbf{S}_n^2 \subset \mathbf{S}_n$ denote the subset of involutive permutations, that is, permutations $\sigma \in \mathbf{S}_n$ such that $\sigma^2 = 1$, and let $\mathbf{S}_n^2(k) \subset \mathbf{S}_n^2$ denote the subset of permutations that can be written as product of k pairwise disjoint transpositions like $\sigma = (i_1 : j_1) \cdots (i_k : j_k)$ with $i_1 < j_1, \ldots, i_k < j_k$ all distinct. The graph P_σ with vertices $1, \ldots, n$ and with k arcs joining (i_l, j_l) (for $l = 1, \ldots, k$) is called the *link pattern associated to* σ. E.g., if $\sigma = (1:4)(2:6)(5:7) \in S_8^2(3)$, then

$P_\sigma =$ 

The vertices $i_1, \ldots, i_k$ (resp. $j_1, \ldots, j_k$) are called the *left* (resp. *right*) *end points* of σ or of P_σ. The other vertices are called the *fixed points*. We will also need some auxiliary notation:

- Two arcs (i_l, j_l), (i_m, j_m) are said to have a *crossing* if $i_l < i_m < j_l < j_m$ (i.e., the arcs intersect in P_σ); let $c(\sigma)$ denote the number of crossings of P_σ. A pair formed by an arc (i_l, j_l) and a fixed point p is said to be a *bridge* if $i_l < p < j_l$ (i.e., the arc spans the vertex p in P_σ); let $b(\sigma)$ denote the number of bridges of P_σ. For example, if σ is as above, then $b(\sigma) = c(\sigma) = 2$: the arcs $(1,4), (2,6)$, resp. $(2,6), (5,7)$, have a crossing, and the pairs $\{(1,4), 3\}$ and $\{(2,6), 3\}$ are bridges.
- Finally, given $1 \leq s < t \leq n$, we let $R_{s,t}(\sigma)$ denote the number of arcs of P_σ that are contained between s and t, i.e., the number of indices $l \in \{1, \ldots, k\}$ such that $s \leq i_l < j_l \leq t$. Furthermore, we define an order on $\mathbf{S}_n^2(k)$ by writing $\sigma' \preceq \sigma$ if we have $R_{s,t}(\sigma') \leq R_{s,t}(\sigma)$ for all s, t.

We associate a subset of flags $\mathcal{Z}_\sigma \subset \mathcal{F}_x$ to any element $\sigma \in \mathbf{S}_n^2(k)$, as follows.

Definition 2. Let $\sigma \in \mathbf{S}_n^2(k)$. A basis $(e_1, \ldots, e_n)$ of V is called a *σ-basis* if it satisfies the following property:

$$x(e_i) = \begin{cases} e_{\sigma(i)} & \text{if } \sigma(i) < i, \\ 0 & \text{otherwise.} \end{cases}$$

A complete flag $F \in \mathcal{F}$ of the form $F = (\langle e_1, \ldots, e_i \rangle_{\mathbb{K}})_{i=0}^n$, where $(e_1, \ldots, e_n)$ is a σ-basis, is called a *σ-flag*. We then denote by $\mathcal{Z}_\sigma$ the set of all the σ-flags.

It is clear that we have $\mathcal{Z}_\sigma \subset \mathcal{F}_x$. Moreover, one can see that the group Z_x acts transitively on the set of σ-bases, thus $\mathcal{Z}_\sigma$ consists of a single Z_x-orbit. Furthermore, for each $F = (V_0, \ldots, V_n) \in \mathcal{Z}_\sigma$, one easily checks that

$$\min\{j = 0, \ldots, n : x(V_i) \subset x(V_{i-1}) + V_j\} = \begin{cases} \sigma(i) & \text{if } \sigma(i) < i, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $\mathcal{Z}_\sigma \cap \mathcal{Z}_{\sigma'} \neq \emptyset$ holds only for $\sigma = \sigma'$. Therefore, $\sigma \mapsto \mathcal{Z}_\sigma$ is an injection from $\mathbf{S}_n^2(k)$ to the set of Z_x-orbits of $\mathcal{F}_x$. Actually this map is bijective, as mentioned among other properties in the following statement (see [4], [8]):

Proposition 1. *As above, $x \in \mathrm{End}(V)$ satisfies $x^2 = 0$ and $\operatorname{rank} x = k$.*

(a) *The map $\sigma \mapsto \mathcal{Z}_\sigma$ is a bijection between $\mathbf{S}_n^2(k)$ and the set of Z_x-orbits of the Springer fiber $\mathcal{F}_x$.*

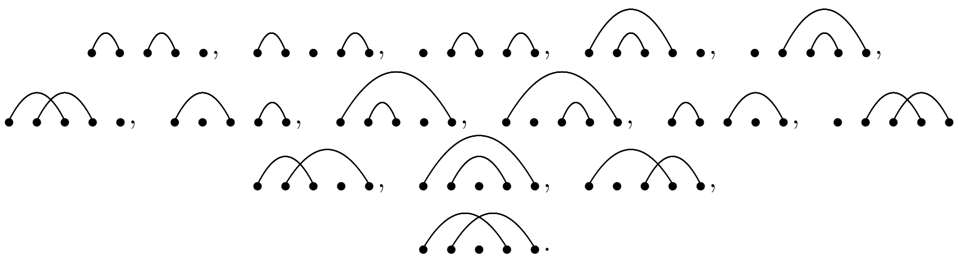
(b) *For each $\sigma \in \mathbf{S}_n^2(k)$, we have $\dim \mathcal{Z}_\sigma = \frac{(n-k)(n-k-1)}{2} + \frac{k(k-1)}{2} - b(\sigma) - c(\sigma)$. In particular, the irreducible components of $\mathcal{F}_x$ are the closures of the orbits*

$\mathcal{Z}_\sigma$ corresponding to the elements $\sigma \in \mathbf{S}_n^2(k)$ such that P_σ has no crossings and no bridges.

(c) *Given $\sigma, \sigma' \in \mathbf{S}_n^2(k)$, we have $\mathcal{Z}_{\sigma'} \subset \overline{\mathcal{Z}_\sigma}$ if and only if $\sigma' \preceq \sigma$.*

(d) *In particular, if $\sigma_0 = (1 : n{-}k{+}1)(2 : n{-}k{+}2) \cdots (k : n)$, then $\mathcal{Z}_{\sigma_0}$ is the only closed Z_x-orbit of $\mathcal{F}_x$, and we have $\dim \mathcal{Z}_{\sigma_0} = \frac{k(k-1)}{2} + \frac{(n-2k)(n-2k-1)}{2}$.*

Example 1. Assume that $n = 5$, $k = 2$. Then, $\mathcal{F}_x$ consists of fifteen Z_x-orbits, corresponding to the link patterns of the following list.



