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UNIQUENESS OF SOLUTIONS FOR A NONLOCAL ELLIPTIC EIGENVALUE PROBLEM

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Abstract. We examine equations of the form

$$\begin{cases}(-\Delta)^{\frac{1}{2}}u = \lambda g(x)f(u) & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $\lambda > 0$ is a parameter and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. Here g is a positive function and f is an increasing, convex function with $f(0) = 1$ and either f blows up at 1 or f is superlinear at infinity. We show that the extremal solution u^* associated with the extremal parameter λ^* is the unique solution. We also show that when f is suitably supercritical and Ω satisfies certain geometrical conditions then there is a unique solution for small positive λ.

1. Introduction

We are interested in the following nonlocal eigenvalue problem

$$(P)_\lambda \qquad \begin{cases}(-\Delta)^{\frac{1}{2}}u = \lambda g(x)f(u) & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $(-\Delta)^{\frac{1}{2}}$ is the square root of the Laplacian operator, $\lambda > 0$ is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$ where $N \geq 2$, and where $0 < g(x) \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha$. The nonlinearity f satisfies one of the following two conditions:

(R) f is smooth, increasing and convex on $\mathbb{R}$ with $f(0) = 1$ and f is superlinear at ∞ (i.e., $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$), or

(S) f is smooth, increasing, convex on $[0, 1)$ with $f(0) = 1$ and $\lim_{t\nearrow 1} f(t) = +\infty$.

In this paper, we prove there is a unique solution of $(P)_\lambda$ for two parameter ranges: for small λ and for $\lambda = \lambda^*$ where λ^* is the so called extremal parameter associated with $(P)_\lambda$. First, let us to recall various known facts concerning the second order analog of $(P)_\lambda$.

Some notations: Let $F(t) := \int_0^t f(\tau)d\tau$ and $C_f := \int_0^{a_f} f(t)^{-1}dt$ where $a_f = \infty$ (resp. $a_f = 1$) when f satisfies (R) (resp. f satisfies (S)). We say a positive function f defined on an interval I is logarithmically convex (or log convex) provided $u \mapsto \log(f(u))$ is convex on I. Also, Ω will always denote a smooth bounded domain in $\mathbb{R}^N$ where $N \geq 2$.

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1.1. The local eigenvalue problem. For a nonlinearity f which satisfies (R) or (S), the following second order analog of $(P)_\lambda$ with the Dirichlet boundary conditions

$$(Q)_\lambda \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is by now quite well understood whenever Ω is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [2–5, 14–16, 18, 20, 21]. We now list the properties one comes to expect when studying $(Q)_\lambda$.

It is well known that there exists a critical parameter $\lambda^* \in (0, \infty)$ such that for all $0 < \lambda < \lambda^*$ there exists a smooth, minimal solution u_λ of $(Q)_\lambda$. Here the minimal solution means in the pointwise sense. In addition for each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing in $(0, \lambda^*)$. This allows one to define the pointwise limit $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ which can be shown to be a weak solution, in a suitably defined sense, of $(Q)_{\lambda^*}$. It is also known that for $\lambda > \lambda^*$ there are no weak solutions of $(Q)_\lambda$. Also, one can show that the minimal solution u_λ is a semi-stable solution of $(Q)_\lambda$ in the sense that

$$\int_\Omega \lambda f'(u_\lambda)\psi^2 \le \int_\Omega |\nabla\psi|^2, \qquad \forall \psi \in H_0^1(\Omega).$$

We now come to the results known for $(Q)_\lambda$ that we are interested in extending to $(P)_\lambda$. In [18] it was shown that the extremal solution u^* is the unique weak solution of $(Q)_{\lambda^*}$. Some of the techniques involve using concave cut offs which do not seem to carry over to the nonlocal setting. Here, we use some techniques developed in [1] that were used in studying a fourth order analogue of $(Q)_\lambda$. In [11] the uniqueness of the extremal solution for $\Delta^2 u = \lambda e^u$ on radial domains with Dirichlet boundary conditions was shown and this was extended to log convex (see below) nonlinearities in [17]. Some of the methods used in [17] were inspired by the techniques of [1] and so will ours in the case where f satisfies (R). In [8] it was shown that the extremal solution associated with $\Delta^2 u = \lambda(1-u)^{-2}$ on radial domains is unique and our methods for nonlinearities satisfying (S) use some of their techniques.

