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FINITE INJECTIVE DIMENSION OVER RINGS WITH NOETHERIAN COHOMOLOGY

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Abstract. We study rings that have Noetherian cohomology over a ring of cohomology operators. Examples of such rings include commutative complete intersection rings and finite-dimensional cocommutative Hopf algebras. The main result is a criterion for a complex of modules over a ring with Noetherian cohomology to have finite injective dimension. The criterion implies in particular that for any module over such a ring, if all higher self-extensions of the module vanish, then it must have finite injective dimension. This generalizes a theorem of Avramov and Buchweitz for complete intersection rings, and a well-known theorem in the representation theory of finite groups from finitely generated to arbitrary modules.

1. Introduction

Let R be an associative ring and S a ring of cohomology operators on R. Thus, S is a commutative graded ring and there exists a family of homogeneous maps of graded rings, indexed by complexes of R-modules M,

$$\zeta_M : S \to \operatorname{Ext}^*_R(M, M),$$

that satisfies a certain commutativity condition. See Section 3 for the full definition. We say R has *Noetherian cohomology* over S if $\operatorname{Ext}^*_R(M, M)$ is a Noetherian S-module via ζ_M for all complexes M, which have Noetherian cohomology over R.

In this paper, we prove the following:

Theorem. *Let R be a ring with Noetherian cohomology over a ring of cohomology operators S, and let M be a complex of R-modules with $H^n(M) = 0$ for $n \gg 0$. Let S^+ be the ideal $\oplus_{i\geq 1} S^i$. If the S-module $\operatorname{Ext}^*_R(M, M)$ is S^+-torsion, then M has finite injective dimension.*

Recall that $\operatorname{Ext}^*_R(M, M)$ is S^+-torsion if for every $x \in \operatorname{Ext}^*_R(M, M)$ there exists an integer n such that $(S^+)^n x = 0$. There is, for instance, an integer l depending on the degrees of the generators of S^+, such that if $\operatorname{Ext}^{nl}_R(M, M) = 0$ for some $n \geq 1$, then $\operatorname{Ext}^*_R(M, M)$ is S^+-torsion; see 4.5. A complex has finite injective dimension if it has a bounded above semi-injective resolution; see 2.4. If the complex in question is a module, then a semi-injective resolution is an injective resolution in the classical sense. To compute $\operatorname{Ext}^*_R(M, M)$ for a complex M, one may use a semi-injective resolution, and so if M is a module then $\operatorname{Ext}^*_R(M, M)$ agrees with the classical notion. Thus, a special case of the theorem is that if M is an R-module with $\operatorname{Ext}^n_R(M, M) = 0$ for $n \gg 0$, then M has finite injective dimension.

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There are many rings with Noetherian cohomology and hence to which the result above applies. First, assume that R is a ring of the form $Q/(f_1, \dots, f_c)$, where Q is a commutative Noetherian regular ring of finite Krull dimension and $f_1, \dots, f_c$ is a Q-regular sequence. The graded polynomial ring $S = R[\chi_1, \dots, \chi_c]$, where the degree of each χ_i is 2, is a ring of cohomology operators for R and R has Noetherian cohomology over S by [11]. In this context, the theorem above generalizes a key instance of [2, Theorem 4.2] from finitely generated modules to a large class of complexes, including all modules:

Corollary A. *Let $R = Q/(f_1, \dots, f_c)$, where Q is a commutative Noetherian regular ring of finite Krull dimension and $f_1, \dots, f_c$ is a Q-regular sequence. Let M be a complex of R-modules with $H^n(M) = 0$ for $n \gg 0$. If $\operatorname{Ext}^{2n}_R(M, M) = 0$ for some $n \geq 1$, then M has finite injective dimension.*

Indeed, if $\operatorname{Ext}^{2n}_R(M, M) = 0$ for some n, then $\operatorname{Ext}^*_R(M, M)$ must be S^+-torsion since the degree of χ_i is 2. Thus $l = 2$ in the notation above; see 5.1 for further details.

Now let R be a Hopf algebra over a field k. Any commutative subring of $\operatorname{Ext}^*_R(k, k)$ is a ring of cohomology operators on R; see 5.5. Let S be the center of $\operatorname{Ext}^*_R(k, k)$. It follows from the main result of [10] that every finite-dimensional cocommutative Hopf algebra has Noetherian cohomology over S. Thus we have the following:

Corollary B. *Let R be a finite-dimensional cocommutative Hopf algebra over a field k and let S be the center of $\operatorname{Ext}^*_R(k, k)$. For an R-complex M with $H^n(M) = 0$ for all $n \gg 0$, if $\operatorname{Ext}^*_R(M, M)$ is S^+-torsion, then M has finite injective dimension.*

In particular, the result applies to the group ring of a finite group over a field where it generalizes a well-known result for finite-dimensional representations to, in particular, arbitrary representations.