The link patterns in the first (resp. second) (resp. third) line correspond to the orbits of dimension 4 (resp. 3) (resp. 2). The only closed orbit has dimension 1 and it is represented by the link pattern in the fourth line.

Remark 1. The numbers $R_{s,t}(\sigma)$ of Definition 1 can be interpreted as follows. Given $(V_0, \ldots, V_n) \in \mathcal{Z}_\sigma$, by definition of $\mathcal{Z}_\sigma$, we have $R_{s,t}(\sigma) = \dim(V_{s-1} + x(V_t)) - s + 1$ for all $1 \leq s < t \leq n$. Then, Proposition 1(c) means that the orbit closure $\overline{\mathcal{Z}_\sigma}$ can be described as the set of flags $F = (V_0, \ldots, V_n) \in \mathcal{F}_x$ such that

$$\dim(V_{s-1} + x(V_t)) \leq s - 1 + R_{s,t}(\sigma) \quad \text{for all } 1 \leq s < t \leq n.$$

Remark 2. Fix $(e_1, \ldots, e_n)$ a basis of V such that $x(e_i) = 0$ for each $i = 1, \ldots, n-k$, and $x(e_{n-k+i}) = e_i$ for $i = 1, \ldots, k$. In this remark, we relate the orbits $\mathcal{Z}_\sigma$ to the special flags F_w ($w \in \mathbf{S}_n$) of Section 1.1. Note that, by construction of F_w, we have $F_w \in \mathcal{F}_x$ if and only if $w^{-1}(i) < w^{-1}(n-k+i)$ for all $i = 1, \ldots, k$. Moreover, in this case, we have $F_w \in \mathcal{Z}_\sigma$, where $\sigma \in \mathbf{S}_n^2(k)$ is the element defined by $\sigma = \prod_{i=1}^k (w^{-1}(i) : w^{-1}(n-k+i))$. It easily follows that each orbit $\mathcal{Z}_\sigma$ contains at least one special flag F_w, though two different special flags F_w, $F_{w'}$ may belong to the same $\mathcal{Z}_\sigma$.

3. Statement of the results

In this section, we formulate our results that describe the singular locus of the Z_x-orbit closures of the Springer fiber $\mathcal{F}_x$. The results rely on an analogue of the Bruhat graph for link patterns, that we introduce now.

Definition 3. Given $\sigma \in \mathbf{S}_n^2(k)$, we let $X(\sigma)$ be the set of elements $\sigma' \in \mathbf{S}_n^2(k)$ such that the link pattern $P_{\sigma'}$ is obtained from P_σ by one of the following operations:

- Either by interchanging two end points of two arcs which have a crossing (that is, $\sigma' = (i : i')\sigma(i : i')$ where i, i' are (left or right) end points of two arcs of P_σ which have a crossing);
- Or by interchanging a fixed point with the end point of an arc over it (that is, $\sigma' = (i : p)\sigma(i : p)$ where p is a fixed point of P_σ and i is (left or right) end point of an arc spanning p).

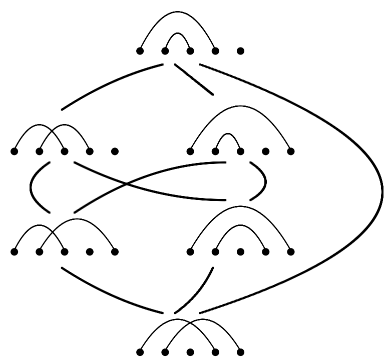


FIGURE 1. The graph $\mathcal{G}(\sigma)$ for σ as in Example 2

Note that we clearly have $\sigma \prec \sigma'$ in this case, hence $\mathcal{Z}_\sigma \subset \overline{\mathcal{Z}_{\sigma'}} \setminus \mathcal{Z}_{\sigma'}$ (cf. Proposition 1(c)). Two elements $\sigma, \sigma' \in \mathbf{S}_n^2(k)$ are said to be *adjacent* if we have either $\sigma' \in X(\sigma)$ or $\sigma \in X(\sigma')$. Finally, given $\sigma \in \mathbf{S}_n^2(k)$, we let $\mathcal{G}(\sigma)$ be the graph whose set of vertices is $\{\tau \in \mathbf{S}_n^2(k) : \tau \preceq \sigma\}$ and with an edge between τ and τ' whenever τ, τ' are adjacent.

Example 2. Let $\sigma = (1:4)(2:3) \in \mathbf{S}_5^2(2)$. The graph $\mathcal{G}(\sigma)$ is drawn in Figure 1, where we represent the vertices τ by the corresponding link patterns P_τ.

Remark 3. (a) By [8, §3], if τ' is in the cover of τ (i.e., $\tau' \prec \tau$ and τ' is maximal for this property), then $\tau \in X(\tau')$, so that τ, τ' are adjacent. In particular, the graph $\mathcal{G}(\sigma)$ is always connected.
(b) Given $\tau \preceq \sigma$, the graph $\mathcal{G}(\tau)$ is the full subgraph of $\mathcal{G}(\sigma)$ whose vertices are the $\tau' \in \mathbf{S}_n^2(k)$ such that $\tau' \preceq \tau$.

Our main result is the following:

Theorem 1. *Let $x \in \mathrm{End}(V)$ be such that $x^2 = 0$ and $\mathrm{rank}\, x = k$. Let $\sigma \in \mathbf{S}_n^2(k)$ and let $\mathcal{Z}_\sigma \subset \mathcal{F}_x$ be the corresponding Z_x-orbit. Let $d_0 := \frac{k(k-1)}{2} + \frac{(n-2k)(n-2k-1)}{2}$ be the dimension of the unique closed orbit of $\mathcal{F}_x$ (cf. Proposition 1(d)).*

(a) *Let $\tau \in \mathbf{S}_n^2(k)$ be such that $\tau \preceq \sigma$, so that $\mathcal{Z}_\tau \subset \overline{\mathcal{Z}_\sigma}$. Let $a(\sigma,\tau)$ denote the number of edges at the vertex τ in the graph $\mathcal{G}(\sigma)$. Then*
$$\dim T_F \overline{\mathcal{Z}_\sigma} = a(\sigma,\tau) + d_0 \quad \textit{for all } F \in \mathcal{Z}_\tau.$$
In particular, $a(\sigma,\tau) \geq \dim \mathcal{Z}_\sigma - d_0$ for each vertex τ.

(b) *The orbit closure $\overline{\mathcal{Z}_\sigma}$ is smooth if and only if the graph $\mathcal{G}(\sigma)$ is regular (i.e., $a(\sigma,\tau) = \dim \mathcal{Z}_\sigma - d_0$ for each vertex τ). Otherwise, the singular locus of $\overline{\mathcal{Z}_\sigma}$ is the union of the orbits $\mathcal{Z}_\tau$ which correspond to the singular vertices τ of $\mathcal{G}(\sigma)$ (i.e., those such that $a(\sigma,\tau) > \dim \mathcal{Z}_\sigma - d_0$).*

In the statement, $T_F\overline{\mathcal{Z}_\sigma}$ stands for the tangent space of $\overline{\mathcal{Z}_\sigma}$ at the point F. Observe that part (b) of the theorem is an immediate consequence of part (a). Thus, our only task is to prove part (a), and this will be done in the next section.