In [19] and [23] a generalization of $(Q)_\lambda$ was examined. They showed that if f is suitably supercritical at infinity and if Ω is a star-shaped domain, then for small $\lambda > 0$ the minimal solution is the unique solution of $(Q)_\lambda$. In [13] this was done for a particular nonlinearity f which satisfies (S). One can weaken the star-shaped assumption and still have uniqueness, see [22], but we do not pursue this approach here. In Section 3, we extend these results to $(P)_\lambda$. For more results the on uniqueness of solutions for various elliptic problems involving parameters, see [12].

For questions on the regularity of the extremal solution in fourth order problems, we direct the interested reader to [10]. We also mention the recent preprint [9] which examines the same issues as this paper but for equations of the form $\Delta^2 u = \lambda f(u)$ in Ω with either the Dirichlet boundary conditions $u = |\nabla u| = 0$ on $\partial\Omega$ or the Navier boundary conditions $u = \Delta u = 0$ on $\partial\Omega$. Elliptic systems of the form $-\Delta u = \lambda f(v)$, $-\Delta v = \gamma g(u)$ in Ω with $u = v = 0$ on $\partial\Omega$ are also examined.

1.2. The nonlocal eigenvalue problem. We first give the needed background regarding $(-\Delta)^{\frac{1}{2}}$ to examine $(P)_\lambda$, for a more detailed background see [6]. In [7] they examined the problem $(P)_\lambda$ with $(-\Delta)^s$ replacing $(-\Delta)^{\frac{1}{2}}$ and with $g(x) = 1$. They did not investigate the questions, we are interested in but they did develop much of the needed theory to examine $(P)_\lambda$ and so we will use many of their results.

There are various ways to make sense of $(-\Delta)^{\frac{1}{2}}u$. Suppose that $u(x)$ is a smooth function defined in Ω which is zero on $\partial\Omega$ and suppose that $u(x) = \sum_k a_k\phi_k(x)$ where (ϕ_k, λ_k) are the eigenpairs of $-\Delta$ in $H_0^1(\Omega)$ which are L^2 normalized. Then one defines

$$(-\Delta)^{\frac{1}{2}}u(x) = \sum_k a_k\sqrt{\lambda_k}\phi_k(x).$$

Another way is to suppose we are given $u(x)$ which is zero on $\partial\Omega$ and we let $u_e = u_e(x, y)$ denote a solution of

$$\begin{cases} \Delta u_e = 0 & \text{in } \mathcal{C} := \Omega \times (0, \infty) \\ u_e = 0 & \text{on } \partial_L\mathcal{C} := \partial\Omega \times (0, \infty) \\ u_e = u(x) & \text{in } \Omega \times \{0\}. \end{cases}$$

Then we define

$$(-\Delta)^{\frac{1}{2}}u(x) = \partial_\nu u_e(x, y)\big|_{y=0},$$

where ν is the outward pointing normal on the bottom of the cylinder, $\mathcal{C}$. We call u_e the harmonic extension of u. We define $H_{0,L}^1(\mathcal{C})$ to be the completion of $C_c^\infty(\Omega\times[0, \infty))$ under the norm $\|u\|^2 := \int_{\mathcal{C}} |\nabla u|^2$. When working on the cylinder generally we will write integrals of the form $\int_{\Omega\times\{y=0\}} \gamma(u_e)$ as $\int_\Omega \gamma(u)$.

Some of our results require one to examine quite weak notions of solutions to $(P)_\lambda$ and so we begin with our definition of a weak solution.

Definition 1. Given $h(x) \in L^1(\Omega)$ we say that $u \in L^1(\Omega)$is a weak solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

provided that

$$\int_\Omega u\psi = \int_\Omega h(x)(-\Delta)^{-\frac{1}{2}}\psi \qquad \forall\psi \in C_c^\infty(\Omega).$$

Here $(-\Delta)^{-\frac{1}{2}}\psi$ is given by the function ϕ where

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\phi = \psi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

The following is a weakened special case of a lemma taken from [7].

Lemma 1. *Suppose that $h \in L^1(\Omega)$. Then there exists a unique weak solution u of (1.1). Moreover if $0 \le h$ a.e. then $u \ge 0$ in Ω.*

Definition 2. Let f be a nonlinearity satisfying (R).