For the proof of the main theorem, we work in an "infinite completion" of the bounded derived category of Noetherian R-modules. This allows us to avoid finiteness conditions on the complexes to which the criterion is applied. By [13], such a completion is given by the homotopy category of injective R-modules. We recall relevant facts about this category in Section 2. In Section 3, we give the precise definition of a ring of cohomology operators and prove a preliminary result. The proof of the main theorem occupies Section 4 and in Section 5 we apply it to the cases discussed above.

The techniques in this paper are inspired by [7]. We have minimized the use of machinery from that paper to make this one closer to being self-contained.

2. Background

Throughout R denotes an associative ring. By "R-module" we mean a left-module over R. An R-complex is a complex of R-modules.

In this section, we briefly recall some definitions and results on triangulated categories. We then review the homological algebra of complexes that we will need.

2.1. Let M be an R-complex. We write $\mathrm{H}^n(M)$ for the nth cohomology group of M and $\mathrm{H}(M)$ for the graded R-module that in degree n is $\mathrm{H}^n(M)$. We say M has *finite cohomology* if $\mathrm{H}(M)$ is a Noetherian R-module. This implies, in particular, that $\mathrm{H}^n(M) = 0$ for $|n| \gg 0$. The complex M is *acyclic* if $\mathrm{H}(M) = 0$.

Let N be another R-complex. We denote the *Hom-complex* between M and N by $\mathrm{Hom}_R(M,N)$. This has components and differential given by

$$\mathrm{Hom}_R(M,N)^n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R\left(M^i, N^{i+n}\right) \qquad \partial(f) = \partial^N \circ f - (-1)^{|f|} f \circ \partial^M,$$

where $|f|$ is the degree of f. A *morphism* $f : M \to N$ is a degree zero cycle of $\mathrm{Hom}_R(M,N)$, i.e., $|f| = 0$ and $\partial(f) = 0$. It is a *quasi-isomorphism* when $\mathrm{H}(f) : \mathrm{H}(M) \to \mathrm{H}(N)$ is an isomorphism.

2.2. The *homotopy category of injective R-modules*, denoted by $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$, has as objects complexes X such that X^i is an injective R-module for all i. The morphisms between objects X, Y are given by

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X,Y) := \mathrm{H}^0(\mathrm{Hom}_R(X,Y)).$$

In other words, morphisms in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ are homotopy equivalence classes of morphisms of complexes.

The standard shift functor on $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is denoted Σ. Thus for a complex

$$X = \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots,$$

we have that $(\Sigma X)^n = X^{n+1}$ and $\partial_{\Sigma X} = -\partial_X$. By $\mathrm{Hom}^*_{\mathsf{K}}(X,Y)$ we denote the $\mathbb{Z}$-graded abelian group that in degree n is $\mathrm{Hom}_{\mathsf{K}}(X, \Sigma^n Y)$. With multiplication given by composition $\mathrm{Hom}^*_{\mathsf{K}}(X,X)$ is a graded ring, whereas $\mathrm{Hom}^*_{\mathsf{K}}(X,Y)$ is a bimodule under the left action by $\mathrm{Hom}^*_{\mathsf{K}}(Y,Y)$ and the right action by $\mathrm{Hom}^*_{\mathsf{K}}(X,X)$.

2.3. The category $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is triangulated. For a proof and reference on triangulated categories see e.g., [17]. A triangulated subcategory of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is *thick* if it is closed under direct summands. It is *localizing* when it is closed under set-indexed direct sums. Every localizing subcategory in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is automatically thick; see e.g., the proof of [12, 1.4.8].

For a subclass of objects C in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$, we denote by $\mathsf{thick}_{\mathsf{K}}(\mathsf{C})$, respectively $\mathsf{loc}_{\mathsf{K}}(\mathsf{C})$, the smallest thick, respectively localizing, subcategory containing C. One may realize these by taking the intersection of all thick, respectively localizing, subcategories containing C.

An object $C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is *compact* if the natural map

$$\bigoplus_{i\in I} \mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C, X_i) \to \mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C, \bigoplus_{i\in I} X_i)$$

is an isomorphism for any set of objects $\{X_i\}_{i\in I}$ of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. We denote the collection of compact objects of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ by $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}}$.

When R is left-Noetherian, [13, 2.3.1] shows that $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is *compactly generated*, i.e., an object $X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is nonzero if and only if there exists a compact object $C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ such that $\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C,X) \neq 0$.