Note that, if we apply the theorem to the element $\sigma \in \mathbf{S}_5^2(2)$ of Example 2, then we get that the orbit closure $\overline{\mathcal{Z}_\sigma}$ is smooth, since the graph $\mathcal{G}(\sigma)$ is regular.

The next corollary proposes a reformulation of the criterion in Theorem 1(b).

Corollary 1. *Let $\sigma \in \mathbf{S}_n^2(k)$ and let $\tau \preceq \sigma$, so that $\mathcal{Z}_\tau \subset \overline{\mathcal{Z}_\sigma}$. Then,*

$$\operatorname{codim}_{\overline{\mathcal{Z}_\sigma}} \mathcal{Z}_\tau \leq |\{\tau' \text{ adjacent to } \tau : \tau \prec \tau' \preceq \sigma\}|,$$

with strict inequality if and only if $\mathcal{Z}_\tau$ lies in the singular locus of $\overline{\mathcal{Z}_\sigma}$.

Proof. In view of Remark 3(b), we have

$$a(\sigma, \tau) = a(\tau, \tau) + |\{\tau' \text{ adjacent to } \tau : \tau \prec \tau' \preceq \sigma\}|.$$

Moreover, since $\mathcal{Z}_\tau$ lies in the regular locus of $\overline{\mathcal{Z}_\tau}$, Theorem 1(b) implies

$$a(\tau, \tau) = \dim \mathcal{Z}_\tau - d_0.$$

This yields

$$|\{\tau' \text{ adjacent to } \tau : \tau \prec \tau' \preceq \sigma\}| - \operatorname{codim}_{\overline{\mathcal{Z}_\sigma}} \mathcal{Z}_\tau = a(\sigma, \tau) - \dim \mathcal{Z}_\sigma + d_0.$$

Thereby, the relations in Theorem 1(b) imply the desired conclusion. □

This reformulation allows us to show that the orbit closures $\overline{\mathcal{Z}_\sigma}$ are regular in codimension one:

Corollary 2. *Let $\sigma \in \mathbf{S}_n^2(k)$ and let $\operatorname{Sing}(\overline{\mathcal{Z}_\sigma})$ be the singular locus of the orbit closure $\overline{\mathcal{Z}_\sigma}$. Then, $\operatorname{codim}_{\overline{\mathcal{Z}_\sigma}} \operatorname{Sing}(\overline{\mathcal{Z}_\sigma}) \geq 2$.*

Proof. We have to check that each 1-codimensional orbit $\mathcal{Z}_\tau \subset \overline{\mathcal{Z}_\sigma}$ lies outside of the singular locus of $\overline{\mathcal{Z}_\sigma}$. Such $\mathcal{Z}_\tau$ lies in the cover of $\overline{\mathcal{Z}_\sigma}$, thus (by Proposition 1(c)) there is no $\tau' \neq \sigma$ such that $\tau \prec \tau' \preceq \sigma$. This implies that

$$|\{\tau' \text{ adjacent to } \tau : \tau \prec \tau' \preceq \sigma\}| \leq 1 = \operatorname{codim}_{\overline{\mathcal{Z}_\sigma}} \mathcal{Z}_\tau,$$

which, by Corollary 1, ensures that $\mathcal{Z}_\tau$ does not lie in the singular locus of $\overline{\mathcal{Z}_\sigma}$. □

In the case where $\overline{\mathcal{Z}_\sigma}$ has maximal dimension (i.e., is a component of $\mathcal{F}_x$), N. Perrin and E. Smirnov [9] have proved that $\overline{\mathcal{Z}_\sigma}$ is a normal variety, which already implies that it is regular in codimension one in this case. In view of Corollary 2, we can speculate that $\overline{\mathcal{Z}_\sigma}$ is normal even if it is not of maximal dimension.

One can deduce from Corollary 1 a simpler criterion of singularity for the closures of the Z_x-orbits of $\mathcal{F}_x$, in the following manner. Let σ_0 be as in Proposition 1(d), so that $\mathcal{Z}_{\sigma_0}$ is the only closed Z_x-orbit of $\mathcal{F}_x$. Then, the singular locus of $\overline{\mathcal{Z}_\sigma}$ is nonempty only when it contains $\mathcal{Z}_{\sigma_0}$. Thus, Corollary 1 implies:

Corollary 3. *Let $\sigma \in \mathbf{S}_n^2(k)$. Then,*

$$\dim \mathcal{Z}_\sigma - d_0 \leq |\{\tau \text{ adjacent to } \sigma_0 : \tau \preceq \sigma\}|$$

with strict inequality if and only if $\overline{\mathcal{Z}_\sigma}$ is singular.

In the case, where $\overline{\mathcal{Z}_\sigma}$ has maximal dimension, Corollary 3 retrieves the conclusion of Theorem 3.1 of [2].

The next examples illustrate our results in the particular situation where $k \leq 2$.

Example 3. (a) Let $n \geq 2$ and suppose that $k = 1$. Then, every Z_x-orbit closure in $\mathcal{F}_x$ is smooth. (b) Let $n \geq 4$ and suppose that $k = 2$. Let $\sigma = (a : b)(c : d) \in \mathbf{S}_n^2(2)$

with $a < b$, $c < d$ and, say, $a < c$. Then, the orbit closure $\overline{\mathcal{Z}_\sigma}$ is singular exactly in the following three cases:

(i) $a < b < c < d$ and ($b > 2$ and $c < n - 1$);
(ii) $a < c < b < d$ and ($a > 1$ and $d < n$);
(iii) $a < c < d < b$ and (($a > 1$ and $c - a > 1$) or ($b < n$ and $b - d > 1$)).

Claim (a) can be shown by applying Corollary 3, or directly as follows. Take $\sigma = (a : b) \in \mathbf{S}_n^2(1)$ (with $a < b$). By Proposition 1(c), we can see that $\overline{\mathcal{Z}_\sigma}$ is the union of the orbits $\mathcal{Z}_{(i:j)}$ for $i \leq a < b \leq j$, hence $\overline{\mathcal{Z}_\sigma}$ is the set of flags $(V_0, \ldots, V_n)$ such that $\operatorname{Im} x \subset V_a$, $V_{b-1} \subset \ker x$. Then, it is straightforward to check that $\overline{\mathcal{Z}_\sigma}$ is an iterated bundle of base type $(\mathrm{Grass}_{b-2}(\mathbb{K}^{n-2}), \mathrm{Grass}_{a-1}(\mathbb{K}^{b-2}), \mathcal{F}^{(a)}, \mathcal{F}^{(b-a-1)}, \mathcal{F}^{(n-b+1)})$, which implies that it is smooth (here, $\mathrm{Grass}_l(\mathbb{K}^m)$ is the variety of l-dimensional subspaces of $\mathbb{K}^m$, and $\mathcal{F}^{(m)}$ denotes the variety of complete flags of $\mathbb{K}^m$).

