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ON FAMILIES OF FILTERED (φ, N)-MODULES

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ABSTRACT. In this paper, as a generalization of Berger's construction, we give a functor from the category of families of filtered (φ, N)-modules (with certain condition) to the category of families of (φ, Γ)-modules. Combining this with Kedlaya and Liu's theorem we show that, when the base is a reduced affinoid space, every family of weakly admissible filtered (φ, N)-modules can locally be converted into a family of semistable Galois representations.

Introduction

In p-adic Hodge theory one considers (φ, Γ)-modules as the category of semilinear algebra data describing p-adic Galois representations, and considers weakly admissible filtered (φ, N)-modules as the category of semilinear algebra data describing semistable Galois representations. See [6, 7] for the constructions of these equivalences.

Recently mathematicians are interested in families of these modules.

In [3] Berger and Colmez defined a functor from the category of families of p-adic Galois representations to the category of families of overconvergent étale (φ, Γ)-modules. But the functor of Berger–Colmez fails to be an equivalence of categories, in contrast with the classical case.

However Kedlaya and Liu [10] showed that, when the base is an affinoid space, every family of overconvergent étale (φ, Γ)-modules can locally be converted into a family of p-adic Galois representations.

Theorem 0.1 ([10, Theorem 0.2]). *Let $\mathcal{L}$ be a reduced affinoid algebra over $\mathbb{Q}_p$, and let $\mathscr{M}_{\mathcal{L}}$ be a family of (φ, Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ in the sense of [10]. If $\mathscr{M}_x$ is étale for some $x \in \mathrm{Max}(\mathcal{L})$, then there exists an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a $\mathcal{B}$-linear representation $V_{\mathcal{B}}$ of G_K whose associated (φ, Γ)-module is isomorphic to $\mathcal{B}\widehat{\otimes}_{\mathcal{L}}\mathscr{M}_{\mathcal{L}}$. Moreover $V_{\mathcal{B}}$ is unique for this property.*

Berger and Colmez [3] also defined a functor from the category of families of semistable Galois representations to the category of families of weakly admissible filtered (φ, N)-modules, which also fails to be equivalent.

In this paper we show that, when the base is an affinoid space, every family of weakly admissible filtered (φ, N)-modules can locally be converted into a family of semistable Galois representations. Following an ideal mentioned in [10], this is done by generalizing Berger's construction in [2] to families of filtered (φ, N)-modules and then applying Theorem 0.1.

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Based on Schneider and Teitelbaum's notions of Fréchet–Stein algebras and coadmissible modules over a Fréchet–Stein algebra, we introduce a category of coadmissible (φ,Γ)-modules. This category is studied by Kedlaya *et al.* [11]. As a generalization of Berger's functor given in [2], we construct a functor from the category of families of filtered (φ,N)-modules satisfying Condition (Gr) to the category of coadmissible (φ,Γ)-modules. For a families of filtered (φ,N)-modules, we mean a filtered (φ,N)-module over a coefficient algebra $\mathcal{L}$. See Section 2 for the precise definition. Such a filtered (φ,N)-module D (over $\mathcal{L}$) is said to satisfies (Gr) if the filtration $\mathrm{Fil}^\bullet$ has the following property:

> For every $i\in\mathbb{Z}$, $\mathrm{Gr}^i D_K=\mathrm{Fil}^i D_K/\mathrm{Fil}^{i+1}D_K$ is locally free over $\mathcal{L}\otimes_{\mathbb{Q}_p} K$ of constant rank.

When the base $\mathcal{L}$ is a reduced affinoid algebra, a coadmissible (φ,Γ)-module is essentially a family of (φ,Γ)-modules (in the sense of [10]), so that we can apply Theorem 0.1. As a result, we obtain that, when the base is a reduced affinoid space, every family of weakly admissible filtered (φ,N)-modules satisfying (Gr) locally comes from some family of semistable Galois representations.

Our main result is the following

Theorem 0.2. *Let $\mathcal{L}$ be a reduced affinoid algebra and let D be a filtered (φ,N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$ that satisfies* (Gr)*. If D_x is weakly admissible for some $x\in$* $\mathrm{Max}(\mathcal{L})$*, then there exists an affinoid neighborhood* $\mathrm{Max}(\mathcal{B})$ *of x and a semi-stable $\mathcal{B}$-representation $V_{\mathcal{B}}$ of G_K whose associated filtered (φ,N)-module is isomorphic to $D_{\mathcal{B}}$. Moreover, $V_{\mathcal{B}}$ is unique for this property.*

In the case of $N=0$ some related results were obtained by Hellmann [9]. We explain Hellmann's results as follows. Fix an integer $d>0$ and (a conjugacy class of) a dominant coweight ν of the algebraic group $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d$ with reflex field E. Let $\mathfrak{D}_\nu$ be the fpqc-stack on the category of rigid spaces over K (or slightly more generally, on the category of adic spaces locally of finite type) whose X-valued points are triples $(D,\varphi,\mathrm{Fil}^\bullet)$ with a locally free $\mathcal{O}_X\otimes_{\mathbb{Q}_p}K_0$-module D, a semilinear automorphism φ, and a filtration of $D\otimes_{\mathbb{Q}_p}K$ that is of type ν.

Theorem 0.3 (**[9, Theorem 1.1]**)**.** *The weakly admissible locus is an open substack $\mathfrak{D}_\nu^{\mathrm{wa}}$ of $\mathcal{D}_\nu$ on the category of adic spaces locally of finite type over E.*

Theorem 0.4 (**[9, Theorem 1.3, Theorem 8.26]**)**.** *Let ν be a miniscule cocharacter of* $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d$.

(a) *Then the groupoid of families of crystalline representations of G_K with Hodge-Tate weights ν is an open substack $\mathfrak{D}_\nu^{\mathrm{adm}}$ of $\mathfrak{D}_\nu^{\mathrm{wa}}$.*
(b) *The groupoid*

$$\mathrm{Res}^{\nu}_{\mathrm{cris}}: X\mapsto\{\textit{ families of crystalline representations on } X \textit{ with Hodge-Tate weights } \nu\}$$

on the category of adic spaces locally of finite type over E is isomorphic to the stack $\mathfrak{D}_\nu^{\mathrm{adm}}$ and thus is an open substack of $\mathfrak{D}_\nu^{\mathrm{wa}}$.

That a cocharacter is miniscule means that the Hodge-Tate weights are in $\{0,1\}$.

Now let D be any filtered φ-module of miniscule type ν (with reflex field $E=\mathbb{Q}_p$). Since D_x is weakly admissible and thus comes from a crystalline representation according to Colmez–Fontaine theorem, by Theorem 0.3 and Theorem 0.4, there exist a neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a crystalline $\mathcal{B}$-representation $V_{\mathcal{B}}$ of G_K whose associated filtered φ-module is isomorphic to $D_{\mathcal{B}}$. So Theorem 0.2 is recovered for filtered φ-modules of miniscule type.

We outline the structure of this paper. In Section 1 we recall the rings coming from p-adic Hodge theory. In Section 2 we recall the notion of families of filtered (φ, N)-modules. In Section 3.1 we recall the notions of free families and locally free families of (φ, Γ)-modules, and in Section 3.2 we recall Berger and Colmez's construction in [3]. In Section 4 we introduce the category of coadmissible (φ, Γ)-modules and give a functor from the category of families of filtered (φ, N)-modules satisfying (Gr) to the category of coadmissible (φ, Γ)-modules. In Section 5 we prove Theorem 0.2.

1. Rings of p-adic Hodge theory

We recall the construction of Fontaine's period rings. Please consult [1, 8] for more details.

Throughout this paper let K be a finite extension of $\mathbb{Q}_p$, K_0 the maximal absolutely unramified subfield of K. Let φ be the $\mathbb{Q}_p$-automorphism of K_0 that reduces to the absolutely Frobenius of the residue field. Let μ_{p^n} be the set of p^n-th roots of unity in $\bar{\mathbb{Q}}_p$, $\mu_{p^\infty} = \cup_{n\geq 0}\, \mu_{p^n}$. For a finite extension K of $\mathbb{Q}_p$, put $K_n = K(\mu_{p^n})$ and $K_\infty = K(\mu_{p^\infty}) = \bigcup_{n>0} K_n$. Set $\Gamma = \Gamma_K = \mathrm{Gal}(K_\infty/K)$ and $H_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K_\infty)$.

Let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$ with valuation subring $\mathcal{O}_{\mathbb{C}_p}$ and p-adic valuation v_p normalized such that $v_p(p) = 1$.

Let $\widetilde{\mathrm{E}}$ be $\{(x^{(i)})_{i\geq 0} \mid x^{(i)} \in \mathbb{C}_p,\ (x^{(i+1)})^p = x^{(i)}\ \forall\, i \in \mathbb{N}\}$, and let $\widetilde{\mathrm{E}}^+$ be the subset of $\widetilde{\mathrm{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. If $x, y \in \widetilde{\mathrm{E}}$, we define $x+y$ and xy by

$$(x+y)^{(i)} = \lim_{j\to\infty} (x^{(i+j)} + y^{(i+j)})^{p^j}, \quad (xy)^{(i)} = x^{(i)}y^{(i)}.$$

Then $\widetilde{\mathrm{E}}$ is a field of characteristic p. Define a function $v_{\mathrm{E}} : \widetilde{\mathrm{E}} \to \mathbb{R} \cup \{+\infty\}$ by putting $v_{\mathrm{E}}((x^{(n)})) = v_p(x^{(0)})$. This is a valuation under which $\widetilde{\mathrm{E}}$ is complete and $\widetilde{\mathrm{E}}^+$ is the ring of integers in $\widetilde{\mathrm{E}}$. If we let $\epsilon = (\epsilon^{(n)})$ be an element of $\widetilde{\mathrm{E}}^+$ with $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$, then $\widetilde{\mathrm{E}}$ is a completed algebraic closure of $\mathbb{F}_p((\epsilon - 1))$.

Let $\widetilde{\mathrm{A}}^+$ be the ring $\mathrm{W}(\widetilde{\mathrm{E}}^+)$ of Witt vectors with coefficients in $\widetilde{\mathrm{E}}^+$, $\widetilde{\mathrm{A}}$ the ring of Witt vectors $\mathrm{W}(\widetilde{\mathrm{E}})$ and $\widetilde{\mathrm{B}} = \widetilde{\mathrm{A}}[1/p]$. Put $\pi = [\epsilon] - 1 \in \widetilde{\mathrm{A}}^+$, where $[\epsilon]$ denotes the Teichmüller lifting of ϵ. Let A be the completion of the maximal unramified extension of $\mathbb{Z}_p((\pi))$ in $\widetilde{\mathrm{A}}$, $\mathrm{B} = \mathrm{A}[1/p]$.