2.4. A complex of injective modules I is *semi-injective* if for all acyclic complexes A, the complex $\mathrm{Hom}_R(A,I)$ is acyclic. When I is semi-injective it has the following lifting property: for every morphism $\alpha : M \to I$ and every quasi-isomorphism $\beta : M \to N$

there exists a unique up to homotopy map $\gamma : N \to I$ making the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{\beta} & N \,. \\ {\scriptstyle\alpha}\big\downarrow & \swarrow{\scriptstyle\gamma} & \\ I & & \end{array}$$

A *semi-injective resolution* of a complex M is a quasi-isomorphism $\eta_M : M \to \mathsf{i}M$, where $\mathsf{i}M$ is semi-injective. Every complex has a semi-injective resolution; this was first proven in [16]. Moreover, by the lifting property, a semi-injective resolution is unique up to isomorphism in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$.

When M is a module, viewed as a complex concentrated in degree 0, a semi-injective resolution of M is just an injective resolution in the usual sense.

2.5. Let $\mathsf{i}M, \mathsf{i}N$ be semi-injective resolutions of complexes M, N, respectively. Define the derived Hom functors as

$$\operatorname{Ext}^n_R(M,N) := \operatorname{Hom}_{\mathsf{K}}(\mathsf{i}M, \Sigma^n \mathsf{i}N) \cong \mathrm{H}^n \operatorname{Hom}_R(\mathsf{i}M, \mathsf{i}N)\,.$$

Set $\operatorname{Ext}^*_R(M,N)$ to be the graded R-module which in degree n is $\operatorname{Ext}^n_R(M,N)$. The lifting property of semi-injective complexes shows that $\operatorname{Ext}^*_R(M,N)$ is independent of the choice of resolutions, up to isomorphism.

If there exists a semi-injective resolution $\eta_M : M \to \mathsf{i}M$ such that $(\mathsf{i}M)^n = 0$ for all $n \gg 0$, then we say M has *finite injective dimension* and write $\operatorname{inj\,dim}_R M < \infty$.

2.6. Let $\mathsf{D}(R)$ be the unbounded derived category of R-modules; see e.g., [17] for the definition. We denote by Q the localization functor $Q : \mathsf{K}(\mathsf{Inj}\,\mathsf{R}) \to \mathsf{D}(R)$ that sends a complex to its image in the derived category. When R is left-Noetherian [13, 2.3.2] shows that Q restricts to an equivalence

$$Q : \mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}} \xrightarrow{\cong} \mathsf{D}^{\mathsf{f}}(R),$$

where $\mathsf{D}^{\mathsf{f}}(R)$ is the full subcategory of $\mathsf{D}(R)$ of objects with finite cohomology. By [13, 3.9] the functor Q has a right adjoint, denoted by Q_ρ, which takes any complex to a semi-injective resolution, viewed as an object of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$.

When restricted to $\mathsf{D}^{\mathsf{f}}(R)$, Q_ρ gives an inverse to the equivalence above. Thus *the compact objects of* K *are exactly the semi-injective resolutions of objects in* $\mathsf{D}^{\mathsf{f}}(R)$.

The following construction is a key part of the proof of the main theorem.

2.7. Let $\mathsf{S} = \mathsf{loc}_{\mathsf{K}}(\mathsf{C})$, for a set of compact objects C in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. For any object X in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ there is a triangle

$$\Gamma X \to X \to \mathsf{L}X \to$$

such that $\Gamma X \in \mathsf{S}$ and $\mathsf{L}X \in \mathsf{S}^{\perp}$, where

$$\mathsf{S}^{\perp} = \{\, Y \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R}) \mid \operatorname{Hom}_{\mathsf{K}}(Z, Y) = 0 \text{ for all } Z \in \mathsf{S}\}.$$

This is a form of *Bousfield localization*; see [15, 1.7] for a proof.

3. Cohomology operators

Throughout this section $S = \oplus_{i\geq 0} S^i$ denotes a commutative graded ring.

3.1. We say S is a *ring of cohomology operators* for R if for every $X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ there is a map of graded rings

$$\zeta_X : S \to \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X),$$

such that the two S-module structures on $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y)$ via ζ_X and ζ_Y agree. Thus, for each $\alpha \in \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y)$, and all homogeneous $s \in S$, we require

$$\zeta_Y(s) \cdot \alpha = (-1)^{|s|} \alpha \cdot \zeta_X(s). \tag{3.1}$$

We say R has *Noetherian cohomology* over S if S is a Noetherian ring of finite Krull dimension, R has finite injective dimension as a left module, and $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C, C)$ is a Noetherian S-module for all compact objects C in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$.