To show Claim (b), we proceed as follows. First, by Proposition 1(b), we find

$$\dim \mathcal{Z}_\sigma - d_0 = \begin{cases} 2n + a + c - b - d - 5 & \text{if } a < b < c < d, \\ 2n + a + c - b - d - 4 & \text{if } a < c < b < d, \\ 2n + a + c - b - d - 3 & \text{if } a < c < d < b. \end{cases}$$

Next, we enumerate the set $X(\sigma_0)$ of elements $\tau \in \mathbf{S}_n^2(2)$ that are adjacent to σ_0: we can see that $X(\sigma_0) = \{(1 : 2)(n-1 : n), (1 : n)(2 : n-1), (i : n-1)(2 : n), (1 : j)(2 : n), (1 : n-1)(l : n), (1 : n-1)(2 : m) \colon i, j, l, m = 3, \ldots, n-2\}$. A straightforward calculation allows us to determine the elements $\tau \in X(\sigma_0)$, which are $\preceq \sigma$ (according to the different possible configurations of a, b, c, d), and then we can apply the criterion in Corollary 3 in order to reach the conclusion of Claim (b).

Example 4. Let $n \geq 6$ and, as in Example 3(b), suppose $k = 2$. Let $\sigma \in \mathbf{S}_n^2(2)$ be such that $\overline{\mathcal{Z}_\sigma} \subset \mathcal{F}_x$ has maximal dimension and is singular. Thus, $\sigma = (a : a+1)(b : b+1)$ with $a + 1 < b$, $1 < a$, $b < n - 1$. Then (as justified below), the singular locus of $\overline{\mathcal{Z}_\sigma}$ is the closure of the Z_x-orbit $\mathcal{Z}_{(a-1:b+2)(a:b+1)}$. In particular, $\mathrm{Sing}(\overline{\mathcal{Z}_\sigma})$ is irreducible, of codimension $2(b - a)$, and, by Example 3(b), it is smooth.

Let $\tau = (a - 1 : b + 2)(a : b + 1)$. The claimed equality $\mathrm{Sing}(\overline{\mathcal{Z}_\sigma}) = \overline{\mathcal{Z}_\tau}$ is checked in two steps:

(1) We show that the orbit $\mathcal{Z}_\tau$ lies in the singular locus. To do this we describe the set $X(\tau)$ of elements $\tau' \succ \tau$ adjacent to τ: $X(\tau) = \{(i : b+2)(a : b+1), (a-1 : j)(a : b+1), (a-1 : b+2)(l : b+1), (a-1 : b+2)(a : m) : i, j, l, m = a+1, \ldots, b\}$. We can see that $\tau' \preceq \sigma$ for each $\tau' \in X(\tau)$. Thus, $|\{\tau' \text{ adjacent to } \tau : \tau \prec \tau' \preceq \sigma\}| = 4(b - a) > 2(b - a) = \operatorname{codim}_{\overline{\mathcal{Z}_\sigma}} \mathcal{Z}_\tau$. Then, Corollary 1 implies that $\mathcal{Z}_\tau \subset \mathrm{Sing}(\overline{\mathcal{Z}_\sigma})$. Thereby, $\overline{\mathcal{Z}_\tau} \subset \mathrm{Sing}(\overline{\mathcal{Z}_\sigma})$.
(2) We show that, if $\tau' \in \mathbf{S}_n^2(2)$ satisfies $\tau' \preceq \sigma$ and $\tau' \not\preceq \tau$ (that is, $\mathcal{Z}_{\tau'} \subset \overline{\mathcal{Z}_\sigma} \setminus \overline{\mathcal{Z}_\tau}$), then $\mathcal{Z}_{\tau'}$ lies outside of $\mathrm{Sing}(\overline{\mathcal{Z}_\sigma})$. Such τ' takes the form $\tau' = (i_1 : j_1)(i_2 : j_2)$ with $i_1 \leq a < j_1$ and ($a < i_2 \leq b < j_2$ or $i_2 \leq a < \min(j_1, j_2) \leq b < \max(j_1, j_2)$). For each τ' of this type, we consider the set $X(\tau')$ of elements $\tau'' \succ \tau'$ adjacent to τ'. For every configuration of i_1, j_1, i_2, j_2, we can check that $|\{\tau'' \in X(\tau') : \tau'' \preceq \sigma\}| = \operatorname{codim}_{\overline{\mathcal{Z}_\sigma}} \mathcal{Z}_{\tau'}$; thus Corollary 1 implies that $\mathcal{Z}_{\tau'} \not\subset \mathrm{Sing}(\overline{\mathcal{Z}_\sigma})$.

4. Proof of Theorem 1

The purpose of this section is to prove Theorem 1(a). The proof that we propose generalizes the ideas of the proof of the main theorem in [2]. We present our strategy in the next subsection.

4.1. Outline of the proof. Let $\sigma, \tau \in \mathbf{S}_n^2(k)$ such that $\tau \preceq \sigma$, and let $F \in \mathcal{Z}_\tau$. Thus, F is a τ-flag, and there is a τ-basis $(e_1, \ldots, e_n)$ such that $F = (\langle e_1, \ldots, e_i \rangle)_{i=0}^n$. Our goal is then to compute the dimension of the tangent space $T_F \overline{\mathcal{Z}_\sigma}$.

Let $B = \{g \in GL(V) : ge_i \in \langle e_j : j = i, \ldots, n \rangle_{\mathbb{K}}\}$ be the group of automorphisms that are lower triangular in the basis, and let $\Omega = B \cdot F \subset \mathcal{F}$. Thus, Ω is an open neighborhood of F in $\mathcal{F}$. Moreover, for $F' \in \Omega$, there are unique scalars $(\varphi_{i,j}(F'))_{1 \leq i < j \leq n}$ such that $F' = (\langle f_1, \ldots, f_i \rangle_{\mathbb{K}})_{i=0}^n$ with $f_i = e_i + \sum_{j>i} \varphi_{i,j}(F') e_j$. The maps $\varphi_{i,j} : \Omega \to \mathbb{K}$ induce an isomorphism of algebraic varieties $\Omega \cong \mathbb{K}^{\frac{n(n-1)}{2}}$. This equips Ω with a structure of vector space with $(\varphi_{i,j})_{1 \leq i < j \leq n}$ as basis of its dual, and such that F identifies with the zero vector. Let $(\varepsilon_{i,j})_{1 \leq i < j \leq n} \subset \Omega$ be the dual basis. The tangent space $\mathcal{T} := T_F \overline{\mathcal{Z}_\sigma}$ identifies to a vector subspace of Ω. Let $\mathcal{T}^\perp = \{\varphi \in \Omega^* : \varphi(\varepsilon) = 0 \ \ \forall \varepsilon \in \mathcal{T}\}$.