If r, s are two elements in $\mathbb{N}[1/p] \cup \{+\infty\}$, we put $\widetilde{\mathrm{A}}^{[r,s]} = \widetilde{\mathrm{A}}^+\{\frac{p}{[\bar{\pi}^r]}, \frac{[\bar{\pi}^s]}{p}\}$ and $\widetilde{\mathrm{B}}^{[r,s]} = \widetilde{\mathrm{A}}^{[r,s]}[1/p]$ with the convention that $p/[\bar{\pi}^{+\infty}] = 1/[\bar{\pi}]$ and $[\bar{\pi}^{+\infty}]/p = 0$. If I is a subinterval of $\mathbb{R} \cup \{+\infty\}$, we put $\widetilde{\mathrm{B}}_I = \cap_{[r,s]\subseteq I} \widetilde{\mathrm{B}}^{[r,s]}$. If $I \subseteq J$ are two closed intervals, then $\widetilde{\mathrm{B}}^J \subseteq \widetilde{\mathrm{B}}^I$, and we define a valuation v_I on $\widetilde{\mathrm{B}}^J$ by demanding $v_I(x) = 0$ if and only if $x \in \widetilde{\mathrm{A}}^I - p\widetilde{\mathrm{A}}^I$. Then $\widetilde{\mathrm{B}}^I$ is a Banach space for the valuation v_I and the completion of $\widetilde{\mathrm{B}}^J$ for the valuation $\widetilde{\mathrm{B}}^I$ is identified with $\widetilde{\mathrm{B}}^I$. Put

$$\widetilde{\mathrm{B}}^{\dagger,r} = \widetilde{\mathrm{B}}^{[r,+\infty]}, \quad \widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{[r,+\infty[} \quad \text{and} \quad \widetilde{\mathrm{B}}^{+}_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{\dagger,0}_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{[0,+\infty[}.$$

Note that $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$ is a Fréchet space for the valuations $v^{[r,s]}$ with $s \in [r, +\infty[$, and $\widetilde{\mathrm{B}}^{\dagger,r}$ is dense in $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$. Put $\widetilde{\mathrm{B}}^{\dagger} = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}$ and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}} = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$. Equip $\widetilde{\mathrm{B}}^{\dagger}$ and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$ with the inductive limit topology. Set $\widetilde{\mathrm{B}}^{+}_{\log} = \widetilde{\mathrm{B}}^{+}_{\mathrm{rig}}[\ell_\pi]$ and $\widetilde{\mathrm{B}}^{\dagger}_{\log} = \widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}[\ell_\pi]$, where $\ell_\pi = \log(\pi)$. Our notations π and ℓ_π coincide with the notations X and $\ell_X = \log(X)$ in [2] respectively. As in [2, Section II.2], we extend the actions of φ and Γ to $\widetilde{\mathrm{B}}^{+}_{\log}$ and $\widetilde{\mathrm{B}}^{\dagger}_{\log}$ by the formulas $\varphi(\ell_\pi) = p\ell_\pi + \log(\varphi(\pi)/\pi^p)$ and $\gamma(\ell_\pi) = \ell_\pi + \log(\gamma(\pi)/\pi)$.

All of the above rings admit actions of G_K. Put $\mathrm{B}_K = \mathrm{B}^{H_K}$, $\widetilde{\mathrm{B}}_K = \widetilde{\mathrm{B}}^{H_K}$, $\widetilde{\mathrm{B}}^I_K = (\widetilde{\mathrm{B}}^I)^{H_K}$, $\widetilde{\mathrm{B}}^{\dagger}_K = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}_K$ and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig},K} = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig},K}$. Put $\mathrm{B}^{\dagger,r}_K = \widetilde{\mathrm{B}}^{\dagger,r}_K \cap \mathrm{B}$. We equip $\mathrm{B}^{\dagger,r}_K$ with the weak topology (see [3]). Let $\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ be the Fréchet completion of $\mathrm{B}^{\dagger,r}_K$ for the topology induced from that on $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig},K}$. Put $\mathrm{B}^{\dagger}_K = \cup_{r\geq 0}\mathrm{B}^{\dagger,r}_K$ and $\mathrm{B}^{\dagger}_{\mathrm{rig},K} = \cup_{r\geq 0}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$. The G-actions on $\mathrm{B}^{\dagger}_K$ and $\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ factor through Γ. For $s \geq r$ let $\mathrm{B}^{[r,s]}_K$ be the completion of $\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ for the valuation $v^{[r,s]}$.

All of $\mathrm{B}^{\dagger}$, $\widetilde{\mathrm{B}}^{\dagger}$, $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$, $\mathrm{B}^{\dagger}_K$ and $\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ admit actions of φ.

There exists a sufficiently large $r(K)$ such that, if $s \geq r \geq r(K)$, then $\mathrm{B}^{[r,s]}_K$ is isomorphic to the ring consisting of $f = \sum_{i=-\infty}^{+\infty} a_i T^i$ with coefficients $a_i \in K'_0$ and convergent on the domain $p^{-1/r} \leq |T| \leq p^{-1/s}$. Here we use K'_0 to denote the maximal absolutely unramified subfield of K_∞.

If $\mathcal{L}$ is a Banach space over $\mathbb{Q}_p$ and B is a locally convex space over $\mathbb{Q}_p$, let $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ be the completion of $\mathcal{L}\otimes_{\mathbb{Q}_p}B$ for the projective tensor product topology [12, Section 17]. Note that, if $\mathcal{L}$ or B is finite over $\mathbb{Q}_p$, then $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B = \mathcal{L}\otimes_{\mathbb{Q}_p}B$.

Lemma 1.1. *If $\mathcal{L}$ is a Banach space over $\mathbb{Q}_p$ and B is a locally convex space over $\mathbb{Q}_p$ with an action of a group G, then the G-action can be extended $\mathcal{L}$-linearly and continuously to $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ in a unique way, and $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B)^G = \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B^G$.*

Proof. By [12, Proposition 10.1] there exists a set X such that $\mathcal{L}$ is topologically isomorphic to the Banach space $c_0(X)$ defined by

$$c_0(X) := \{f : X \to \mathbb{Q}_p \text{ such that for any } \varepsilon > 0 \text{ the set } \{x \in X : |f(x)| > \varepsilon\} \text{ is finite}\}.$$

Therefore $\mathcal{L}$ has a topological basis $\{e_x\}_{x\in X}$ if we identify $\mathcal{L}$ with $c_0(X)$ via the above isomorphism. From the definition of completion topological tensor product, we see that $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ consists of $\sum_{x\in X} a_x e_x$ with $a_x \in B$, such that for any open neighborhood U of 0 in B, the set $\{x \in X : a_x \notin U\}$ is finite. We can extend the G-action to $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$ by letting $g(\sum_{x\in X} a_x e_x) = \sum_{x\in X} g(a_x)e_x$. The uniqueness of such an extension follows from the continuity. It is clear that $g(\sum_{x\in X} a_x e_x) = \sum_{x\in X} a_x e_x$ if and only if a_x are all in B^G. In other words $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B)^G = \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B^G$. □

Definition 1.2. A *coefficient algebra* means a commutative Banach algebra $\mathcal{L}$ over $\mathbb{Q}_p$ satisfying the following conditions:

(a) The norm on $\mathcal{L}$ restricts to the norm on $\mathbb{Q}_p$;
(b) For each maximal ideal $\mathfrak{m}$ of $\mathcal{L}$, the residue field $L_{\mathfrak{m}} := \mathcal{L}/\mathfrak{m}$ is finite over $\mathbb{Q}_p$;
(c) The Jacobson radical of $\mathcal{L}$ is zero; in particular, $\mathcal{L}$ is reduced.

For example, any reduced affinoid algebra over $\mathbb{Q}_p$ is a coefficient algebra. In particular, any finite extension of $\mathbb{Q}_p$ is a coefficient algebra.

As $\widetilde{\mathrm{B}}^I$ and B^I are Fréchet algebras and thus are locally convex, for any coefficient algebra $\mathcal{L}$ we can form $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\mathrm{I}}$ and $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\mathrm{I}}$. Then we define $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ to be $\cup_{r\geq 0}\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ and equip it with the inductive limit topology. We define $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}$ and $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{K}$ similarly. Then we put $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{+}_{\log} := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{+}_{\mathrm{rig}})[\ell_\pi]$ and $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log} := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}})[\ell_\pi]$. From the proof of Lemma 1.1 we see that, if $B = \widetilde{\mathrm{B}}^{\mathrm{I}}, \mathrm{B}^{\mathrm{I}}$, etc, and if ξ is an endomorphism on B, $1\otimes\xi : \mathcal{L}\otimes_{\mathbb{Q}_p} B \to \mathcal{L}\otimes_{\mathbb{Q}_p} B$ can be uniquely extended to a continuous endomorphism on $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}B$. By abuse of notations, we always denote the resulting endomorphism by the same notation ξ.

Notation 1.3. For $\mathcal{L}$ a coefficient algebra over $\mathbb{Q}_p$ and I a subinterval of $\mathbb{R}\cup\{+\infty\}$, let $\mathcal{R}^{\mathrm{I}}_{\mathcal{L}}$ be the ring of Laurent series over $\mathcal{L}$ in the variable T that is convergent if $v_p(T)^{-1}\in I$. Write $\mathcal{R}^r_{\mathcal{L}}$ for $\mathcal{R}^{[r,+\infty[}_{\mathcal{L}}$.

When $r \geq r(K)$, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ is isomorphic to $\mathcal{R}^r_{\mathcal{L}}\otimes_{\mathbb{Q}_p} K'_0$ via $\pi_K \mapsto T$, and so $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ is isomorphic to $\mathcal{R}_{\mathcal{L}}\otimes_{\mathbb{Q}_p} K'_0$.

2. Filtered (φ, N)-modules

Definition 2.1. Let $\mathcal{L}$ be a coefficient algebra. A *filtered (φ, N)-module* over $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$ is a locally free $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$-module D of finite rank together with the following structures:

(a) a φ-semilinear automorphism on D which is again denoted by φ;
(b) a linear endomorphism N on D satisfying $N\varphi = p\varphi N$;
(c) a descending, separated and exhaustive $\mathbb{Z}$-filtration $\mathrm{Fil}^\bullet D_K$ on $D_K := K\otimes_{K_0} D$ by $\mathcal{L}\otimes_{\mathbb{Q}_p} K$-submodules.

Let $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$ be the category of filtered (φ, N)-modules over $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$. When $\mathcal{L} = \mathbb{Q}_p$, $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$ is denoted by $\mathrm{FilM}^{\varphi,N}_{K}$ for shortness.

If $\mathcal{L}'$ is another coefficient algebra and there is a continuous map $\mathcal{L}\to\mathcal{L}'$, then we have a functor

$$\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}} \to \mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}'}, \quad D \mapsto D_{\mathcal{L}'} := \mathcal{L}'\otimes_{\mathcal{L}} D.$$

In particular, if $\mathfrak{m}$ is a maximal ideal of $\mathcal{L}$, then $D_{\mathfrak{m}} = L_{\mathfrak{m}}\otimes_{\mathcal{L}} D$ is a filtered (φ, N)-module over $L_{\mathfrak{m}}\otimes_{\mathbb{Q}_p} K_0$. Hence a filtered (φ, N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p} K_0$ can be considered as a family of filtered (φ, N)-modules on $\mathrm{Max}(\mathcal{L})$, the maximal spectrum of $\mathcal{L}$.