Remark 3.2. Equivalently, S is a ring of cohomology operators for R if there is a ring map $S \to \mathsf{Z}(\mathsf{K}(\mathsf{Inj}\,\mathsf{R}))$, where $\mathsf{Z}(-)$ denotes the graded center of a triangulated category; see e.g., [7, Section 4].

A ring of cohomology operators for R has been defined previously in [5] to be a ring map $S \to \mathsf{Z}(\mathsf{D}(R))$. The essentially surjective functor $\mathsf{Q} : \mathsf{K}(\mathsf{Inj}\,\mathsf{R}) \to \mathsf{D}(R)$ induces a ring map $\mathsf{Z}(\mathsf{K}(\mathsf{Inj}\,\mathsf{R})) \to \mathsf{Z}(\mathsf{D}(R))$ and thus a ring of cohomology operators in our sense gives rise to a ring of cohomology operators in the sense of [5].

In the rest of the section, we assume that S is Noetherian, has finite Krull dimension, and is a ring of cohomology operators for R. We set $S^+ = \oplus_{i\geq 1} S^i$.

We will need the following result on the structure of a ring with Noetherian cohomology.

3.3. Assume R has Noetherian cohomology over S. Then the following hold:

(1) R is left-Noetherian;
(2) $\operatorname{inj\,dim}_{R_{\mathfrak{p}}} R_{\mathfrak{p}} < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R^c$;
(3) An R-complex with finite cohomology M has finite projective dimension if and only if $\operatorname{Ext}^n_R(M, M) = 0$ for all $n \gg 0$ if and only if M has finite injective dimension.

This is contained in [4], where less assumptions are placed on S. For the rings in Section 5 to which we apply the main theorem the properties above are well known.

The following construction was introduced in [7]:

3.4. Let s be a homogeneous element of S of degree n and let X be an object of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. The *Koszul object* of s on X, denoted $X/\!\!/s$, is the mapping cone of $\zeta_X(s) \in \operatorname{Hom}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, \Sigma^n X)$. Thus there is an exact triangle

$$X \xrightarrow{\zeta_X(s)} \Sigma^n X \to X/\!\!/s \to, \tag{3.2}$$

and $X/\!\!/s$ is unique up to isomorphism. For $\mathsf{s} = s_1, \dots, s_r$ a sequence of homogeneous elements of S, the Koszul object of s on X, denoted $X/\!\!/\mathsf{s}$, is defined inductively as the Koszul object of s_r on $X/\!\!/(s_1, \dots, s_{r-1})$.

Let Y be another object of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. We need the following properties of Koszul objects:

(1) If X is compact, then so is $X/\!\!/\mathsf{s}$; this follows by induction and the triangle (3.2) above.
(2) There exists an integer $n \geq 0$, independent of X and Y, such that

$$(\mathsf{s})^n \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(Y, X/\!\!/\mathsf{s}) = 0 = (\mathsf{s})^n \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X/\!\!/\mathsf{s}, Y),$$

where $(\mathsf{s}) = (s_1, \ldots, s_n)$ is the ideal in S generated by $s_1, \ldots, s_n$.
(3) If $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X/\!\!/\mathsf{s}, Y) = 0$ and the S-module $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y)$ is s-torsion then

$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y) = 0.$$

The last two results are contained in [7, 5.11].

The next result shows that every compact object of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ can be cut down to an object with finite projective dimension using the above construction.

Proposition 3.5. *Assume R has Noetherian cohomology over S. Let $\mathsf{s} = s_1, \ldots, s_r$ be a set of generators of the ideal $S^+ = \oplus_{i>0} S^i$ and let $\mathsf{i}R \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ be an injective resolution of R. For every compact object C of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ the object $C/\!\!/\mathsf{s}$ is in $\mathsf{thick}_{\mathsf{K}}(\mathsf{i}R)$. In particular there is an inclusion of subcategories:*

$$\mathsf{thick}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}}) \subseteq \mathsf{thick}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(\mathsf{i}R).$$

Proof. By 3.4(2) there exists $n \geq 1$ such that $(\mathsf{s})^n \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}) = 0$. Since $C/\!\!/\mathsf{s}$ is compact, the S-module $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s})$ is finitely generated by the definition of Noetherian cohomology. A standard argument now shows that

$$\operatorname{Hom}^m_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}) = 0 \text{ for } m \gg 0. \tag{3.3}$$

Since $C/\!\!/\mathsf{s}$ is compact, by 2.6, the complex $C/\!\!/\mathsf{s}$ is semi-injective. Thus,

$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}) \cong \operatorname{Ext}^*_R(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}).$$