- We say that $u(x) \in L^1(\Omega)$ is a weak solution of $(P)_\lambda$ provided $g(x)f(u) \in L^1(\Omega)$, and

$$\int_\Omega u\psi = \lambda\int_\Omega g(x)f(u)(-\Delta)^{-\frac{1}{2}}\psi \qquad \forall\psi \in C_c^\infty(\Omega).$$

- We say $u$is a regular energy solution of $(P)_\lambda$ provided that u is bounded, the harmonic extension u_e of u, is an element of $H^1_{0,L}(\mathcal{C})$ and satisfies

$$\int_{\mathcal{C}} \nabla u_e \cdot \nabla \phi = \lambda \int_{\Omega} g(x) f(u)\phi, \tag{1.1}$$

  for all $\phi \in H^1_{0,L}(\mathcal{C})$.
- We say $\overline{u}$ is a regular energy supersolution of $(P)_\lambda$ provided that $0 \le \bar{u}$ is bounded, the harmonic extension of $\bar{u}$ is an element of $H^1_{0,L}(\mathcal{C})$ and satisfies

$$\int_{\mathcal{C}} \nabla \overline{u}_e \cdot \nabla \phi \ge \lambda \int_{\Omega} g(x) f(\overline{u})\phi, \tag{1.2}$$

  for all $0 \le \phi \in H^1_{0,L}(\mathcal{C})$.

In the case where f satisfies (S) a few minor changes are needed in the definition of solutions. For a weak solution u one requires that $u \le 1$ a.e. in Ω. For u to be a regular energy solution one requires that $\sup_\Omega u < 1$.

We will need the following monotone iteration result, see [7]. Suppose that $\underline{u}$ and $\overline{u}$ are regular energy sub and supersolutions of $(P)_\lambda$. Then there exists a regular energy solution u of $(P)_\lambda$ and $\underline{u} \le u \le \overline{u}$ in Ω. By a regular energy subsolution, we are using the natural analog of regular energy supersolution.

We define the extremal parameter

$$\lambda^* := \sup \{0 \le \lambda : (P)_\lambda \text{ has a regular energy solution}\},$$

and we now show some basic properties.

Lemma 2. (1) *Then* $0 < \lambda^*$.

(2) *Then* $\lambda^* < \infty$.

(3) *For* $0 < \lambda < \lambda^*$ *there exists a regular energy solution* u_λ *of* $(P)_\lambda$ *which is minimal and semi-stable.*

(4) *For each* $x \in \Omega$ *the map* $\lambda \mapsto u_\lambda(x)$ *is increasing on* $(0, \lambda^*)$ *and hence the pointwise limit* $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ *is well defined. Then* u^* *is a weak solution of* $(P)_{\lambda^*}$ *and satisfies* $\int_\Omega g(x) f'(u^*) f(u^*) dx < \infty$.

In this paper, we do not need the notion of a semi-stable solution other than for the proof of (4). For the definition of a semi-stable solution one can either use a nonlocal notion, see [7] or instead work on the cylinder which is what we choose to do. We say that a regular energy solution u of $(P)_\lambda$ is semi-stable provided that

$$\int_{\mathcal{C}} |\nabla \phi|^2 \ge \lambda \int_{\Omega} g(x) f(u) \phi^2 \qquad \forall \phi \in H^1_{0,L}(\mathcal{C}). \tag{1.3}$$

We now prove the lemma.

Proof: (1) Let $\overline{u}$ denote a solution of $(-\Delta)^{\frac{1}{2}} \overline{u} = t g(x)$ with $\overline{u} = 0$ on $\partial\Omega$ where $t > 0$ is small enough such that $\sup_\Omega \overline{u} < 1$. One sees that $\overline{u}$ is a regular energy supersolution of $(P)_\lambda$ provided $t \ge \lambda \sup_\Omega f(\overline{u})$ which clearly holds for small positive λ. Zero is clearly a regular energy subsolution and so we can apply the monotone iteration procedure to obtain a regular energy solution and hence $\lambda^* > 0$.

(2) Suppose that either f satisfies (R) and $C_f < \infty$ or f satisfies (S) and so trivially $C_f < \infty$.