In what follows, we aim to construct a family of $d_0 + a(\sigma, \tau)$ linearly independent vectors of $\mathcal{T}$, and a family of $\frac{n(n-1)}{2} - d_0 - a(\sigma, \tau)$ linearly independent elements of $\mathcal{T}^\perp$. In view of the relation $\dim \mathcal{T} + \dim \mathcal{T}^\perp = \dim \Omega = \frac{n(n-1)}{2}$, this will imply the equality $\dim \mathcal{T} = d_0 + a(\sigma, \tau)$, and the proof will be complete.

4.2. Auxiliary statement. In this subsection, we formulate an explicit statement which describes the construction of the linearly independent elements of $\mathcal{T}$ and $\mathcal{T}^\perp$ outlined above, and which will imply Theorem 1(a).

We first need some notation. Given $1 \leq i < j \leq n$, we let $\omega_{i,j}(\tau) \in \mathbf{S}_n^2(k)$ be the element whose link pattern $P_{\omega_{i,j}(\tau)}$ is obtained from P_τ by interchanging the vertices i and j (i.e., $\omega_{i,j}(\tau) = (i : j)\tau(i : j)$). Let I be the set of pairs (i, j) with $1 \leq i < j \leq n$. We introduce four subsets I_0, I_1^-, I_1^+, I_2.

- Let I_0 be the set of pairs $(i, j) \in I$ such that i, j are both fixed points of P_τ (i.e., $\tau_i = i$ and $\tau_j = j$) or both right end points (i.e., $\tau_i < i$ and $\tau_j < j$).
- Let I_1^- be the set of pairs $(i, j) \in I$ satisfying one of the following conditions:
  (a) i is a fixed point and j is the left end point of an arc in the link pattern P_τ, that is, $\tau_i = i < j < \tau_j$;
  (b) i is right end point and j is fixed point, that is, $\tau_i < i < j = \tau_j$;
  (c) i is right end point and j is left end point, that is, $\tau_i < i < j < \tau_j$;
  (d) i and j are left end points and the arc starting at i is over the arc starting at j, that is, $i < j < \tau_j < \tau_i$.
- Let I_1^+ be the set of pairs $(i, j) \in I$ that satisfy one of the conditions:
  (a) i is a fixed point and j is the right end point of an arc over i, that is, $\tau_j < \tau_i = i < j$;
  (b) j is fixed point and i is left end point of an arc over j, that is, $i < j = \tau_j < \tau_i$;
  (c) i and j are the left end points of two arcs that have a crossing, that is, $i < j < \tau_i < \tau_j$;

  - (d) i and j are respectively left- and right end points of two arcs that have a crossing and j is under the arc starting at i, that is, $\tau_j < i < j < \tau_i$.
- Let $I_2 = I \setminus (I_0 \cup I_1^- \cup I_1^+)$.

Note that $|I_0| = d_0$. A careful comparison with Definition 3 shows that the maps $I_1^- \to \{\tau' \text{ adjacent to } \tau : \tau' \prec \tau\}$, $(i,j) \mapsto \omega_{i,j}(\tau)$ and $I_1^+ \to \{\tau' \text{ adjacent to } \tau : \tau' \succ \tau\}$, $(i,j) \mapsto \omega_{i,j}(\tau)$ are bijective.

Proposition 2. *Let* $(i,j) \in I$.

- (i) *If* $(i,j) \in I_0$, *then* $\varepsilon_{i,j} \in \mathcal{T}$, *or* $\varepsilon_{i,j} + \varepsilon_{i',j'} \in \mathcal{T}$ *for some* $i' < i$, $j' < j$.
- (ii) *If* $(i,j) \in I_1^- \cup I_1^+$ *and* $\omega_{i,j}(\tau) \preceq \sigma$, *then* $\varepsilon_{i,j} \in \mathcal{T}$.
- (iii) *If* $(i,j) \in I_1^- \cup I_1^+$ *and* $\omega_{i,j}(\tau) \not\preceq \sigma$, *then* $\varphi_{i,j} \in \mathcal{T}^\perp$, *or* $\varphi_{i,j} - \varphi_{i',j'} \in \mathcal{T}^\perp$ *for some* $i' > i$, $j' > j$.
- (iv) *If* $(i,j) \in I_2$, *then* $\varphi_{i,j} \in \mathcal{T}^\perp$, *or* $\varphi_{i,j} - \varphi_{i',j'} \in \mathcal{T}^\perp$ *for some* $i' > i$, $j' > j$.

Observe that $|\{(i,j) \in I_1^- \cup I_1^+ : \omega_{i,j}(\tau) \preceq \sigma\}| = a(\sigma,\tau)$. Thus, parts (i) and (ii) of the proposition provide $d_0 + a(\sigma,\tau)$ linearly independent elements of $\mathcal{T}$. Parts (iii) and (iv) provide $\frac{n(n-1)}{2} - d_0 - a(\sigma,\tau)$ linearly independent elements of $\mathcal{T}^\perp$. Thereby, in view of what is explained in Section 4.1, it suffices to show Proposition 2 in order to get Theorem 1(a). The proof of the proposition is cut into the following subsections.

4.3. Proof of Proposition 2(i). Let $(i,j) \in I_0$. Thus, we have either ($i = \tau_i$ and $j = \tau_j$) or ($\tau_i < i$ and $\tau_j < j$). For each $t \in \mathbb{K}$, we define an element $g_t \in Z_x$ by letting $g_t(e_i) = e_i + te_j$, $g_t(e_{\tau_i}) = e_{\tau_i} + te_{\tau_j}$, and $g_t(e_l) = e_l$ for each $l \in \{1,\ldots,n\} \setminus \{i,\tau_i\}$. The curve $\{g_t(F) : t \in \mathbb{K}\}$ lies in $\mathcal{Z}_\tau$, hence in $\overline{\mathcal{Z}_\sigma}$, and the tangent vector at $t = 0$ to this curve is thereby an element of $\mathcal{T}$.

If ($\tau_i < i$, $\tau_j < j$ and $\tau_i < \tau_j$), then we have $g_t(F) = (\langle f_1,\ldots,f_l\rangle_{\mathbb{K}})_{l=0}^n$ with $f_i = e_i + te_j$, $f_{\tau_i} = e_{\tau_i} + te_{\tau_j}$, and $f_l = e_l$ for each $l \notin \{i,\tau_i\}$. In other words, $g_t(F) = t(\varepsilon_{i,j} + \varepsilon_{\tau_i,\tau_j})$. Thus, the tangent vector at $t = 0$ to the curve $\{g_t(F) : t \in \mathbb{K}\}$ is $\varepsilon_{i,j} + \varepsilon_{\tau_i,\tau_j}$. This implies that $\varepsilon_{i,j} + \varepsilon_{\tau_i,\tau_j} \in \mathcal{T}$ in this case.