Lemma 2.2. *Suppose that L is a finite extension of $\mathbb{Q}_p$. If D is an object in $\mathrm{FilM}^{\varphi,N}_{K;L}$, then D is free over $L\otimes_{\mathbb{Q}_p} K_0$.*

Proof. Assume that $L\otimes_{\mathbb{Q}_p} K_0 = \prod_i L_i$. Put $D_i = L_i\otimes_{L\otimes_{\mathbb{Q}_p}K_0} D$. Then $D = \bigoplus_i D_i$. As φ acts transitively on the set $\{L_i\}$, it also acts transitively on the set $\{D_i\}$. This implies that for any two indices i, j we have $\dim_{L_i} D_i = \dim_{L_j} D_j$ which ensures the freeness of D. ☐

Proposition 2.3. *Suppose that $\mathcal{L}$ is a reduced affinoid algebra over $\mathbb{Q}_p$. Let D be an object in* $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$. *Then for any $x\in\mathrm{Max}(\mathcal{L})$ there exists a neighborhood* $\mathrm{Max}(\mathcal{B})$ *of x such that $D_{\mathcal{B}}=\mathcal{B}\otimes_{\mathcal{L}}D$ is free over $\mathcal{B}\otimes_{\mathbb{Q}_p}K_0$.*

Proof. By Lemma 2.2, D_x is free over $L_x\otimes_{\mathbb{Q}_p}K_0$. Let $\{v_i\}$ be a basis of D_x over $L_x\otimes_{\mathbb{Q}_p}K_0$, and let $\{e_j\}$ be a basis of K_0 over $\mathbb{Q}_p$. Then $\{e_jv_i\}_{i,j}$ is a basis of D_x over L_x. For any i let $\tilde{v}_i$ be a lifting of v_i in D. Let M be the $\mathcal{L}$-submodule of D generated by $\{e_j\tilde{v}_i\}_{i,j}$. Since D/M is finitely generated over $\mathcal{L}$, the support of D/M is a closed subset of $\mathrm{Spec}(\mathcal{L})$. Thus D/M vanishes on a (Zariski) open neighborhood of x in $\mathrm{Spec}(\mathcal{L})$. So there is some $f\in\mathcal{L}$ with $f(x)\neq0$ such that D/M vanishes on $\mathrm{Spec}(\mathcal{L}_f)$, or equivalently $\mathcal{L}_f\otimes_{\mathcal{L}}D$ is generated by $\{e_j\tilde{v}_i\}_{i,j}$ over $\mathcal{L}_f$. Let $\mathrm{Max}(\mathcal{B})$ be the Laurent subdomain of $\mathrm{Max}(\mathcal{L})$ defined by $|f|\geq|f(x)|$. Then $\mathrm{Max}(\mathcal{B})$ is an open neighborhood of x in $\mathrm{Max}(\mathcal{L})$, and $\mathcal{B}\otimes_{\mathcal{L}}D$, as a $\mathcal{B}$-module, is generated by $\{e_j\tilde{v}_i\}_{i,j}$, which implies that $\mathcal{B}\otimes_{\mathcal{L}}D$ is free over $\mathcal{B}\otimes_{\mathbb{Q}_p}K_0$. □

Let B_{dR} be Fontaine's de Rham period ring. Put

$$\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^+_{\mathrm{dR}}:=\varprojlim_i\mathcal{L}\otimes_{\mathbb{Q}_p}(\mathrm{B}^+_{\mathrm{dR}}/t^i\mathrm{B}^+_{\mathrm{dR}})\quad\text{and}\quad\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}}:=\cup_{i\geq0}t^{-i}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^+_{\mathrm{dR}}).$$

Then G_K acts continuously on $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}}$ in the way that G_K acts on $\mathcal{L}$ trivially.

Recall that

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\mathrm{rig}}[1/t])^{G_K}=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\log}[1/t])^{G_K}=\mathcal{L}\otimes_{\mathbb{Q}_p}K_0,\ \ (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}})^{G_K}=\mathcal{L}\otimes_{\mathbb{Q}_p}K.$$

Let V be an $\mathcal{L}$-representation of G_K, which means that V is a finite locally free $\mathcal{L}$-module (of constant rank) together with a continuous action of G_K.

Definition 2.4. We say that V is *semistable* (resp. *crystalline*) if

$$\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)=((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\log}[1/t])\otimes_{\mathcal{L}}V)^{G_K}$$
$$\big(\text{resp. }\mathrm{D}_{\mathrm{cris},\mathcal{L}}(V)=((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^+_{\mathrm{rig}}[1/t])\otimes_{\mathcal{L}}V)^{G_K}\big)$$

is a locally free $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$-module of rank $d=\mathrm{rank}_{\mathcal{L}}V$. Similarly we say that V is *de Rham* if

$$\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V):=((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}})\otimes_{\mathcal{L}}V)^{G_K}$$

is a locally free $\mathcal{L}\otimes_{\mathbb{Q}_p}K$-module of rank d.

Write $\mathrm{Rep}^{\mathrm{cris}}_{\mathcal{L}}(G_K)$ (resp. $\mathrm{Rep}^{\mathrm{st}}_{\mathcal{L}}(G_K)$, $\mathrm{Rep}^{\mathrm{dR}}_{\mathcal{L}}(G_K)$) for the category of crystalline (resp. semistable, de Rham) $\mathcal{L}$-representations of G_K.

Now we suppose that $\mathcal{L}$ is a reduced affinoid algebra till the end of this section.

In this case, by a result of Berger and Colmez [3, Corollary 6.3.3], V is crystalline (resp. semi-stable) if and only if so are $V_{\mathfrak{m}}=L_{\mathfrak{m}}\otimes_{\mathcal{L}}V$ for all $\mathfrak{m}\in\mathrm{Max}(\mathcal{L})$. Furthermore

$$\mathrm{D}_{\mathrm{cris},L_{\mathfrak{m}}}(V_{\mathfrak{m}})=L_{\mathfrak{m}}\otimes_{\mathcal{L}}\mathrm{D}_{\mathrm{cris},\mathcal{L}}(V)\quad(\text{resp. }\mathrm{D}_{\mathrm{st},L_{\mathfrak{m}}}(V_{\mathfrak{m}})=L_{\mathfrak{m}}\otimes_{\mathcal{L}}\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)).$$

If V is semi-stable, then $\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)$ is an object in $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$ with $\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)_K=\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)$. So $\mathrm{D}_{\mathrm{st},\mathcal{L}}$ is a functor from the category $\mathrm{Rep}^{\mathrm{st}}_{\mathcal{L}}(G_K)$ to the category $\mathrm{FilM}^{\varphi,N}_{K;\mathcal{L}}$.

In the case when $\mathcal{L}$ is a finite extension of $\mathbb{Q}_p$, the image of the functor $\mathrm{D}_{\mathrm{st},\mathcal{L}}$ can be determined explicitly. An object D in $\mathrm{FilM}_{K;\mathcal{L}}^{\varphi,N}$ can also be considered as an object in $\mathrm{FilM}_{K}^{\varphi,N}$ by forgetting the $\mathcal{L}$-module structure. We say that D is *weakly admissible* if it is so as an object in $\mathrm{FilM}_{K}^{\varphi,N}$. Let $\mathrm{FilM}_{K;\mathcal{L}}^{\varphi,N,\mathrm{wa}}$ be the full subcategory of $\mathrm{FilM}_{K;\mathcal{L}}^{\varphi,N}$ consisting of weakly admissible objects.

Proposition 2.5. *If L is a finite extension of $\mathbb{Q}_p$, then the functor $\mathrm{D}_{\mathrm{st},L}$ is an equivalence of categories between $\mathrm{Rep}_L^{\mathrm{st}}(G_K)$ and $\mathrm{FilM}_{K;L}^{\varphi,N,\mathrm{wa}}$; the quasi-inverse is the functor $\mathrm{V}_{\mathrm{st},L}$ defined by*

$$\mathrm{V}_{\mathrm{st},L}(D) = (\widetilde{\mathrm{B}}^+_{\log}[1/t] \otimes_{K_0} D)^{\varphi=1,N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}} \otimes_K D_K).$$

Proof. According to Colmez–Fontaine theorem [6], $\mathrm{D}_{\mathrm{st},\mathbb{Q}_p}$ is an equivalence of categories between $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K)$ and $\mathrm{FilM}_{K;\mathbb{Q}_p}^{\varphi,N,\mathrm{wa}}$. But an object V in $\mathrm{Rep}_L^{\mathrm{st}}(G_K)$ is equivalent to an object $\tilde{V}$ in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_K)$ together with a homomorphism $L \to \mathrm{End}(\tilde{V})$, while an object D in $\mathrm{FilM}_{K;L}^{\varphi,N,\mathrm{wa}}$ is equivalent to an object $\tilde{D}$ in $\mathrm{FilM}_{K;\mathbb{Q}_p}^{\varphi,N,\mathrm{wa}}$ together with a homomorphism $L \to \mathrm{End}(\tilde{D})$. Here, $\mathrm{End}(\tilde{V})$ denotes $\mathrm{End}_{G_K}(\tilde{V})$, and $\mathrm{End}(\tilde{D})$ denotes the ring of endomorphisms of $\tilde{D}$ in the category $\mathrm{FilM}_{K;\mathbb{Q}_p}^{\varphi,N,\mathrm{wa}}$. □

In the general case, it is difficult to determine the image of the functor $\mathrm{D}_{\mathrm{st},\mathcal{L}}$.

The main result of this paper is the following

Theorem 2.6 (=**Theorem 0.2**)**.** *Let $\mathcal{L}$ be a reduced affinoid algebra and let D be a filtered (φ, N)-module over $\mathcal{L} \otimes_{\mathbb{Q}_p} K_0$ that satisfies* (Gr). *If D_x is weakly admissible for some $x \in \mathrm{Max}(\mathcal{L})$, then there exists an affinoid neighborhood* $\mathrm{Max}(\mathcal{B})$ *of x and a semi-stable $\mathcal{B}$-representation $V_{\mathcal{B}}$ of G_K whose associated filtered (φ, N)-module is isomorphic to $D_{\mathcal{B}}$. Moreover, $V_{\mathcal{B}}$ is unique for this property.*

The proof of Theorem 2.6 will be given in Section 5.

3. (φ, Γ)-modules

3.1. Free and locally free (φ, Γ)-modules. By a (*locally*) *free* (φ, Γ)*-module* over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{\dagger}$ (resp. $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}$) we mean a (locally) free $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{\dagger}$(resp. $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}$)-module D of finite rank equipped with a semilinear action of φ such that the map $\varphi^*\mathrm{D} \to \mathrm{D}$ is an isomorphism, and a semilinear action of Γ that commutes with the φ-action and is continuous for the profinite topology on Γ and the topology on $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{\dagger}$ (resp. $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}$) given in Section 1.