Let u denote a regular energy solution of $(P)_\lambda$ and let u_e denote the harmonic extension. Let ϕ denote the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ and let ϕ_e be its harmonic extension; so $\phi_e(x,y) = \phi(x)e^{-\sqrt{\lambda_1}y}$. Multiply $0 = -\Delta u_e$ by $\frac{\phi_e}{f(u_e)}$ and integrate this over the cylinder $\mathcal{C}$ to obtain

$$\int_\Omega \lambda g(x)\phi = \int_{\mathcal{C}} \frac{\nabla u_e \cdot \nabla \phi_e}{f(u_e)} - \int_{\mathcal{C}} \frac{|\nabla u_e|^2 \phi_e f'(u_e)}{f(u_e)^2},$$

and note that the second integral on the right is nonpositive and hence we can rewrite this as

$$\int_\Omega \lambda g(x)\phi \le \int_{\mathcal{C}} \nabla \phi_e \cdot \nabla h(u_e),$$

where $h(t) = \int_0^t \frac{1}{f(\tau)}d\tau$. Integrating the right-hand side by parts, we have that it is equal to $\int_\Omega (-\Delta)^{\frac{1}{2}}\phi h(u)$ which is equal to $\sqrt{\lambda_1}\int_\Omega \phi h(u)$. So $h(u) \le C_f$ and hence we have

$$\lambda \int_\Omega g(x)\phi \le \sqrt{\lambda_1} C_f \int_\Omega \phi.$$

This shows that $\lambda^* < \infty$. The case where f satisfies (R) and where $C_f = \infty$ needs a separate proof, see the proof of (4). Note that there are examples of f which satisfy (R) and for which $C_f = \infty$, for example $f(t) := (t+1)\log(t+1) + 1$.

(3) The proof in the case where $g(x) = 1$ also works here, see [7].

(4) Again the proof used in the case where $g(x) = 1$ works to show the monotonicity of u_λ, see [7], and hence u^* is well defined. One should note that our notion of a weak solution is more restrictive than what is typically used, i.e., we require $g(x)f(u) \in L^1(\Omega)$ where typically one would only require that $\delta(x)g(x)f(u) \in L^1(\Omega)$ where $\delta(x)$ is the distance from x to $\partial\Omega$. Hence here our proof will differ from [7].

Claim: There exist some $C < \infty$ such that

$$\int_\Omega g(x)f'(u_\lambda)f(u_\lambda) \le C, \tag{1.4}$$

for all $0 < \lambda < \lambda^*$ (at this point we are allowing for the possibility of $\lambda^* = \infty$). We first show that the claim implies that $\lambda^* < \infty$. Note that if $(-\Delta)^{\frac{1}{2}}\phi = g(x)$ with $\phi = 0$ on $\partial\Omega$ then an application of the maximum principle along with the fact that $f(u_\lambda) \ge 1$ gives $u_\lambda \ge \lambda\phi$ in Ω. This along with (1.4) rules out the possibility of $\lambda^* = \infty$. Using a proof similar to the one in [7] one sees that u^* is a weak solution to $(P)_{\lambda^*}$ except for the extra integrability condition $g(x)f(u^*) \in L^1(\Omega)$ that we require. But sending $\lambda \nearrow \lambda^*$ in (1.4) gives us the desired regularity and we are done.

We now prove the claim. Let $u = u_\lambda$ denote the minimal solution of $(P)_\lambda$ and let u_e denote its harmonic extension. Take $\psi := f(u_e) - 1$ in (1.3) (ψ can be shown to be an admissible test function) and write the right-hand side as

$$\int_{\mathcal{C}} \nabla(f(u_e) - 1)f'(u_e) \cdot \nabla u_e,$$

and integrate this by parts. Using $(P)_\lambda$ and after some cancellation one arrives at

$$\int_{\mathcal{C}} (f(u_e) - 1)f''(u_e)|\nabla u_e|^2 \le \lambda \int_\Omega g(x)f'(u)f(u). \tag{1.5}$$

Define $H(t) := \int_0^t f''(\tau)(f(\tau)-1)d\tau$ and so the left-hand side of (1.5) can be written as $\int_{\mathcal{C}} \nabla H(u_e)\cdot \nabla u_e$ and integrating this by parts gives

$$\lambda \int_\Omega g(x) f(u) H(u).$$

Combining this with (1.5) gives

$$\int_\Omega g(x)f(u)H(u) \le \int_\Omega g(x) f(u) f'(u). \tag{1.6}$$

To complete the proof, we show that $H(u)$ dominates $f'(u)$ for big u (resp. u near 1) when f satisfies (R) (resp. (S)). If $0 < T < t$ then one easily sees that

$$H(t) \ge (f(T)-1)(f'(t) - f'(T)).$$

Using this along with (1.6) and dividing the domain of Ω into regions $\{u \ge T\}$ and $\{u < T\}$ one obtains the claim.