If ($i = \tau_i$ and $j = \tau_j$) or ($\tau_i < i$, $\tau_j < j$ and $\tau_i > \tau_j$), then we have $g_t(F) = (\langle f_1,\ldots,f_l\rangle_{\mathbb{K}})_{l=0}^n$ with $f_i = e_i + te_j$ and $f_l = e_l$ for each $l \neq i$. Thereby, $g_t(F) = t\varepsilon_{i,j}$. We therefore obtain that $\varepsilon_{i,j} \in \mathcal{T}$ in this case. This shows Proposition 2(i).

4.4. Proof of Proposition 2(ii). We first concentrate on the subset I_1^-. Since $\tau \preceq \sigma$, each $(i,j) \in I_1^-$ satisfies $\omega_{i,j}(\tau) \prec \sigma$. Thus, we have to show that $\varepsilon_{i,j} \in \mathcal{T}$ for each $(i,j) \in I_1^-$.

Recall the conditions (a) to (d) of the definition of I_1^-. If (i,j) satisfies one of the conditions (a) to (c), then the automorphism $g_t \in GL(V)$ defined by $g_t(e_l) = e_l$ for $l \neq i$ and $g_t(e_i) = e_i + te_j$ satisfies $g_t \in Z_x$. If (i,j) satisfies condition (d), then we rather consider the map $g_t \in GL(V)$ given by $g_t(e_l) = e_l$ for all $l \notin \{i,\tau_i\}$, $g_t(e_i) = e_i + te_j$, and $g_t(e_{\tau_i}) = e_{\tau_i} + te_{\tau_j}$; again, we have $g_t \in Z_x$. Thus, in both cases, the curve $\{g_t(F) : t \in \mathbb{K}\}$ lies in $\overline{\mathcal{Z}_\sigma}$, and the tangent vector at $t = 0$ to this curve is an element of $\mathcal{T}$. In both cases, we can see that $g_t(F) = (\langle f_1,\ldots,f_l\rangle_{\mathbb{K}})_{l=0}^n$ with $f_i = e_i + te_j$ and $f_l = e_l$ for each $l \neq i$. So, $g_t(F) = t\varepsilon_{i,j}$. Differentiating at $t = 0$, we obtain that $\varepsilon_{i,j} \in \mathcal{T}$, as claimed.

Next, let us focus on the subset I_1^+: we take $(i,j) \in I_1^+$ satisfying $\omega_{i,j}(\tau) \preceq \sigma$, and we have to show that $\varepsilon_{i,j} \in \mathcal{T}$.

Set $e'_i = e_j$, $e'_j = e_i$, and $e'_l = e_l$ for $l \notin \{i, j\}$. Using that $(i, j) \in I_1^+$, we can check that $(e'_1, \ldots, e'_n)$ is a $\omega_{i,j}(\tau)$-basis. So $F' := (\langle e'_1, \ldots, e'_l \rangle_{\mathbb{K}})_{l=0}^n$ is a $\omega_{i,j}(\tau)$-flag, that is, $F' \in \mathcal{Z}_{\omega_{i,j}(\tau)}$. Since $\omega_{i,j}(\tau) \preceq \sigma$, by Proposition 1, we deduce that $F' \in \overline{\mathcal{Z}_\sigma}$.

Recall the conditions (a) to (d) of the definition of I_1^+. If (i, j) satisfies (a), (b) or (d), then we consider $g_t \in GL(V)$ defined by $g_t(e_j) = e_j + te_i$, and $g_t(e_l) = e_l$ for all $l \neq j$. If (i, j) satisfies (c), then we rather take $g_t \in GL(V)$ such that $g_t(e_j) = e_j + te_i$, $g_t(e_{\tau_j}) = e_{\tau_j} + te_{\tau_i}$, and $g_t(e_l) = e_l$ for all $l \notin \{j, \tau_j\}$. In both cases, we see that $g_t \in Z_x$, hence the curve $\{g_t(F') : t \in \mathbb{K}\}$ is contained in $\overline{\mathcal{Z}_\sigma}$. In both cases, we can see that $F = \lim_{t\to\infty} g_t(F')$. In fact, for each $t \in \mathbb{K}^\times$, we have $g_t(F') = (\langle f_1, \ldots, f_l \rangle_{\mathbb{K}})_{l=0}^n$ where $f_l = e_l$ for $l \neq i$ and $f_i = e_i + t^{-1}e_j$. In other words, $g_t(F') = t^{-1}\varepsilon_{i,j}$. In particular, this yields $t\varepsilon_{i,j} \in \overline{\mathcal{Z}_\sigma}$ for all $t \in \mathbb{K}$. Differentiating at $t = 0$, we obtain $\varepsilon_{i,j} \in \mathcal{T}$. Proposition 2(ii) ensues.

4.5. Proof of Proposition 2(iii). Let $(i, j) \in I_1^- \cup I_1^+$ be such that $\omega_{i,j}(\tau) \not\preceq \sigma$. We have necessarily $(i, j) \in I_1^+$ (because each $(l, m) \in I_1^-$ satisfies $\omega_{l,m}(\tau) \prec \tau$ and so $\omega_{l,m}(\tau) \preceq \sigma$), hence (i, j) satisfies one of the conditions (a) to (d) in the definition of I_1^+ formulated in Section 4.2.

Recall that $R_{s,t}(\tau)$ (for $1 \leq s < t \leq n$) denotes the number of arcs between s and t in the link pattern P_τ. On one hand, since τ is such that $\tau \preceq \sigma$, one has $R_{s,t}(\tau) \leq R_{s,t}(\sigma)$ for each s, t, and moreover, in view of the definition of $\omega_{i,j}(\tau)$ and of the set I_1^+, one has $R_{s,t}(\tau) \leq R_{s,t}(\omega_{i,j}(\tau)) \leq R_{s,t}(\tau) + 1$. On the other hand, the assumption that $\omega_{i,j}(\tau) \not\preceq \sigma$ implies that there are $s < t$ such that $R_{s,t}(\omega_{i,j}(\tau)) > R_{s,t}(\sigma)$. We fix a pair (s, t) with the latter property. For this pair (s, t), we therefore have

$$R_{s,t}(\sigma) = R_{s,t}(\tau) \quad \text{and} \quad R_{s,t}(\omega_{i,j}(\tau)) = R_{s,t}(\tau) + 1.$$

We abbreviate $r = R_{s,t}(\sigma)$. By the equality $R_{s,t}(\tau) = R_{s,t}(\sigma)$, there are indices $i_1 < \cdots < i_r$ such that $s \leq \tau_{i_p} < i_p \leq t$ for each p. By Remark 1, every element $F' = (V_0, \ldots, V_n) \in \Omega \cap \overline{\mathcal{Z}_\sigma}$ satisfies

$$(*) \qquad \dim(V_{s-1} + x(V_t)) \leq s + r - 1.$$

In what follows, we will rely on relation $(*)$ and on the indices $i_1, \ldots, i_r$ in order to deduce certain relations in the tangent space $\mathcal{T}$.