Now 3.3 and 3.3(3) show that $C/\!\!/\mathsf{s}$ has finite projective dimension. One checks, by induction on projective dimension for instance, that this implies that $C/\!\!/\mathsf{s} \in \mathsf{thick}_{\mathsf{D}(R)}(R)$. Since triangulated functors preserve thick subcategories we have that

$$Q_\rho(C/\!\!/\mathsf{s}) \in \mathsf{thick}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(Q_\rho R).$$

As semi-injective resolutions are unique in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ and $C/\!\!/\mathsf{s}$ and $Q_\rho(C/\!\!/\mathsf{s})$ are semi-injective, we have that $Q_\rho(C/\!\!/\mathsf{s}) \cong C/\!\!/\mathsf{s}$ and $Q_\rho R \cong \mathsf{i}R$. Stringing together the above shows that $C/\!\!/\mathsf{s}$ is in $\mathsf{thick}_{\mathsf{K}}(\mathsf{i}R)$. □

4. Finite injective dimension

In this section, we prove the theorem in the introduction. To do this we need the following:

Proposition 4.1. *Let R be a left-Noetherian ring that has finite injective dimension as a left R-module and let M be an R-complex with $\mathrm{H}^n(M) = 0$ for $n \gg 0$. Let $\mathsf{i}R$*

and $\mathsf{i}M$ *be semi-injective resolutions of* R *and* M*, respectively. If* $\mathsf{i}M$ *is in* $\mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$*, then* M *has finite injective dimension.*

Proof. Since M has right-bounded cohomology, we may pick a projective resolution $P \xrightarrow{\simeq} M$, i.e., a quasi-isomorphism such that P^j is projective and $P^j = 0$ for $j \gg 0$. Each P^j has finite injective dimension bounded by the injective dimension of the ring, which we denote by d.

Fix an injective resolution of each P^j of length at most d. By the comparison theorem there are maps between the resolutions which form a bicomplex. Taking the total sum complex of this bicomplex gives a complex L and a quasi-isomorphism $P \xrightarrow{\simeq} L$, such that each L^j is injective and $L^j = 0$ for $j \gg 0$. Now let $L \to \mathsf{i}L$ be a semi-injective resolution. We have a diagram

$$\begin{array}{ccccc} P & \xrightarrow{\simeq} & L & \xrightarrow{\simeq} & \mathsf{i}L \\ & \searrow^{\simeq} & & & \\ & & M & \xrightarrow{\simeq} & \mathsf{i}M \end{array}$$

By the lifting property of semi-injective resolutions, described in 2.4, we see that $\mathsf{i}M \cong \mathsf{i}L$ in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. In particular $\mathsf{i}L$ is a semi-injective resolution of M and $\mathsf{i}L \in \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$.

Let T be the mapping cone of $L \to \mathsf{i}L$. We have a triangle

$$L \to \mathsf{i}L \xrightarrow{v} T \to$$

in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. Note that T is acyclic since $L \to \mathsf{i}L$ is a quasi-isomorphism. Thus, we have isomorphisms

$$\operatorname{Hom}^*_{\mathsf{K}}(\mathsf{i}R, T) \cong \operatorname{Hom}^*_{\mathsf{K}(R)}(R, T) \cong \operatorname{H}^*(T) = 0.$$

The first is [13, 2.1], the second is clear, and the third is the fact that T is acyclic.

The full subcategory whose objects are

$$\{\, X \mid \operatorname{Hom}^*_{\mathsf{K}}(X, T) = 0\}$$

is a localizing subcategory of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. Thus, since $\mathsf{i}R$ is in this subcategory, so is $\mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$. In particular $\mathsf{i}L \in \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$, and thus $\operatorname{Hom}^*_{\mathsf{K}}(\mathsf{i}L, T) = 0$. This shows that the map v above is nullhomotopic. We will show that this forces $\mathsf{i}L$ to have an injective cokernel in a high degree.

Since v is nullhomotopic there exists a map $s : \mathsf{i}L \to T$ such that $\partial s + s\partial = v$. Let k be an integer such that $L^n = 0$ for all $n \geq k$, which exists by assumption. Thus v^n is bijective for all $n \geq k$ and we have that $(v^n)^{-1}\partial s + (v^n)^{-1}s\partial = 1_{\mathsf{i}L^n}$. One checks that v^{-1} commutes with the differentials in the degrees for which it is defined; this gives

$$\partial(v^{n-1})^{-1}s + (v^n)^{-1}s\partial = 1_{\mathsf{i}L^n}.$$

Thus $v^{-1}s$ is a contracting homotopy of $1_{\mathsf{i}M}$ in high degrees. A simple diagram chase now shows that $\operatorname{Im}(\partial^k)$ splits as a submodule of $(\mathsf{i}L)^{k+1}$ and hence is injective.