2. Uniqueness of the extremal solution

Theorem 1. *Suppose that either f satisfies (R) and is log convex or satisfies (S) and is strictly convex. Then the followings hold.*

1. *There are no weak solutions for $(P)_\lambda$ for any $\lambda > \lambda^*$.*
2. *The extremal solution u^* is the unique weak solution of $(P)_{\lambda^*}$.*

The following are some properties that the nonlinearity f satisfies.

Proposition 1. (1) *Let f be a log convex nonlinearity which satisfies (R).*

(i) *For all $0 < \lambda < 1$ and $\delta > 0$ there exists $k > 0$ such that*

$$f(\lambda^{-1}t) + k \ge (1+\delta) f(t) \quad \textit{for all} \quad 0 \le t < \infty.$$

(ii) *Given $\varepsilon > 0$ there exists $0 < \mu < 1$ such that*

$$\mu^2 (f(\mu^{-1}t) + \varepsilon) \ge f(t) + \frac{\varepsilon}{2} \quad \textit{for all} \quad 0 \le t < \infty.$$

(iii) *Then f is strictly convex.*

(2) *Let f be a nonlinearity which satisfies (S).*

(i) *Given $\varepsilon > 0$ there exists $0 < \mu < 1$ such that*

$$\mu (f(\mu^{-1}t) + \varepsilon) \ge f(t) + \frac{\varepsilon}{2} \quad \textit{for all} \quad 0 \le t \le \mu.$$

(ii) *Then $\lim_{t\nearrow 1} \frac{f(t)}{F(t)} = \infty$ where $F(t) := \int_0^t f(\tau)d\tau$.*

Proof. See [1,17] for the proof of (1)-(i) and (1)-(ii). Part (1)-(iii) is trivial.
(2)-(i) Set $h(t) := \mu\{f(\mu^{-1}t) + \varepsilon\} - f(t) - \frac{\varepsilon}{2}$ and note that $h'(t) \ge 0$ for all $0 \le t \le \mu$ and that $h(0) > 0$ for μsufficiently close to 1, which gives us the desired result.
(2)-(ii) Let $0 < t < 1$ and we use a Riemann sum with right-hand endpoints to approximate $F(t)$. So for any positive integer n we have

$$F(t) \le \frac{t}{n} \sum_{k=1}^n f\left(\frac{kt}{n}\right) \le \frac{t(n-1)}{n} f\left(\frac{(n-1)t}{n}\right) + \frac{t}{n} f(t),$$

and so

$$\limsup_{t \nearrow 1} \frac{F(t)}{f(t)} \leq \frac{1}{n},$$

but since n is arbitrary we have the desired result. □

The following is an essential step in proving Theorem 1. We give the proof of this lemma later.

Lemma 3. *Suppose that f is log convex and satisfies (R) or f satisfies (S). Suppose $\varepsilon > 0$ and that $0 \leq \tau$ is a weak solution of*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} \tau = l(x) & \text{in } \Omega \\ \tau = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g(x)(f(\tau) + \varepsilon) \leq l(x) \in L^1(\Omega)$. Then there exists a regular energy solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = g(x)\left(f(u) + \frac{\varepsilon}{2}\right) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof of Theorem 1: Without loss of generality assume that $\lambda^* = 1$ and let u^* denote the extremal solution of $(P)_{\lambda^*}$. Suppose that v is also a weak solution of $(P)_{\lambda^*}$ and v is not equal to u^*. Set $\Omega_0 := \{x \in \Omega : u^*(x) \neq v(x),\ u^*(x), v(x) \in \mathbb{R}\}$ (resp. $\Omega_0 = \{x \in \Omega : u^*(x) \neq v(x),\ u^*(x), v(x) < 1\}$) when f satisfies (R) (resp. (S)) and note that $|\Omega_0| > 0$. Define

$$h(x) := \begin{cases} \dfrac{f(u^*(x)) + f(v(x))}{2} - f\left(\dfrac{u^*(x) + v(x)}{2}\right) & x \in \Omega_0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that by the strict convexity of f, which we obtain either by hypothesis or by Proposition 1, we have $0 \leq h$ in Ω and $h > 0$ in Ω_0. Also note that $h \in L^1(\Omega)$. Define $z := \frac{u^*+v}{2}$. Since u^* and v are weak solutions of $(P)_{\lambda^*}$, z is a weak solution of