Definition 3.1. A locally free (φ, Γ)-module D over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{\dagger}$ is called *étale* if it admits a finite (φ, Γ)-stable $(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}_K^{\dagger})$-submodule N such that $\varphi^*N \to N$ is isomorphic and the induced map $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{\dagger}) \otimes_{\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}_K^{\dagger}} N \to \mathrm{D}$ is isomorphic. We say that a locally free (φ, Γ)-module D over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}$ is *étale* if it arises from an étale (φ, Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{\dagger}$ by base extension.

By [10, Proposition 6.5] the natural base change functor from the category of étale (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_K$ to the category of étale (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$ is fully faithful.

The following property of free (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$ is very useful.

Proposition 3.2. *Let* D *be a free* φ*-module over* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$. *Then there exists a sufficiently large* $r(\mathrm{D}) > r(K)$ *such that for any* $r \geq r(\mathrm{D})$ *there exists a unique free* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$*-submodule* D_r *of* D *satisfying the following conditions*

(a) $\mathrm{D} = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r$;

(b) $\varphi(\mathrm{D}_r)$ *is contained in* $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pr}_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r$ *and generates the latter as an* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pr}_{\mathrm{rig},K}$*-module.*

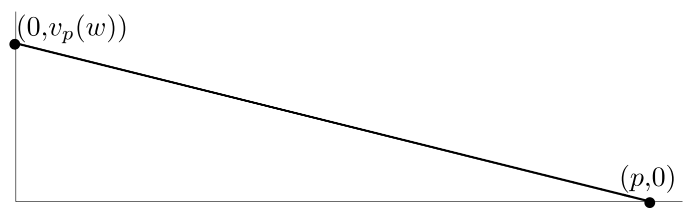
In particular, we have $\mathrm{D}_s = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,s}_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r$ *for any* $s > r$, *and if* D *is a* (φ,Γ)*-module, then* $\gamma(\mathrm{D}_r) = \mathrm{D}_r$ *for all* $\gamma \in \Gamma$.

In the case when $\mathcal{L} = \mathbb{Q}_p$, this is exactly [2, Theorem I.3.3]. For the proof of Proposition 3.2 we need Lemma 3.3 and Proposition 3.4 below.

Let $F(T)$ be a formal series such that $F(T) = \varphi(T)$. Since φ is a lifting of the absolute Frobenus, we can write $F(T) = T^p + pf(T)$ with $f \in \mathcal{O}_{K_0'}[[T]]$.

Lemma 3.3. *When* $r > p$, *the map* $z \mapsto F(z)$ *induces a surjection from* $\{z \in \mathbb{C}_p \mid p^{-1/r} \leq |z| < 1\}$ *to* $\{w \in \mathbb{C}_p \mid p^{-p/r} \leq |w| < 1\}$.

Proof. When $0 < v_p(w) \leq 1$, the Newton polygon of $F(T) - w$ is



So any root z of $F(T) - w$ satisfies $v_p(z) = v_p(w)/p$. □

Proposition 3.4. *Let* $\mathcal{L}$ *be a coefficient algebra. When* $r \gg 0$, *we have*

$$\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K} \cap \varphi(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}) = \varphi(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r/p}_{\mathrm{rig},K})$$

Proof. We choose a $\mathbb{Q}_p$-basis $\{v_1, \cdots, v_m\}$ of K_0'. Then $\{\varphi(v_1), \cdots \varphi(v_m)\}$ is again a $\mathbb{Q}_p$-basis of K_0'. When $r \gg 0$, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ is isomorphic to $\mathcal{R}^r_{\mathcal{L}} \otimes_{\mathbb{Q}_p} K_0'$. Thus, if G is in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$, we may express G in the form

$$G = \sum_{j=1}^{m}\left(\sum_i x_{ij}T^i\right) \otimes v_j,$$

where $\sum_i x_{ij} z^i$ is convergent on the domain $p^{-1/r} \le |z| < 1$ for any $j \in \{1, \cdots, m\}$. Put

$$H = \varphi(G) = \sum_{j=1}^{m} \left(\sum_i x_{ij} \varphi(T)^i \right) \otimes \varphi(v_j) = \sum_{j=1}^{m} \left(\sum_i x_{ij} F(T)^i \right) \otimes \varphi(v_j).$$

If H is again in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$, then by Lemma 3.3, $\sum_i x_{ij} w^i$ is convergent on the domain $p^{-p/r} \le |w| < 1$ for any $j \in \{1, \cdots, m\}$, which implies that G is in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r/p}_{\mathrm{rig},K}$. $\square$

Proof of Proposition 3.2. The argument is similar to the proof of [2, Theorem I.3.3]. We give the details for completeness.

Since D is a free $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$-module, it has an $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}$-basis $\{e_1, \dots, e_d\}$. As

$$\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K} = \bigcup_{r>0} \mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K},$$

there exists $r_0 = r(\mathrm{D})$ such that the matrix of φ relative to this basis is contained in $\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r_0}_{\mathrm{rig},K})$. For any $r \ge r_0$ put $\mathrm{D}_r = \oplus_{i=1}^d (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}) e_i$. Obviously

$$\mathrm{D} = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r.$$

Further

$$\mathrm{D}_{pr} = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,pr}_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r$$

has a basis in $\varphi(\mathrm{D}_r)$. Indeed, $\{\varphi(e_i) \mid i = 1 \dots, d\}$ is such a basis. This proves the existence of D_r.

Let $\mathrm{D}^{(1)}_r$ and $\mathrm{D}^{(2)}_r$ be two $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$-submodules of D satisfying Conditions (a) and (b). We choose bases for these two submodules. Let P_1 and P_2 be respectively the matrices of φ relative to these two bases, so P_1, P_2 are in $\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,pr}_{\mathrm{rig},K})$. Let $M \in \mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\mathrm{rig},K})$ be the transformation matrix from the basis of $\mathrm{D}^{(1)}_r$ to that of $\mathrm{D}^{(2)}_r$. Then $\varphi(M) = P_1^{-1} M P_2$. We show that M is in $\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. If M is in $\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,s}_{\mathrm{rig},K})$ with $s \ge pr$, then $\varphi(M) = P_1^{-1} M P_2$ is also in $\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,s}_{\mathrm{rig},K})$. By Proposition 3.4 we see that M is in $\mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,s/d}_{\mathrm{rig},K})$, and by induction we finally get $M \in \mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. Similarly we have $M^{-1} \in \mathrm{M}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. So M is in $\mathrm{GL}_d(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$, which implies that $\mathrm{D}^{(1)}_r = \mathrm{D}^{(2)}_r$. This proves the uniqueness of D_r.

If $s > r$, the module $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,s}_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r$ satisfies (a) and (b) (in which take r to be s). Thus by the uniqueness of D_s we have

$$\mathrm{D}_s = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,s}_{\mathrm{rig},K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r.$$

If D is a (φ, Γ)-module, from the uniqueness of D_r we obtain $\gamma(\mathrm{D}_r) = \mathrm{D}_r$ for any $\gamma \in \Gamma$. $\square$

3.2. Locally free (φ, Γ)-modules associated to $\mathcal{L}$-linear representations. We recall the functor of Berger and Colmez from the category of $\mathcal{L}$-representations of G_K to the category of étale (φ, Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_K$ and the functor of Kedlaya and Liu from the the category of $\mathcal{L}$-representations of G_K to the category of étale (φ,Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$.

For any finite extension K' of K put $\mathrm{A}^{\dagger,r}_{K',n}=\varphi^{-n}(\mathrm{A}^{\dagger,p^n r}_{K'})$.

Proposition 3.5 ([**3, Proposition 4.2.8**]). *Let $\mathcal{L}$ be a coefficient algebra over $\mathbb{Q}_p$. Let $T_{\mathcal{L}}$ be a free $\mathcal{O}_{\mathcal{L}}$-linear representation of rank d. Let K' be a finite Galois extension of K such that $G_{K'}$ acts trivially on $T_{\mathcal{L}}/12pT_{\mathcal{L}}$. Then there exists $n(K',T_{\mathcal{L}})\geq 0$ such that for $n\geq n(K',T_{\mathcal{L}})$, $(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\widetilde{\mathrm{A}}^{\dagger,(p-1)/p})\otimes_{\mathcal{O}_{\mathcal{L}}}T_{\mathcal{L}}$ has a unique $(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger,(p-1)/p}_{K',n})$-submodule $\mathrm{D}^{\dagger,(p-1)/p}_{\mathcal{L};K',n}(T_{\mathcal{L}})$ that is free of rank d and fixed by $H_{K'}$, has a basis almost invariant under $\Gamma_{K'}$ (i.e., for any $\gamma\in\Gamma_{K'}$ the matrix of action of $\gamma-1$ on this basis has positive valuation) and satisfies*

$$(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\widetilde{\mathrm{A}}^{\dagger,(p-1)/p})\otimes_{\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger,(p-1)/p}_{K',n}}\mathrm{D}^{\dagger,(p-1)/p}_{\mathcal{L};K',n}(T_{\mathcal{L}})=(\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\widetilde{\mathrm{A}}^{\dagger,(p-1)/p})\otimes_{\mathcal{O}_{\mathcal{L}}}T_{\mathcal{L}}.$$

Here $\Gamma_{K'}=\mathrm{Gal}(K'K_\infty/K')$ and $H_{K'}=\mathrm{Gal}(\bar{\mathbb{Q}}_p/K'K_\infty)$.

Theorem 3.6 ([**3, Théorème 4.2.9**]). *Let $\mathcal{L}$ be a coefficient algebra over $\mathbb{Q}_p$. Let V be an $\mathcal{L}$-representation admitting a free Galois stable $\mathcal{O}_{\mathcal{L}}$-lattice T. Then there exists some n such that for any $r\geq r(V)=(p-1)p^{n-1}$ we may define*

$$\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V):=((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{K'})\otimes_{\mathcal{O}_{\mathcal{L}}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{A}^{\dagger,r(V)}_{K'}}\varphi^n(\mathrm{D}^{\dagger,(p-1)/p}_{\mathcal{L};K',n}(T))^{H_K}$$

for some K' and n, so that the construction does not depend on the choices of T, K', n, and the following statements hold.

- (a) *The $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{K})$-module $\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)$ is locally free of rank d.*
- (b) *The natural map $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger,r})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_K}\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)\to(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger,r})\otimes_{\mathcal{L}}V$ is an isomorphism.*
- (c) *For any maximal ideal $\mathfrak{m}$ of $\mathcal{L}$, writing $V_{\mathfrak{m}}:=L_{\mathfrak{m}}\otimes_{\mathcal{L}}V$, the natural map $L_{\mathfrak{m}}\otimes_{\mathcal{L}}\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)\to\mathrm{D}^{\dagger,r}_{L_{\mathfrak{m}}}(V_{\mathfrak{m}})$ is an isomorphism.*

Put

$$\mathrm{D}^{\dagger}_{\mathcal{L}}(V):=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_K}\mathrm{D}^{\dagger,r}_{\mathcal{L}}(V)$$

and

$$\mathrm{D}^{\dagger}_{\mathrm{rig},\mathcal{L}}(V):=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K}\mathrm{D}^{\dagger}_{\mathcal{L}}(V).$$

Then $\mathrm{D}^{\dagger}_{\mathcal{L}}(V)$ (resp. $\mathrm{D}^{\dagger}_{\mathrm{rig},\mathcal{L}}(V)$) is an étale (φ,Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K$ (resp. $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$).