Since v is a bijection in degrees $n \geq k$ and T is acyclic, this implies that $\operatorname{H}^n(\mathsf{i}L) = 0$ for $n \geq k$. Thus $\mathsf{i}L$ has an injective cokernel in a degree higher than its last nonzero cohomology; by [3, 2.4.I] this implies that M has finite injective dimension. One may also verify this directly by noting that we have shown that $\mathsf{i}L \cong X \oplus Y$ with $X^i = 0$ for $i \gg 0$ and Y nullhomotopic. □

Theorem 4.2. *Let R be an associative ring and S a Noetherian graded ring of finite Krull dimension. Assume that S is a ring of cohomology operators on R and that R has Noetherian cohomology over S. For an R-complex M with $\mathrm{H}^n(M) = 0$ for $n \gg 0$, if the S-module $\mathrm{Ext}_R^*(M, M)$ is $S^+ = \oplus_{i \geq 1} S^i$-torsion, then M has finite injective dimension.*

Proof. Let $X = \mathsf{i}M$ be a semi-injective resolution of M. Then, by 2.5,

$$\mathrm{Ext}_R^*(M, M) \cong \mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(X, X).$$

Let s be a finite set of generators of the ideal S^+. By 3.3, R is left-Noetherian and it has finite injective dimension by the assumption of Noetherian cohomology. Thus by 4.1 it is enough to show that $\mathsf{i}M \in \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$. Since every localizing subcategory in $\mathsf{K(Inj\,R)}$ is thick (see 2.3), Proposition 3.5 shows that

$$\mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K}^{\mathsf{c}}) \subseteq \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R). \tag{4.1}$$

Thus to prove the theorem it is enough to show that $X \in \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K(Inj\,R)}^{\mathsf{c}})$. Let us set $\mathsf{C} := \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K(Inj\,R)}^{\mathsf{c}})$.

Fix a compact object D. By hypothesis $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(X, X)$ is S^+-torsion. By the definition of cohomology operators, the action of S on $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, X)$ factors through $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(X, X)$ and hence $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, X)$ is also S^+-torsion.

Now consider the full subcategory T of $\mathsf{K(Inj\,R)}$ with objects those $Z \in \mathsf{K(Inj\,R)}$ such that $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, Z)$ is S^+-torsion. It is clearly closed under suspension. Given a triangle $Y \to Z \to W \to \Sigma Y$ in $\mathsf{K(Inj\,R)}$, there is an exact sequence of S-modules:

$$\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, Y) \to \mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, Z) \to \mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, W).$$

From this we see that if $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, Y)$ and $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, W)$ are S^+-torsion then $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, Z)$ is as well. This shows that T is triangulated. For a family of objects $\{Z_i\}$ in T, we have that

$$\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}\left(D, \bigoplus_i Z_i\right) \cong \bigoplus_i \mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, Z_i),$$

since D is compact. Thus T is closed under direct sums and hence is localizing. By 3.4(2), for every object C the module $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, C/\!\!/\mathsf{s})$ is S^+-torsion. Thus

$$\mathsf{C} = \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K(Inj\,R)}^{\mathsf{c}}) \subseteq \mathsf{T}$$

since T is localizing and each $C/\!\!/\mathsf{s}$ is in T.

Since C is compactly generated there is a triangle

$$\Gamma X \to X \to \mathsf{L}X \to \tag{4.2}$$

with $\Gamma X \in \mathsf{C}$ and $\mathsf{L}X \in \mathsf{C}^{\perp}$; see 2.7. We have that $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, \Gamma X)$ is S^+-torsion since $\Gamma X \in \mathsf{C} \subseteq \mathsf{T}$. We have shown above that $X \in \mathsf{T}$. Thus $\mathsf{L}X \in \mathsf{T}$ since T is triangulated. By definition this means $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, \mathsf{L}X)$ is S^+-torsion. Since $D/\!\!/\mathsf{s} \in \mathsf{C}$ and $\mathsf{L}X \in \mathsf{C}^{\perp}$, we have that

$$\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D/\!\!/\mathsf{s}, \mathsf{L}X) = 0.$$

By 3.4(3) this implies that $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, \mathsf{L}X) = 0$. But since D was an arbitrary compact object and $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is compactly generated (see 2.3), this shows that $\mathsf{L}X = 0$. By the triangle (4.2) this implies that $\Gamma X \cong X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ and hence X is an object of $\mathsf{C} = \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}})$. □

Remark 4.3. The hypothesis that $\mathrm{H}^n(M) = 0$ for $n \gg 0$ is necessary. Indeed, from the definition of finite injective dimension, recalled in 2.4, if a complex M has finite injective dimension, then $\mathrm{H}^n(M) = 0$ for $n \gg 0$.