In the next step, we determine all the possible configurations of $s, t, i, \tau_i, j, \tau_j$.

- Assume first that (i, j) satisfies condition (a) of the definition of I_1^+, i.e., $\tau_j < \tau_i = i < j$. In this case, $\tau, \omega_{i,j}(\tau)$ only differ by the fact that the link pattern P_τ contains (τ_j, j) as an arc and i as a fixed point, whereas $P_{\omega_{i,j}(\tau)}$ contains (τ_j, i) as an arc and j as a fixed point. Thus, the relation $R_{s,t}(\omega_{i,j}(\tau)) > R_{s,t}(\tau)$ holds only if $\tau_j, i \in \{s, \ldots, t\}$ and $j \notin \{s, \ldots, t\}$, that is, only if $s \leq \tau_j < i = \tau_i \leq t < j$.
- If (i, j) satisfies condition (b) of the definition of I_1^+, that is, $i < j = \tau_j < \tau_i$, then similarly the fact that $R_{s,t}(\omega_{i,j}(\tau)) > R_{s,t}(\tau)$ implies that $j, \tau_i \in \{s, \ldots, t\}$ and $i \notin \{s, \ldots, t\}$. Thereby, $i < s \leq j = \tau_j < \tau_i \leq t$ in this case.
- Assume that (i, j) satisfies condition (c) of the definition of I_1^+, that is, $i < j < \tau_i < \tau_j$. In this case, $\tau, \omega_{i,j}(\tau)$ differ by the fact that $(i, \tau_i), (j, \tau_j)$ are arcs of P_τ, whereas $(i, \tau_j), (j, \tau_i)$ are arcs of $P_{\omega_{i,j}(\tau)}$. Then, we can see that we

have $R_{s,t}(\omega_{i,j}(\tau)) > R_{s,t}(\tau)$ only for $j, \tau_i \in \{s, \ldots, t\}$ and $i, \tau_j \notin \{s, \ldots, t\}$. Whence, $i < s \leq j < \tau_i \leq t < \tau_j$.

- Next, assume that (i, j) satisfies condition (d) of the definition of I_1^+, so $\tau_j < i < j < \tau_i$. Here, the only difference between τ and $\omega_{i,j}(\tau)$ is that $(i, \tau_i), (\tau_j, j)$ are arcs of P_τ, whereas $(\tau_j, i), (j, \tau_i)$ are arcs of $P_{\omega_{i,j}(\tau)}$. Thus, the fact that $R_{s,t}(\omega_{i,j}(\tau)) > R_{s,t}(\tau)$ implies that we have either ($\tau_j, i \in \{s, \ldots, t\}$ and $j, \tau_i \notin \{s, \ldots, t\}$) or ($j, \tau_i \in \{s, \ldots, t\}$ and $\tau_j, i \notin \{s, \ldots, t\}$). Therefore, in this case, we have either $s \leq \tau_j < i \leq t < j < \tau_i$ or $\tau_j < i < s \leq j < \tau_i \leq t$.

Finally, this analysis reveals that only two situations can happen: either

(1) $s \leq \tau_j < i \leq t < j$ and $\tau_i \geq i$, or
(2) $i < s \leq j < \tau_i \leq t$ and ($\tau_j = j$ or $\tau_j \notin \{s, \ldots, t\}$).

Now, let us consider successively the situations (1), (2). In both cases, we determine through formula $(*)$ a linear equation satisfied in the tangent space $\mathcal{T}$, that is, we construct an element of the space $\mathcal{T}^\perp$. To this end, we start with an arbitrary element $F' = (V_0, \ldots, V_n) \in \Omega \cap \overline{\mathcal{Z}_\sigma}$ with $V_l = \langle f_1, \ldots, f_l \rangle_{\mathbb{K}}$ for all l, where $f_l = e_l + \sum_{m>l} \varphi_{l,m}(F') e_m$ as in Section 4.1. Recall that we have fixed indices $i_1 < \ldots < i_r$ with $s \leq \tau_{i_p} < i_p \leq t$ for each p.

(1) We consider the family of vectors $(f_1, \ldots, f_{s-1}, x(f_{i_1}), \ldots, x(f_{i_r}), x(f_i))$. This family comprises $s+r$ vectors that all lie in the subspace $V_{s-1} + x(V_t)$. In view of formula $(*)$, the family is linearly dependent. Thus, the matrix of the family in the basis $(e_1, \ldots, e_n)$ has rank $< s+r$. In particular, the minor determinant with respect to the sub-basis $(e_1, \ldots, e_{s-1}, e_{\tau_{i_1}}, \ldots, e_{\tau_{i_r}}, e_{\tau_j})$ is equal to zero. The nullity of this minor determinant yields a polynomial relation satisfied on $\Omega \cap \overline{\mathcal{Z}_\sigma}$. Let us describe this relation more precisely. Note that $x(f_{i_p}) = e_{\tau_{i_p}} + \sum_{l \in L_{i_p}} \varphi_{i_p,l}(F') e_{\tau_l}$ for each p, where $L_{i_p} := \{l > i_p : \tau_l < l\}$. Furthermore, since $\tau_i \geq i$ and $\tau_j < j$, we have $x(e_i) = 0$, $x(e_j) = e_{\tau_j}$ and thus $x(f_i) = \varphi_{i,j}(F') e_{\tau_j} + \sum_{l \in L_i \setminus \{j\}} \varphi_{i,l}(F') e_{\tau_l}$ where $L_i := \{l > i : \tau_l < l\}$. Therefore, a relation of the following form holds on $\Omega \cap \overline{\mathcal{Z}_\sigma}$:

$$\begin{vmatrix} 1 & * & \cdots & (*) & * \\ * & 1 & & \vdots & * \\ \vdots & & \ddots & & \vdots \\ & (*) & & 1 & * \\ & & \cdots & * & \varphi_{i,j} \end{vmatrix} = 0,$$

where the coefficients of the matrix marked with the symbol $*$ are either $\varphi_{l,m}$ for some $l < m$, or zero. Developing along the last column, this relation can be written $\varphi_{i,j} = P$, where P is a polynomial in the $\varphi_{l,m}$'s with no term of degree ≤ 1. The differential at 0 of $\varphi_{i,j} - P$ therefore vanishes on the tangent space $\mathcal{T}$. This differential is simply $\varphi_{i,j}$, whence $\varphi_{i,j} \in \mathcal{T}^\perp$.