Proposition 3.7 ([**10**]). *We have*

$$V=\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_K}\mathrm{D}^{\dagger}_{\mathcal{L}}(V)\Big)^{\varphi=1}=\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathrm{D}^{\dagger}_{\mathrm{rig},\mathcal{L}}(V)\Big)^{\varphi=1}.$$

4. Coadmissible (φ,Γ)-modules and filtered (φ,N)-modules

In this section we introduce a notion of coadmissible (φ,Γ)-modules and construct a functor, a family version of Berger's functor [2], from the category of filtered (φ,N)-

modules satisfying the condition (Gr) to the category of coadmissible (φ, Γ)-modules. Throughout this section, we always assume that $\mathcal{L}$ is noetherian and satisfies the condition (FL) given in Section 4.1, and that r, r', s, s' and u are in $\mathbb{N}[1/p]$.

4.1. Coadmissible (φ, Γ)-modules. First we recall the notions of Fréchet–Stein algebras and coadmissible modules defined by Schneider and Teitelbaum [13, Section 3].

Definition 4.1. A (commutative) Fréchet–algebra A over K is called a *Fréchet–Stein algebra* if there is a sequence $q_1 \leq \cdots \leq q_n \leq \cdots$ of continuous algebra seminorms on A that defines the Fréchet topology on A such that

- $A_{q_n} := A/\{x \in A \,|\, q_n(x) = 0\}$ is a noetherian Banach algebra,
- A_{q_n} is a flat $A_{q_{n+1}}$-module for any $n \in \mathbb{N}$.

For $(A, (q_n))$ as above we have $A \xrightarrow{\simeq} \varprojlim_n A_{q_n}$.

Definition 4.2. A *coherent sheaf* for $(A, (q_n))$ is a sequence $\{M_n\}_{n\in\mathbb{N}}$, where M_n is a finite A_{q_n}-module, together with isomorphisms $A_{q_n} \otimes_{A_{q_{n+1}}} M_{n+1} \xrightarrow{\simeq} M_n$.

If $\{M_n\}$ is a coherent sheaf for $(A, (q_n))$, its A-module of "global sections" is defined by $\Gamma(\{M_n\}) := \varprojlim_n M_n$. If $\{M_n\}$ is a coherent sheaf for $(A, (q_n))$ and if $M = \Gamma(\{M_n\})$, then the natural map $A_{q_n} \otimes_A M \to M_n$ is isomorphic for any $n \in \mathbb{N}$.

Definition 4.3. An A-module is called *coadmissible* if it is isomorphic to the module of global sections of some coherent sheaf for $(A, (q_n))$.

The "global sections" functor Γ is an equivalence of categories between the category of coherent sheaves for $(A, (q_n))$ and the category of coadmissible A-modules.

For a coadmissible A-module M associated to a coherent sheaf $\{M_n\}$, we may equip each M_n with its canonical Banach space topology and then equip M with the projective limit topology of these canonical topologies. The resulting topology on M is called the *canonical topology* of M.

Let $(A', (q'_m))$ be another Fréchet–Stein algebra and assume that there is a continuous map $A \to A'$. For a coadmissible A-module M, in general $A' \otimes_A M$ is not a coadmissible A'-module. But $\{A'_{q'_m} \otimes_A M\}$ is a coherent sheaf. Let $(A' \otimes_A M)^{\mathrm{ad}}$ denote the corresponding coadmissible A'-module. Then the natural map $A' \otimes_A M \to (A' \otimes_A M)^{\mathrm{ad}}$ has a dense image.

Until the end of this section we assume that the coefficient algebra $\mathcal{L}$ is noetherian and satisfies the following condition:

(FL) For any two closed subintervals $I' = [r', s'] \subseteq I = [r, s]$ of $[0, +\infty[$ with $r \leq r' \leq s' \leq s$ all in $\mathbb{N}[1/p]$, $\mathcal{R}_{\mathcal{L}}^{I'}$ is flat over $\mathcal{R}_{\mathcal{L}}^{I}$.

Lemma 4.4. *If $\mathcal{L}$ is a reduced affinoid algebra over $\mathbb{Q}_p$, then $\mathcal{L}$ satisfies* (FL)*.*

Proof. Let $I' \subseteq I$ be two closed subintervals of $[0, +\infty[$ with $I = [r, s]$ and $I' = [r', s']$, $r \leq r' \leq s' \leq s$. Since r' and s' are rational number, we may write $r' = a/b$ and

$s' = c/d$ with $a, b, c, d \in \mathbb{N}$. Then $\mathrm{Max}(\mathcal{R}_{\mathcal{L}}^{I'})$ is the Weierstrass subdomain of $\mathrm{Max}(\mathcal{R}_{\mathcal{L}}^{I})$ defined by $|p^b/T^a| \leq 1$ and $|T^c/p^d| \leq 1$. (In the case of $r = r' = 0$, $\mathrm{Max}(\mathcal{R}_{\mathcal{L}}^{I'})$ is the Weierstrass subdomain defined by $|T^c/p^d| \leq 1$.) Thus by [5, Corollary 7.3.2/6] we see that $\mathcal{R}_{\mathcal{L}}^{I'}$ is flat over $\mathcal{R}_{\mathcal{L}}^{I}$. □

The condition (FL) ensures that $\mathcal{R}_{\mathcal{L}}^{r}$ is a Fréchet–Stein algebra and is isomorphic to the projective limit $\varprojlim_s \mathcal{R}_{\mathcal{L}}^{[r,s]}$ of Banach algebras. When r is sufficiently large, $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}$ is isomorphic to $\mathcal{R}_{\mathcal{L}}^{r} \otimes_{\mathbb{Q}_p} K_0'$ and thus is a Fréchet–Stein algebra.

Definition 4.5. A *coadmissible φ-module* (resp. *(φ, Γ)-module*) M over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger}$ means a direct system $\{\mathrm{M}_r\}_{r\geq u}$, where u is a positive rational number in $\mathbb{N}[1/p]$, and M_r is a coadmissible $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}$-module for any $r \geq u$, satisfying the following properties:

(a) For any $s \geq r$, $\mathrm{M}^{[r,s]} := (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} \mathrm{M}_r$ is locally free over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}$ of constant rank;

(b) For any $r' \geq r$, the natural map $\mathrm{M}_r \to \mathrm{M}_{r'}$ induces an isomorphism

$$\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r'}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,r}} M\Big)^{\mathrm{ad}} \xrightarrow{\sim} \mathrm{M}_{r'};$$

(c) For any $r \geq u$ there exists a semilinear map $\varphi : \mathrm{M}_r \to \mathrm{M}_{pr}$ such that $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{rig},K}^{\dagger,pr}) \cdot \varphi(\mathrm{M}_r)$ is dense in M_{pr} for the canonical topology and such that the following diagram

$$\begin{array}{ccc} \mathrm{M}_r & \longrightarrow & \mathrm{M}_{r'} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \mathrm{M}_{pr} & \longrightarrow & \mathrm{M}_{pr'} \end{array}$$

is commutative for any pair $r < r'$ with $r \geq u$, where the horizontal arrows are natural maps.

If M satisfies one more condition

(d) for any $r \geq u$ there exists a semilinear Γ-action on M_r that are compatible with the natural maps $\mathrm{M}_r \to \mathrm{M}_{r'}$ $(r' \geq r)$ and the maps $\varphi : \mathrm{M}_r \to \mathrm{M}_{pr}$,

then M is called a *coadmissible (φ, Γ)-module.*

Remark 4.6. Condition (b) is equivalent to the following condition:

If $r \leq r' \leq s' \leq s$, then the map $\mathrm{M}_r \to \mathrm{M}_{r'}$ induces an isomorphism

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}} \mathrm{M}^{[r,s]} \xrightarrow{\sim} \mathrm{M}^{[r',s']}.$$

Condition (c) is equivalent to the following condition:

For any pair $r \leq s$ with $r \geq u$, there exists a semilinear map $\varphi : \mathrm{M}^{[r,s]} \to \mathrm{M}^{[pr,ps]}$ such that $\varphi(\mathrm{M}^{[r,s]})$ generates $\mathrm{M}^{[pr,ps]}$, and such that if $r \leq r' \leq s' \leq s$, then the following diagram

$$\begin{array}{ccc} \mathrm{M}^{[r,s]} & \longrightarrow & \mathrm{M}^{[r',s']} \\ \Big\downarrow \varphi & & \Big\downarrow \varphi \\ \mathrm{M}^{[pr,ps]} & \longrightarrow & \mathrm{M}^{[pr',ps']} \end{array}$$

is commutative where the horizontal arrows are natural maps.

Proposition 4.7. *Any free φ-module* (*resp.* (φ, Γ)*-module*) *over* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ *is a coadmissible φ-module* (*resp.* (φ, Γ)*-module*).

Proof. This follows from Proposition 3.2. □

If $\mathcal{L} \to \mathcal{L}'$ is a continuous map of coefficient algebras, and if M is a coadmissible φ-module (resp. (φ, Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$, then there exists a unique coadmissible φ-module (resp. (φ, Γ)-module) over $\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$, denoted by $\mathrm{M}_{\mathcal{L}'}$, such that for any pair $s > r$ as in Definition 4.5 one has

$$(\mathrm{M}_{\mathcal{L}'})^{[r,s]} = (\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_{K}) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_{K}} \mathrm{M}^{[r,s]}.$$

To end this subsection, we apply Kedlaya and Liu's result [10] to coadmissible φ-modules and (φ, Γ)-modules.

Definition 4.8. Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{L}$ be a reduced affinoid algebra over K. Recall that $\mathcal{R}^r_K$ denotes the ring of Laurent series with coefficients in K in a variable T convergent on the annulus $0 < v_p(T) \leq 1/r$. By a *vector bundle* over $\mathcal{L}\widehat{\otimes}_K\mathcal{R}^r_K$ we will mean a coherent locally free sheaf over the product of this annulus with $\mathrm{Max}(\mathcal{L})$ in the category of rigid analytic spaces over K. (In the case that $\mathcal{L}$ is disconnected, we insist that the rank be constant.) By a vector bundle over $\mathcal{L}\widehat{\otimes}_K\mathcal{R}_K$ we will mean an object in the direct limit of the categories of vector bundles over $\mathcal{L}\widehat{\otimes}_K\mathcal{R}^r_K$ as $r \to +\infty$.

When $r \gg 0$ one has isomorphisms $\mathrm{B}^{\dagger,r}_{\mathrm{rig},K} \cong \mathcal{R}^r_{K'_0}$. We thus obtain the notion of a vector bundle over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ dependent on the choice of the isomorphism. However, the notion of a vector bundle over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ does not depend on the choice.

Definition 4.9. Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{L}$ be a reduced affinoid algebra over $\mathbb{Q}_p$. By a *family of φ-modules (resp. (φ, Γ)-modules)* over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ we mean a vector bundle $\mathscr{M}$ over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ equipped with an isomorphism $\varphi^*\mathscr{M} \to \mathscr{M}$ viewed as a semilinear φ-action (and a semilinear Γ-action commuting with the φ-action).