We record the following that was contained in the proof of 4.2.

Corollary 4.4. *Under the assumptions and notation of Theorem 4.2, there is an equality*

$$\mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}}) = \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R).$$

Proof. One containment is given by (4.1). For the other direction, note that since the S-module $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(\mathsf{i}R, \mathsf{i}R) \cong \operatorname{Ext}^*_R(R, R)$ is S^+-torsion, the proof of 4.2 above shows that $\mathsf{i}R \in \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}})$. □

Corollary 4.5. *Let R, S and M be as in 4.2. Let $s_1, \dots, s_r$ be a finite set of homogeneous generators of the ideal S^+. Set*

$$d := \max\{\deg s_i \mid 1 \le i \le r\} \text{ and } l := \operatorname{lcm}\{\deg s_i \mid 1 \le i \le r\}.$$

Then $\operatorname{inj\,dim}_R M < \infty$ *if one of the following holds:*

(1) *there exists an integer $n \ge 0$ such that $\operatorname{Ext}^j_R(M, M) = 0$ for all $n \le j \le n + d - 1$; or*
(2) *there exists an integer $m \ge 0$ such that $\operatorname{Ext}^{ml}_R(M, M) = 0$.*

Proof. Either condition forces the S-module $\operatorname{Ext}^*_R(M, M)$ to be S^+-torsion. Indeed, assume that there exists an integer n such that (1) holds. For every i there exists an integer k_i such that

$$n \le k_i(\deg s_i) \le n + d - 1.$$

One way to see this is by induction on n. Consider the ideal $(S^+)^{k_1+\dots+k_r} = (s_1, \dots, s_r)^{k_1+\dots+k_r}$ in S. It is generated by monomials in the s_i of the form $s_1^{n_1} \dots s_r^{n_r}$ for positive integers n_i with $\sum n_i = \sum k_i$. For each such monomial there exists an i such that $n_i \ge k_i$, else $\sum n_i < \sum k_i$; applying ζ_M to the monomial, and using that ζ_M is a map of rings, we see that

$$\begin{aligned}\zeta_M(s_1^{n_1} \dots s_r^{n_r}) &= \zeta_M(s_1^{n_1}) \dots \zeta_M(s_i^{n_i}) \dots \zeta_M(s_r^{n_r}) \\ &= \zeta_M(s_1^{n_1}) \dots \zeta_M(s_i^{k_i})\zeta_M(s_i^{n_i-k_i}) \dots \zeta_M(s_r^{n_r}) = 0,\end{aligned}$$

since $\zeta_M(s_i^{k_i}) \in \operatorname{Ext}_R^{k_i(\deg s_i)}(M, M) = 0$. Thus

$$(S^+)^{k_1+\dots+k_r} \operatorname{Ext}^*_R(M, M) = \zeta_M((S^+)^{k_1+\dots+k_r}) \operatorname{Ext}^*_R(M, M) = 0$$

and hence $\operatorname{Ext}^*_R(M, M)$ is S^+-torsion. By Theorem 4.2 this shows that $\operatorname{inj\,dim}_R M < \infty$.

To prove (2) assume that such an m exists. For every $i = 1, \dots, r$, there exists an integer d_i such that $d_i(\deg s_i) = l$. Letting $\alpha = m(\sum_i d_i)$, a similar proof as above shows that $(s_1, \dots, s_r)^\alpha \operatorname{Ext}_R^*(M, M) = 0$. □

5. Applications

In this section, we apply Theorem 4.2 in the two contexts discussed in the introduction.

5.1. Let R be a commutative ring with a presentation

$$R \cong Q/(\mathbf{f}),$$

where Q is a commutative Noetherian regular ring of finite Krull dimension and $(\mathbf{f}) = (f_1, \dots, f_c)$ is a Q-regular sequence.

Let $S = R[\chi_1, \dots, \chi_c]$ be the polynomial ring in c indeterminates over R, graded by setting $|\chi_i| = 2$. For every $X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ there is a homomorphism of graded R-algebras

$$\zeta_X : S \to \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X)\,.$$

When $X = \mathsf{i}M$ is the injective resolution of a finitely generated R-module M, so that

$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X) \cong \operatorname{Ext}^*_R(M, M),$$

such a map ζ_X may be constructed as in [9, Section 1] using a free resolution of M. The process described in [1, Section 1], which replaces free resolutions with injective resolutions, generalizes to arbitrary objects of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. The results of *loc. cit.* show that the maps ζ_X satisfy the conditions of a ring of cohomology operators.