(2) Here, we consider the family of vectors

$$(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{s-1}, x(f_{i_1}), \ldots, x(f_{i_r}), f_i, x(f_{\tau_i})).$$

As above, this family comprises $r+s$ vectors, all lying in $V_{s-1}+x(V_t)$, so that formula $(*)$ implies that the family is linearly dependent. Thus, the matrix of the family in the basis $(e_1,\ldots,e_n)$ has rank $< s+r$, hence the minor determinant with respect to the sub-basis $(e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_{s-1},e_{\tau_{i_1}},\ldots,e_{\tau_{i_r}},$ $e_i,e_j)$ is equal to zero (here we use the assumption ($\tau_j=j$ or $\tau_j\notin\{s,\ldots,t\}$), which guarantees that $j\notin\{\tau_{i_1},\ldots,\tau_{i_r}\}$, so that the vectors in the tuple are all distinct and form indeed a sub-basis). The nullity of this minor determinant provides a polynomial relation on $\Omega\cap\overline{\mathcal{Z}_\sigma}$ that we describe now more carefully. As in situation (1), one has $x(f_{i_p})=e_{\tau_{i_p}}+\sum_{l\in L_{i_p}}\varphi_{i_p,l}(F')e_{\tau_l}$ for each p. Since $i<\tau_i$, one has $x(e_{\tau_i})=e_i$, thus $x(f_{\tau_i})=e_i+\sum_{l\in L_{\tau_i}}\varphi_{\tau_i,l}(F')e_{\tau_l}$ with $L_{\tau_i}:=\{l>\tau_i:\tau_l<l\}$. Hence a relation of the following form is satisfied on $\Omega\cap\overline{\mathcal{Z}_\sigma}$:

$$\begin{vmatrix} 1 & * & \cdots & (*) & * \\ * & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & 1 & * & * \\ (*) & \cdots & * & 1 & 1 \\ * & \cdots & * & \varphi_{i,j} & \psi \end{vmatrix} = 0,$$

where as before the symbol $*$ is either $\varphi_{l,m}$ for some $l<m$, or zero, and where we have either $\psi=\varphi_{\tau_i,\tau_j}$ (if $\tau_j\in L_{\tau_i}$) or $\psi=0$ (otherwise). Developing the determinant along the last column, the relation becomes $\varphi_{i,j}-\psi=P$, where P is a polynomial in the $\varphi_{l,m}$'s with no term of degree ≤ 1. Differentiating at 0, we deduce that the relation $\varphi_{i,j}-\psi=0$ holds on the tangent space $\mathcal{T}$. Therefore, $\varphi_{i,j}-\psi\in\mathcal{T}^\perp$. This completes the proof of Proposition 2(iii).

4.6. Proof of Proposition 2(iv). We first note that the set I_2 is formed by the pairs (i,j), $1\leq i<j\leq n$, which satisfy at least one of the following conditions:

(a) $i\leq\tau_j<j$ and $i\leq\tau_i$;
(b) $i<\tau_i<j$.

Indeed, these are the only configurations that are not represented in I_0,I_1^-,I_1^+.

Take an arbitrary element $F'=(\langle f_1,\ldots,f_l\rangle_{\mathbb{K}})_{l=0}^n\in\Omega\cap\overline{\mathcal{Z}_\sigma}$, where $f_l=e_l+\sum_{m>l}\varphi_{l,m}(F')e_m$ as before. We will exploit the property of F' to be x-stable.

First, assume that (i,j) satisfies condition (a) above. In this case, one has $x(e_i)=0$ and $x(e_j)=e_{\tau_j}$, so that $x(f_i)=\varphi_{i,j}(F')e_{\tau_j}+\sum_{l\in L_i\setminus\{j\}}\varphi_{i,l}(F')e_{\tau_l}$ where $L_i=\{l>i:\tau_l<l\}$. Since F' is x-stable, the family of vectors $(f_1,\ldots,f_{i-1},x(f_i))$ is linearly dependent. Thus, the matrix of $(f_1,\ldots,f_{i-1},x(f_i))$ with respect to the basis $(e_1,\ldots,e_n)$ has rank $<i$. In particular, the minor determinant with respect to the sub-basis $(e_1,\ldots,e_{i-1},e_{\tau_j})$ is zero. The nullity of this minor determinant yields a polynomial relation that is satisfied on the subset $\Omega\cap\overline{\mathcal{Z}_\sigma}$. This relation takes the form

$$\begin{vmatrix} 1 & 0 & \cdots & (0) & * \\ * & 1 & & \vdots & * \\ \vdots & & \ddots & & \vdots \\ & (*) & & 1 & * \\ & & \cdots & * & \varphi_{i,j} \end{vmatrix} = 0,$$

where the coefficients of the matrix represented with the star $*$ are either $\varphi_{l,m}$ for some l, m, or zero. Developing along the last column, the relation becomes $\varphi_{i,j} = P$, where P is a polynomial in the $\varphi_{l,m}$'s with no term of degree ≤ 1. The differential at 0 of $\varphi_{i,j} - P$ is therefore a linear form which vanishes on the tangent space $\mathcal{T}$, whence $\varphi_{i,j} \in \mathcal{T}^{\perp}$.

Next, assume that (i,j) satisfies condition (b). Hence, $x(e_{\tau_i}) = e_i$, so that $x(f_{\tau_i}) = e_i + \sum_{l \in L_{\tau_i}} \varphi_{\tau_i,l}(F')e_{\tau_l}$, where $L_{\tau_i} = \{l > \tau_i : \tau_l < l\}$. Since F' is x-stable, we see that the family of vectors $(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{\tau_i-1}, f_i, x(f_{\tau_i}))$ is linearly dependent. Thus, its matrix in the basis $(e_1, \ldots, e_n)$ has rank $< \tau_i$, and so the minor determinant with respect to the sub-basis $(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{\tau_i-1}, e_i, e_j)$ is zero. This yields a polynomial relation of the following form satisfied on the subset $\Omega \cap \overline{\mathcal{Z}_\sigma}$:

$$\begin{vmatrix} 1 & & (0) & * & * \\ * & \ddots & & \vdots & * \\ \vdots & \ddots & 1 & * & \vdots \\ (*) & \cdots & * & 1 & 1 \\ * & \cdots & * & \varphi_{i,j} & \psi \end{vmatrix} = 0,$$

where, as above, each star $*$ is either 0 or $\varphi_{l,m}$ for some l, m, and where either $\psi = \varphi_{\tau_i,\tau_j}$ (if $\tau_j \in L_{\tau_i}$) or $\psi = 0$ (otherwise). Developing the determinant along the last column, we get $\varphi_{i,j} - \psi = P$, where P is a polynomial in the $\varphi_{l,m}$'s with no term of degree ≤ 1. Differentiating at 0, we deduce that the relation $\varphi_{i,j} - \psi = 0$ is satisfied on the tangent space $\mathcal{T}$, whence $\varphi_{i,j} - \psi \in \mathcal{T}^{\perp}$. This achieves the proof of Proposition 2(iv).

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