Now let $(\mathrm{M}; \{\mathrm{M}_r\}_{r \geq u})$ be a coadmissible φ-module (resp. (φ, Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. For any $r \geq u$ let $\mathscr{M}_r$ be the coherent sheaf over $\mathrm{Max}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$ associated to M_r. Then $\mathscr{M}_r$ is a vector bundle over the affinoid space $\mathrm{Max}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. Let $\mathscr{M}$ be the direct limit of the system $\{\mathscr{M}_r \mid r \geq u\}$. Conditions (c) and (d) in Definition 4.5 ensure that $\mathscr{M}$ is a family of φ-modules (resp. (φ, Γ)-modules) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. In this way we associate to any coadmissible φ-module (resp. (φ,Γ)-module) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ a family of φ-modules (resp. (φ,Γ)-modules) over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$.

By Theorem 0.1 i.e., [10, Theorem 0.2] we have the following

Corollary 4.10. *Let $\mathcal{L}$ be a reduced affinoid algebra over $\mathbb{Q}_p$, M a coadmissible (φ,Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. If M_x is étale for some $x\in\mathrm{Max}(\mathcal{L})$, then there exist an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x and a $\mathcal{B}$-linear representation $V_{\mathcal{B}}$ of G_K whose associated (φ,Γ)-module is $\mathcal{B}\widehat{\otimes}_{\mathcal{L}}\mathrm{M}$. Furthermore, $V_{\mathcal{B}}$ is unique for this property.*

4.2. Coadmissible φ-modules associated to φ-compatible sequences. Put $r_n=(p-1)p^{n-1}$. For any $r\geq(p-1)/p$, let $n(r)$ be the smallest integer n such that $r_n\geq r$.

For any $n\geq n(r)$, there exists a natural map $\varphi^{-n}:\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}\hookrightarrow K_n[[t]]$. Here $K_n[[t]]$ is equipped with the topology that is the pullback of the product topology on $\prod_{m=0}^{+\infty}K_n$ via the map $K_n[[t]]\to\prod_{m=0}^{+\infty}K_n, \sum_{m=0}a_mt^m\mapsto(a_m)_m$, where each K_n is equipped with the usual p-adic topology. For this topology $\{p^m\mathcal{O}_{K_n}[[t]]+t^mK_n[[t]]\}_{m\geq0}$ is a fundamental system of neighborhoods of 0. We extend φ^{-n} continuously to an $\mathcal{L}$-linear map $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}\to\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$, which is denoted by ι_n. By this map $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is endowed with an $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$-module structure.

If D is a locally free φ-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$, the formula $\iota_n(\lambda)\cdot x=\lambda\,x$ gives an $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$-module structure on D_r, which is denoted as $\iota_n(\mathrm{D}_r)$. By abuse of notations, put

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes_{\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})}\iota_n(\mathrm{D}_r).$$

There is a natural map

$$\begin{aligned}\varphi_n:&\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_{n+1}((t))\Big)\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))}\Big[\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\Big)\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r\Big]\\ &\to\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_{n+1}((t))\Big)\otimes^{\iota_{n+1}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r\end{aligned}$$

defined by $\varphi_n(f\otimes(g\otimes\iota_n(x)))=fg\otimes\iota_{n+1}(\varphi(x))$.

Definition 4.11. Let D be a locally free φ-module over $\mathrm{B}^{\dagger}_{\mathrm{rig},K}$, $u\geq r(\mathrm{D})$. Let $\{M_n\}_{n\geq n(u)}$ be a sequence, where M_n is an $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$-submodule of

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u.$$

We say that $\{M_n\}_{n\geq n(u)}$ is φ-*compatible* if

$$\varphi_n\big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_{n+1}[[t]])\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]}M_n\big)=M_{n+1}.$$

Let D be a locally free (φ,Γ)-module, h a positive integer and u a sufficiently large rational number. Let M_u be a closed flat $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}$-submodule of $t^{-h}\mathrm{D}_u$ satisfying the following conditions:

(a) $t^h\mathrm{D}_u\subseteq\mathrm{M}_u\subseteq t^{-h}\mathrm{D}_u$;
(b) M_u is Γ-invariant;

(c) $\varphi(\mathrm{M}_u)$ is contained in $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pu}_{\mathrm{rig},K})\cdot\mathrm{M}_u$;
(d) $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pu}_{\mathrm{rig},K})\cdot\varphi(\mathrm{M}_u)$ is dense in $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pu}_{\mathrm{rig},K})\cdot\mathrm{M}_u$ for the canonical topology of $t^{-h}\mathrm{D}_{pu}$.

For any $n\geq n(u)$, put

$$M_n=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{M}_u.$$

Then $\{M_n\}_{n\geq n(u)}$ is φ-compatible and satisfies

$$t^h(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u\subseteq M_n\subseteq t^{-h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u$$

for all $n\geq n(u)$.

For the converse we have the following theorem.

Theorem 4.12. *Let* D *be a locally free* φ*-module* (*resp.* (φ,Γ)*-module*) *of rank* d *over* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$, $u\geq r(\mathrm{D})$. *If*

$$\{M_n\mid M_n\subseteq\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\Big)\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u\}_{n\geq n(u)}$$

is a φ*-compatible sequence such that* M_n *is a locally free* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$*-module of rank* d *for any* n*, and there exists a positive integer* h *such that*

$$t^h(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u\subseteq M_n\subseteq t^{-h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{D}_u,\tag{4.1}$$

then there exists a coadmissible φ*-submodule* (*resp.* (φ,Γ)*-submodule*) M *of* $\mathrm{D}[1/t]$ *such that*

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}}\mathrm{M}_u=M_n$$

for any $n\geq n(u)$.

To prove Theorem 4.12 we first give two lemmas and two propositions below.

For any $r\geq u$, we put

$$\mathrm{M}_r=\{x\in t^{-h}\mathrm{D}_r\mid\iota_n(x)\in M_n\text{ for any }n\geq n(r)\}.$$

Lemma 4.13. M_r *is a coadmissible* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$*-submodule of* $t^{-h}\mathrm{D}_r$.

Proof. As the maps ι_n, $n\geq n(r)$, are all continuous, M_r is closed in $t^{-h}\mathrm{D}_r$. But a submodule of a coadmissible $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$-module is itself coadmissible if and only if it is closed. □

Lemma 4.14. *We have* $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r=M_n$ *for any* $n\geq n(r)$.

Proof. Note that $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is isomorphic to $(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)[[t]]$, so $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is a noetherian ring. Put $q=\varphi(\pi)/\pi$. Then $K_n[[t]]$ is the $\varphi^{n-1}(q)$-adic completion of $\varphi^{-n}(\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. It follows that $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]\cong(\mathcal{L}\otimes_{\mathbb{Q}_p}K_n)[[t]]$ is the $\varphi^{n-1}(q)$-adic completion of $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. Thus $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ is flat over $\iota_n(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})$. Hence

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\to(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}t^{-h}\mathrm{D}_r$$

is injective, and by the definition of M_r the image of this map is contained in M_n. On the other hand, as $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r$ and M_n are finite over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$, they are complete for the t-adic topology. So we only need to show that the natural map

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\to M_n/t^hM_n$$

is surjective. By (4.1), for any $x\in M_n$, there exists $y\in t^{-h}\mathrm{D}_r$ such that $\iota_n(y)-x\in t^hM_n$. By [2, Lemma I.2.1] there exists $t_{n,3h}\in\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}$ such that $\iota_n(t_{n,3h})=1 \mod t^{3h}K_n[[t]]$ and $\iota_m(t_{n,3h})\in t^{3h}K_m[[t]]$ if $m\geq n(r)$ and $m\neq n$. Put $z=t_{n,3h}y$. Then

$$\iota_n(z)-\iota_n(y)\in t^{2h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r\subseteq t^hM_n$$

and

$$\iota_m(z)\in t^{2h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_m[[t]])\otimes^{\iota_m}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r\subseteq t^hM_m$$

if $m\neq n$. Thus z is in M_r and the map $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\to M_n/t^hM_n$ is surjective. □

Proposition 4.15. *If $s\geq r\geq u$, then $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K)\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r$ is locally free over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K$ of rank $d=\mathrm{rank}(\mathrm{D})$.*

Proof. Put $\mathrm{M}^{[r,s]}=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K)\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r$. Since $\mathrm{M}^{[r,s]}$ is contained in a locally free module $t^{-h}\mathrm{D}^{[r,s]}$ of rank d and contains a locally free submodule $t^h\mathrm{D}^{[r,s]}$ of rank d, it suffices to show that $\mathrm{M}^{[r,s]}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K$. By Gabber's criterion [4, Section 2.6 Lemma 1] we only need to show the following three assertions. (i) $\mathrm{M}^{[r,s]}[1/t]$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K[1/t]$. (ii) $\mathrm{M}^{[r,s]}$ is t-torsion free. (iii) $\mathrm{M}^{[r,s]}/t\mathrm{M}^{[r,s]}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K/(t)=\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}^{[r,s]}_K/(t))$. The first two are trivial. For (iii), if there is no integer n such that $r\leq r_n\leq s$, then t is invertible in $\mathrm{B}^{[r,s]}_K$ and there is nothing to prove. So we assume that there exists at least one integer n such that $r\leq r_n\leq s$.

In this case the map $\varphi^{-n}:\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}\to K_n[[t]]$ can be extended to a map $\varphi^{-n}:\mathrm{B}^{[r,s]}_K\to K_n[[t]]$. We also use ι_n to denote the map

$$\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_{\mathrm{rig},K}\to\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]].$$

The maps ι_n with $r\leq r_n\leq s$ induces an inclusion

$$\iota^{[r,s]}:\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K\to\prod_{r\leq r_n\leq s}\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]],$$

and an isomorphism

$$\bar{\iota}^{[r,s]}:\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}^{[r,s]}_K/(t))\to\prod_{r\leq r_n\leq s}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n.$$

Here, we use $\prod_{r\le r_n\le s}$ to denote $\prod_{n:\ r\le r_n\le s}$ for shortness. Since

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}}\mathrm{M}^{[r,s]}=M_n$$

is locally free of rank d over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ for any n satisfying $r\le r_n\le s$, we see that

$$\begin{aligned}&\left(\prod_{r\le r_n\le s}\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]\right)\otimes^{\iota^{[r,s]}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}}\mathrm{M}^{[r,s]}\\&=\prod_{r\le r_n\le s}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota^{[r,s]}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}}\mathrm{M}^{[r,s]}\end{aligned}$$

is locally free of rank d over $\prod_{r\le r_n\le s}\left(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]\right)$. Consequently,

$$\left(\prod_{r\le r_n\le s}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n\right)\otimes^{\bar{\iota}^{[r,s]}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r,s]}/(t))}(\mathrm{M}^{[r,s]}/t\mathrm{M}^{[r,s]})$$

is locally free of rank d over $\prod_{r\le r_n\le s}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n$. As $\bar{\iota}^{[r,s]}$ is isomorphic, $\mathrm{M}^{[r,s]}/t\mathrm{M}^{[r,s]}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r,s]}/(t))$. □

Proposition 4.16. (a) *For any $s\ge s'\ge r'\ge r\ge u$ we have a natural isomorphism*

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}}\mathrm{M}^{[r,s]}\xrightarrow{\sim}\mathrm{M}^{[r',s']}.$$

(b) *For any pair $r'\ge r$ with $r\ge u$, $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K})\cdot\mathrm{M}_r$ is contained and dense in $\mathrm{M}_{r'}$.*

(c) *$\varphi(\mathrm{M}_r)$ is contained in M_{pr} and $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,pr}_{\mathrm{rig},K})\cdot\varphi(\mathrm{M}_r)$ is dense in M_{pr}.*

Proof. We first prove (a). As $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}$ is flat over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}$, the natural map

$$\begin{aligned}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r,s]}}\mathrm{M}^{[r,s]}&=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\\&\to(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}t^{-h}\mathrm{D}_r\end{aligned}$$

is an injection. By definition $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K})\cdot\mathrm{M}_r$ is contained in $\mathrm{M}_{r'}$, so the image of the above injection is contained in $\mathrm{M}^{[r',s']}$. Let N_1 denote this image and let N_2 denote $\mathrm{M}^{[r',s']}$.