By [6, 5.1] the S-module $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(\mathsf{i}M, \mathsf{i}M) \cong \operatorname{Ext}^*_R(M, M)$ is finitely generated when M has finite cohomology over R. This was proved first by Gulliksen [11] for modules. It follows from 2.6 that R has Noetherian cohomology over S. Restating Theorem 4.2 in this context, we have:

Corollary 5.2. *Let Q be a commutative Noetherian regular ring of finite Krull dimension, $(\mathbf{f}) = (f_1, \dots, f_c)$ a Q-regular sequence and $R = Q/(\mathbf{f})$. For an R-complex M with $H^n(M) = 0$ for all $n \gg 0$, if $\operatorname{Ext}^*_R(M, M)$ is S^+-torsion, then M has finite injective dimension.*

In the notation of Corollary 4.5 we see that $d = 2 = l$. Since R is a Gorenstein ring of finite Krull dimension, a module has finite projective dimension if and only if it has finite injective dimension. This gives:

Corollary 5.3. *If M is an arbitrary R-module such that $\operatorname{Ext}^{2n}_R(M, M) = 0$ for some $n \geq 1$ then M has finite projective dimension.*

Remark 5.4. In [2, 4.2] the same statement is proved for finitely generated modules of finite complete intersection dimension over a Noetherian ring. All finitely generated modules over the ring R have finite complete intersection dimension. However, complete intersection dimension is not defined for non-finitely generated modules, so we have not generalized completely [2, 4.2].

5.5. Let R be a Hopf algebra over a field k. For two R-modules M, N we view $M \otimes_k N$ as an R-module via the diagonal map $\Delta : R \to R \otimes_k R$. When M, N are injective then so is $M \otimes_k N$. For $X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ the functor $- \otimes_k X$ preserves homotopies of maps.

Thus there is a functor $-\otimes_k X\colon \mathsf{K}(\mathsf{Inj}\,\mathsf{R}) \to \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. Viewing k as an R-module via the augmentation there is an isomorphism

$$\varphi_X : \mathsf{i}k \otimes_k X \xrightarrow{\cong} X,$$

see [8, 5.3] which proof holds in our more general situation. Thus for each X one gets a map

$$\eta_X\colon \operatorname{Hom}^*_{\mathsf{K}}(\mathsf{i}k, \mathsf{i}k) \to \operatorname{Hom}^*_{\mathsf{K}}(X, X)$$

that sends $\alpha : \mathsf{i}k \to \Sigma^n \mathsf{i}k$ to

$$\varphi_{\Sigma^n X}(\alpha \otimes_k X)(\varphi_X)^{-1} : X \to \Sigma^n X.$$

One can check that η_X is a ring map. Let S be the ring $\operatorname{Ext}^*_R(k,k) \cong \operatorname{Hom}^*_{\mathsf{K}}(\mathsf{i}k, \mathsf{i}k)$. By [14, (VIII.4.7), (VIII.4.3)] the ring S is graded-commutative and the maps η_X satisfy the commutativity relations (3.1). Thus setting

$$S^{\text{even}} := \begin{cases} \bigoplus_{i\geq 0} \operatorname{Ext}^{2i}_R(k,k) & \text{if } \operatorname{char} k \neq 2, \\ \operatorname{Ext}^*_R(k,k) & \text{if } \operatorname{char} k = 2, \end{cases}$$

we see that S^{even} is commutative and is a ring of cohomology operators on R.

By the main result of [10], when R is cocommutative and finite-dimensional over k, the ring S is Noetherian and $\operatorname{Ext}^*_R(M,N)$ is a Noetherian S-module (via η_M, or equivalently, η_N) for all complexes M, N with finite cohomology. The ideal of odd degree elements in S is nilpotent when $\operatorname{char} k \neq 2$. Thus when R is a cocommutative finite-dimensional Hopf algebra it has Noetherian cohomology over S^{even}.

Specializing Theorem 4.2 and Corollary 4.5 to this context, and using that R is self-injective, we have:

Corollary 5.6. *Let R be a finite-dimensional cocommutative Hopf algebra and S^{even} the commutative ring defined as above. For an R-complex M with $H^n(M) = 0$ for all $n \gg 0$, if $\operatorname{Ext}^*_R(M,M)$ is S^+-torsion, then M has finite injective dimension.*

Corollary 5.7. *Let R be as above and M an R-module. There exists an integer l such that if $\operatorname{Ext}^{nl}_R(M,M) = 0$ for some $n \geq 1$ then M has finite projective dimension.*

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