We claim that $N_2=N_1+tN_2$. If there is no $n\in\mathbb{Z}$ such that $r'\le r_n\le s'$, then t is invertible in $\mathrm{B}_K^{[r',s']}$ and there is nothing to prove. So we assume that there exists at least one $n\in\mathbb{Z}$ such that $r'\le r_n\le s'$. For any n with $r'\le r_n\le s'$, by Lemma 4.14 we have

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}}N_1=M_n=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']}}N_2.$$

It follows that

$$\left(\prod_{r'\leq r_n\leq s'}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n\right)\otimes^{\bar\iota^{[r',s']}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r',s']}/(t))}N_1/tN_1$$
$$=\left(\prod_{r'\leq r_n\leq s'}\mathcal{L}\otimes_{\mathbb{Q}_p}K_n\right)\otimes^{\bar\iota^{[r',s']}}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}(\mathrm{B}_K^{[r',s']}/(t))}N_2/tN_2.$$

Since $\bar\iota^{[r',s']}$ is an isomorphism, we obtain $N_1/tN_1=N_2/tN_2$. In other words, we have $N_2=N_1+tN_2$.

By induction we obtain that $N_2=N_1+t^\ell N_2$ for any integer $\ell\geq 1$. In particular, $N_2=N_1+t^{2h}N_2$. As $t^{2h}N_2\subseteq t^h\mathrm{D}^{[r',s']}$ is contained in N_1, we get $N_1=N_2$.

We next prove (b). We have already seen that $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K})\cdot\mathrm{M}_r$ is contained in $\mathrm{M}_{r'}$. The closure of $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K})\cdot\mathrm{M}_r$ is exactly the coadmissible $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r'}_{\mathrm{rig},K}$-module associated to the coherent sheaf

$$\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_K^{[r',s']})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{M}_r\Big)_{s'\geq r'},$$

and hence coincides with $\mathrm{M}_{r'}$ by (a).

We can prove (c) similarly. We omit the details. □

Proof of Theorem 4.12. Let M be the inductive system $\{\mathrm{M}_r\}_{r\geq u}$. By Lemma 4.13, Proposition 4.15 and Proposition 4.16, we see that $(\mathrm{M};\{\mathrm{M}_r\}_{r\geq u})$ is a coadmissible φ-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. If D is a (φ,Γ)-module, then by definition M_r is stable under Γ. In this case, $(\mathrm{M};\{\mathrm{M}_r\}_{r\geq u})$ is a coadmissible (φ,Γ)-module. □

4.3. Coadmissible (φ,Γ)-modules associated to filtered (φ,N)-modules. Recall that $\varphi(\ell_\pi)=p\ell_\pi+\log(\varphi(\pi)/\pi^p)$ and $\gamma(\ell_\pi)=\ell_\pi+\log(\gamma(\pi)/\pi)$ for any $\gamma\in\Gamma$. Let N be the $\mathrm{B}^\dagger_{\mathrm{rig},K}$-derivation on $\mathrm{B}^\dagger_{\mathrm{rig},K}[\ell_\pi]$ defined by $N(\ell_\pi)=-p/(p-1)$. We extend these operators $\mathcal{L}$-linearly and continuously to $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K})[\ell_\pi]$. Then we extend the inclusion $\iota_n:\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}\rightarrow\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$ to $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})[\ell_\pi]$ by putting $\iota_n(\ell_\pi)=\log(\varepsilon^{(n)}\exp(t/p^n)-1)\in\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]$.

Let D be a filtered (φ,N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$ of rank d that satisfies Condition (Gr).

Put

$$\mathrm{D}=\big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K})[\ell_\pi]\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0}D\big)^{N=0}.$$

Proposition 4.17. *The following statements hold:*

(a) D *is a locally free* (φ,Γ)*-module over* $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$ *of rank* d.
(b) *We have*

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K})[\ell_\pi]\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}}\mathrm{D}=(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K})[\ell_\pi]\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0}D.$$

The proof is due to Xiao.

Proof. Without loss of generality we may assume that D is free. Let $e_1, \dots, e_d$ be a basis of D. Note that $N^d = 0$ on D. For any $i = 1, \dots, d$, put $f_i = \exp(\frac{p-1}{p}\ell_\pi N)e_i = \sum_{j=0}^{d-1} \frac{(\frac{p-1}{p}\ell_\pi)^j}{j!} N^j(e_i)$. As $e_i = \exp(-\frac{p-1}{p}\ell_\pi N) f_i$ for any i, we see that $\{f_1, \dots, f_d\}$ is a basis of $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K})[\ell_\pi] \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}} D$ over $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K})[\ell_\pi]$. Observe that $N(f_i) = 0$ for all i. It follows that D is a free (φ, Γ)-module and $\{f_1, \dots, f_d\}$ is a basis of D. □

For any $n \geq 0$ we have $\varphi^{-n}(K_0) \subseteq K$. Thus there are $\varphi^{-n}(K_0)$-module structures on K and on D. To avoid misunderstanding we use $\iota_n(D)$ to denote the $\varphi^{-n}(K_0)$-module structure on D. Write $K \otimes^{\iota_n}_{K_0} D$ for $K \otimes_{\varphi^{-n}(K_0)} \iota_n(D)$. There is a map $\xi_n : K \otimes_{K_0} D \to K \otimes^{\iota_n}_{K_0} D$ sending $\mu \otimes x$ to $\mu \otimes \iota_n(\varphi^n(x))$. Then we obtain a filtration on the range $D^n_K = K \otimes^{\iota_n}_{K_0} D$ via the map ξ_n. Define a filtration on $(\mathcal{L} \otimes_{\mathbb{Q}_p} K_n)((t))$ by the formula

$$\mathrm{Fil}^i\big((\mathcal{L} \otimes_{\mathbb{Q}_p} K_n)((t))\big) = t^i(\mathcal{L} \otimes_{\mathbb{Q}_p} K_n)[[t]], \quad i \in \mathbb{Z}.$$

Then for each n we obtain a filtration on $(\mathcal{L} \otimes_{\mathbb{Q}_p} K_n)((t)) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p} K} D^n_K$.

Put

$$M_n(D) = \mathrm{Fil}^0\big((\mathcal{L} \otimes_{\mathbb{Q}_p} K_n)((t)) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p} K} D^n_K\big).$$

Since D satisfies (Gr), for any $n \in \mathbb{N}$, $M_n(D)$ is a locally free $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]$-module of rank d.

Choose an integer $u \geq r(\mathrm{D})$. If $n \geq n(u)$, we may consider M_n as an $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]$-submodule of $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}} \mathrm{D}_u$.

Proposition 4.18. *The family $\{M_n(D)\}_{n \geq n(u)}$ is φ-compatible.*

Proof. This follows from the formula $\xi_{n+1} = \varphi_n \circ \xi_n$ (for all $n \geq n(u)$) on D_K. □

Let h be a positive integer such that the filtration on D_K satisfies $\mathrm{Fil}^{-h} D_K = D_K$ and $\mathrm{Fil}^h D_K = 0$. Then for any $n \geq n(u)$, $M_n(D)$ satisfies

$$t^h(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}} \mathrm{D}_u \subseteq M_n(D) \subseteq t^{-h}(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,u}_{\mathrm{rig},K}} \mathrm{D}_u.$$

Applying Theorem 4.12 we get a coadmissible (φ, Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$ which is denoted by $\mathcal{M}(D)$. Therefore we obtain a functor, denoted by $\mathcal{M}$, from the category of filtered (φ, N)-modules over $\mathcal{L} \otimes_{\mathbb{Q}_p} K_0$ satisfying (Gr) to the category of coadmissible (φ, Γ)-modules over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^\dagger_{\mathrm{rig},K}$.

The functor $\mathcal{M}$ is functorial by the following

Proposition 4.19. *If $\mathcal{L}'$ is another coefficient algebra and $\mathcal{L} \to \mathcal{L}'$ is a continuous map, then $\mathcal{M}(D_{\mathcal{L}'}) = \mathcal{M}(D)_{\mathcal{L}'}$.*

Proof. We have

$$M_n(D_{\mathcal{L}'}) = (\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]) \otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p} K_n[[t]]} M_n(D).$$

Thus by the definitions of $\mathcal{M}(D)_r$ and $\mathcal{M}(D_{\mathcal{L}'})_r$ we have a natural map

$$(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathcal{M}(D)_r\to\mathcal{M}(D_{\mathcal{L}'})_r.$$

What we need to show is that, for any $s\geq r\geq u$ this map induces an isomorphism

$$(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K)\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K}\mathcal{M}(D)^{[r,s]}\to\mathcal{M}(D_{\mathcal{L}'})^{[r,s]}.$$

Let N_1 and N_2 be respectively the domain and the range of this map. Then for any n satisfying $r\leq r_n\leq s$ we have

$$\begin{aligned}
&(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K}N_1\\
&=(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K}\mathcal{M}(D)^{[r,s]}\\
&=(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]}\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K}\mathcal{M}(D)^{[r,s]}\Big)\\
&=(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]}M_n(D)\\
&=M_n(D_{\mathcal{L}'})\\
&=(\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}'\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{[r,s]}_K}N_2
\end{aligned}$$

Now repeating the argument in the proof of Proposition 4.16 (a) we obtain $N_1=N_2$. □

Corollary 4.20. *If $\mathfrak{m}$ is a maximal ideal of $\mathcal{L}$, then $\mathcal{M}(D)_{\mathfrak{m}}$ is the (φ,Γ)-module over $L_{\mathfrak{m}}\otimes_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ associated to the filtered (φ,N)-module $D_{\mathfrak{m}}$ over $L_{\mathfrak{m}}\otimes_{\mathbb{Q}_p}K_0$.*

The following proposition tells us that the functor $\mathcal{M}$ is faithful.

Proposition 4.21. *If D is a filtered (φ,N)-module over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$ satisfying* (Gr), *then*

$$D=\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t,\ell_\pi]\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathcal{M}(D)\Big)^{\Gamma}.$$

To prove Proposition 4.21 we need the following lemma.

Lemma 4.22. *We have*

$$((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t,\ell_\pi])^{\Gamma}=\mathcal{L}\otimes_{\mathbb{Q}_p}K_0.$$

Proof. We define the operators $\nabla=\frac{\log(\gamma)}{\log\chi_{\mathrm{cyc}}(\gamma)}$ (γ sufficiently close to 1) and $\partial=[\varepsilon]\frac{d}{d\pi}$ on $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$ in a way similar to that in [1], and then extend them to $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t,\ell_\pi]$. Note that $\nabla=t\partial$. If $x\in((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t,\ell_\pi])^{\Gamma}$, then $\nabla x=t\partial x=0$ and so x is in $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. As

$$(\mathrm{B}^{[r,s]}_K)^{\Gamma}=(\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})^{\Gamma}=K_0,$$

by Lemma 1.1 we obtain $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K})^{\Gamma}=\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. So we have $(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})^{\Gamma}=\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. □

Proof of Proposition 4.21. From Proposition 4.17 (b) and the relation $\mathrm{D}[1/t] = \mathcal{M}(D)[1/t]$ we obtain

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t,\ell_\pi]\otimes_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}}\mathcal{M}(D) = (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K})[1/t,\ell_\pi]\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K_0} D.$$

Now our conclusion follows from Lemma 4.22. □

5. Proof of Theorem 2.6

Throughout this section let $\mathcal{L}$ be a reduced affinoid algebra over $\mathbb{Q}_p$. Since any reduced affinoid algebra satisfies (FL), we can apply results in Section 4.

Proposition 5.1, Corollary 5.2 and Proposition 5.3 below are useful for the proof of Theorem 2.6. Put

$$\mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V) := ((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{+}_{\log})\otimes_{\mathcal{L}} V)^{G_K}.$$

Proposition 5.1. *If V is a de Rham $\mathcal{L}$-representation of G_K, then the map $\widetilde{\mathrm{B}}^{+}_{\log}\to\widetilde{\mathrm{B}}^{\dagger}_{\log}$ induces an isomorphism*

$$\mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V)\to\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log})\otimes_{\mathcal{L}} V\Big)^{G_K}. \tag{5.1}$$

Proof. The injectivity of (5.1) is clear.

For any $n\in\mathbb{N}$, put $D_n = (\widetilde{\mathrm{B}}^{\dagger,r_n}_{\log}\widehat{\otimes}_{\mathbb{Q}_p}V)^{G_K}$ which is an $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$-module. Note that ι_n induces an inclusion $D_n\hookrightarrow \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)$. As V is de Rham, we see that $\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)$ is finite over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. Thus D_n is finite over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$. There is a sufficiently large n_0 such that the image of $\mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V)$ is contained in D_{n_0}. For any $n\geq n_0$ and any maximal ideal $\mathfrak{m}$ of $\mathcal{L}$, by [1, Proposition 3.4] the map $\mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V)/\mathfrak{m}\mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V)\to D_n/\mathfrak{m}D_n$ is surjective. Combining this with the fact that D_n is finite over $\mathcal{L}\otimes_{\mathbb{Q}_p}K_0$, we see that the map $\mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V)\to D_n$ is surjective. It follows that (5.1) is surjective. □

Corollary 5.2. *If V is a de Rham $\mathcal{L}$-representation of G_K, then the map $\widetilde{\mathrm{B}}^{+}_{\log}\to\widetilde{\mathrm{B}}^{\dagger}_{\log}$ induces an isomorphism*

$$\mathrm{D}_{\mathrm{st},\mathcal{L}}(V)\to\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log}[1/t])\otimes_{\mathcal{L}} V\Big)^{G_K}. \tag{5.2}$$

Proof. If V is of negative Hodge–Tate weights, we have $\mathrm{D}_{\mathrm{st},\mathcal{L}}(V) = \mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V)$ and $\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log}[1/t])\otimes_{\mathcal{L}} V\Big)^{G_K} = \Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log})\otimes_{\mathcal{L}} V\Big)^{G_K}$. In general, we have

$$\mathrm{D}_{\mathrm{st},\mathcal{L}}(V) = t^{-d}\mathrm{D}^{+}_{\mathrm{st},\mathcal{L}}(V(-d))(d)$$

and

$$\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log}[1/t])\otimes_{\mathcal{L}} V\Big)^{G_K} = t^{-d}\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log})\otimes_{\mathcal{L}} V(-d)\Big)^{G_K}(d)$$

when d is sufficiently large. Thus our statement follows from Proposition 5.1. □

Proposition 5.3. *If V is a semistable $\mathcal{L}$-representation and $\mathrm{D} = \mathrm{D}^{\dagger}_{\mathrm{rig}}(V)$, then for any sufficiently large $n\in\mathbb{N}$ we have*

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]])\otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}}\mathrm{D}_r = \mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t)))\otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big).$$

Proof. By [3, Lemma 4.3.1 and Theorem 5.3.2], if $n \in \mathbb{N}$ is sufficiently large, then

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^+_{\mathrm{dR}}) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r = \mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}}) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K} \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big)$$

and

$$\Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\Big) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r = \Big(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\Big) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K} \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V).$$

Combining these two facts and the fact that

$$\begin{aligned}&\mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K} \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big)\\ =&\mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}}) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K} \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big) \cap (\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K} \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V),\end{aligned}$$

we obtain

$$(\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]) \otimes^{\iota_n}_{\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}_r \subseteq \mathrm{Fil}^0\Big((\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))) \otimes_{\mathcal{L}\otimes_{\mathbb{Q}_p}K} \mathrm{D}_{\mathrm{dR},\mathcal{L}}(V)\Big).$$

By [2] this inclusion becomes isomorphic after modulo $\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathcal{L}$. Therefore it is itself isomorphic. □

Proof of Theorem 2.6. Let D be a filtered (φ, N)-module over $\mathcal{L} \otimes_{\mathbb{Q}_p} K_0$ that satisfies (Gr). As D satisfies (Gr), $\mathcal{M}(D)$ is a coadmissible (φ,Γ)-module over $\mathcal{L}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}$. As the functor $\mathcal{M}$ is functorial, we have $\mathcal{M}(D)_x = \mathcal{M}(D_x)$. Because D_x is weakly admissible, by [2], $\mathcal{M}(D)_x$ is étale. Thus by Corollary 4.10 there exist an affinoid neighborhood $\mathrm{Max}(\mathcal{B})$ of x in $\mathrm{Max}(\mathcal{L})$ and a $\mathcal{B}$-linear representation $V_{\mathcal{B}}$ whose associated (φ,Γ)-module is $\mathcal{B}\widehat{\otimes}_{\mathcal{L}}\mathcal{M}(D) = \mathcal{M}(\mathcal{B}\otimes_{\mathcal{L}} D)$.

By Proposition 4.21

$$\begin{aligned}\mathcal{B}\otimes_{\mathcal{L}} D &= \Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\log,K}[1/t]) \otimes_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}} \mathcal{M}(\mathcal{B}\otimes_{\mathcal{L}} D)\Big)^{\Gamma}\\ &= \Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\log,K}[1/t]) \otimes_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D}^{\dagger}_{\mathrm{rig}}(V_{\mathcal{B}})\Big)^{\Gamma}.\end{aligned}$$

It follows that, for any rigid point $y \in \mathrm{Max}(\mathcal{B})$,

$$(\mathcal{B}\otimes_{\mathcal{L}} D)_y = \Big((L_y \otimes_{\mathbb{Q}_p} \mathrm{B}^{\dagger}_{\log,K}[1/t]) \otimes_{L_y\otimes_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D}^{\dagger}_{\mathrm{rig}}(V_{\mathcal{B}}\otimes_{\mathcal{B}} L_y)\Big)^{\Gamma}$$

where $L_y = \mathcal{B}/\mathfrak{m}_y$. Thus $V_{\mathcal{B}}\otimes_{\mathcal{B}} L_y$ is semistable for any $y \in \mathrm{Max}(\mathcal{B})$. Then by [3] $V_{\mathcal{B}}$ is semistable.

Note that

$$\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\log,K}[1/t]) \otimes_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger}_{\mathrm{rig},K}} \mathrm{D}^{\dagger}_{\mathrm{rig}}(V_{\mathcal{B}})\Big)^{\Gamma} \subseteq \Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathrm{B}}^{\dagger}_{\log}[1/t]) \otimes_{\mathcal{B}} V_{\mathcal{B}}\Big)^{G_K}.$$

So, by Corollary 5.2, $\mathcal{B}\otimes_{\mathcal{L}} D$ is contained in $\mathrm{D}_{\mathrm{st},\mathcal{B}}(V_{\mathcal{B}})$. The inclusion $\mathcal{B}\otimes_{\mathcal{L}} D \to \mathrm{D}_{\mathrm{st},\mathcal{B}}(V_{\mathcal{B}})$ is in fact isomorphic, since it induces isomorphisms $D_y \xrightarrow{\sim} \mathrm{D}_{\mathrm{st},L_y}(V_y)$ at all rigid points $y \in \mathrm{Max}(\mathcal{B})$.

By Lemma 4.14 there exists a sufficiently large r such that for any $n \geq n(r)$,

$$(\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]) \otimes^{\iota_n}_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}^{\dagger}_{\mathrm{rig},K}(V_{\mathcal{B}})_r = \mathrm{Fil}^0\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))) \otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K} (\mathcal{B}\otimes_{\mathcal{L}} D)_K\Big).$$

But by Proposition 5.3 we have

$$(\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n[[t]]) \otimes^{\iota_n}_{\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathrm{B}^{\dagger,r}_{\mathrm{rig},K}} \mathrm{D}^{\dagger}_{\mathrm{rig},K}(V_{\mathcal{B}})_r = \mathrm{Fil}^0\Big((\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))) \otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K} \mathrm{D}_{\mathrm{dR},\mathcal{B}}(V_{\mathcal{B}})\Big).$$

Hence

$$\mathrm{Fil}^0\Big(\big(\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\big)\otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K}(\mathcal{B}\otimes_{\mathcal{L}}D)_K\Big) = \mathrm{Fil}^0\Big(\big(\mathcal{B}\widehat{\otimes}_{\mathbb{Q}_p}K_n((t))\big)\otimes_{\mathcal{B}\otimes_{\mathbb{Q}_p}K}\mathrm{D}_{\mathrm{dR},\mathcal{B}}(V_{\mathcal{B}})\Big).$$

It follows that the filtration on $\mathrm{D}_{\mathrm{dR},\mathcal{B}}(V_{\mathcal{B}})$ and the filtration on $(\mathcal{B}\widehat{\otimes}_{\mathcal{L}}D)_K$ coincide. Therefore the filtered (φ, N)-module associated to $V_{\mathcal{B}}$ is $\mathcal{B}\otimes_{\mathcal{L}} D$.

The uniqueness of $V_{\mathcal{B}}$ follows from Corollary 4.10. □

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