$$(-\Delta)^{\frac{1}{2}} z = g(x) f(z) + g(x) h(x) \quad \text{in } \Omega,$$

with $z = 0$ on $\partial\Omega$. From now on we omit the boundary values since they will always be zero unless otherwise mentioned. Let χ and ϕ denote weak solutions of $(-\Delta)^{\frac{1}{2}} \chi = g(x)h(x)$ and $(-\Delta)^{\frac{1}{2}} \phi = g(x)$in Ω. By taking $\varepsilon > 0$ small enough one has that $\chi \geq \varepsilon\phi$ in Ω. Set $\tau := z + \varepsilon\phi - \chi$ and note that τ is a weak solution of

$$(-\Delta)^{\frac{1}{2}} \tau = g(x)(f(z) + \varepsilon) \geq 0 \quad \text{in } \Omega,$$

and by Lemma 1, we have that $0 \leq \tau$. Moreover, from the fact that $\tau \leq z$ in Ω we have

$$g(x)(f(\tau) + \varepsilon) \leq (-\Delta)^{\frac{1}{2}} \tau \in L^1(\Omega).$$

Applying Lemma 3, there exists a regular energy solution u of

$$(-\Delta)^{\frac{1}{2}} u = g(x)\left(f(u) + \frac{\varepsilon}{2}\right) \quad \text{in } \Omega.$$

Set $w := u + \alpha u - \frac{\varepsilon}{2}\phi$ where $\alpha > 0$ is chosen small enough such that $\alpha u \leq \frac{\varepsilon}{2}\phi$ in Ω. A straightforward computation shows that w is a regular energy solution of

$$(-\Delta)^{\frac{1}{2}} w = (1 + \alpha) g(x) f(u) + \frac{\varepsilon}{2} \alpha g(x) \quad \text{in } \Omega,$$

and that $w \le u$ in Ω. By Lemma 1, we also have $0 \le w$ in Ω. From this, we see that w is a regular energy supersolution of

$$(-\Delta)^{\frac{1}{2}} w \ge (1+\alpha) g(x) f(w) \quad \text{in } \Omega,$$

with zero boundary conditions. We now apply the monotone iteration argument to obtain a regular energy solution $\tilde{u}$ of $(-\Delta)^{\frac{1}{2}} \tilde{u} = (1+\alpha) g(x) f(\tilde{u})$ in Ω which contradicts the fact that $\lambda^* = 1$. So, we have shown that $|\Omega_0| = 0$ and so $u^* = v$ a.e. in Ω.

Proof of Lemma 3: Let $\varepsilon > 0$ and suppose that $0 \le \tau \in L^1(\Omega)$ is a weak solution of $(-\Delta)^{\frac{1}{2}} \tau = l(x)$in Ω where $0 \le g(x)(f(\tau)+\varepsilon) \le l(x)$in Ω. As in the proof of Theorem 1, we omit the boundary values since they will always be Dirichlet boundary conditions and we also assume that $\lambda^* = 1$. First, assume that f is a log convex nonlinearity, which satisfies (R). Let $u_0 := \tau$ and let u_1, u_2, u_3 be weak solutions of

$$\begin{aligned}
(-\Delta)^{\frac{1}{2}} u_1 &= \mu g(x)(f(u_0)+\varepsilon) \quad \text{in } \Omega,\\
(-\Delta)^{\frac{1}{2}} u_2 &= \mu g(x)(f(u_1)+\varepsilon) \quad \text{in } \Omega,\\
(-\Delta)^{\frac{1}{2}} u_3 &= \mu g(x)(f(u_2)+\varepsilon) \quad \text{in } \Omega,
\end{aligned}$$

where $0 < \mu < 1$ is the constant given in Proposition 1 such that $\mu^2(f(\frac{t}{\mu})+\varepsilon) \ge f(t) + \frac{\varepsilon}{2}$ for all $t \ge 0$. One easily sees that $u_2 \le u_1 \le \mu u_0$. Now note that

$$\begin{aligned}
(-\Delta)^{\frac{1}{2}} u_1 &= \mu g(x)(f(u_0)+\varepsilon)\\
&\ge \mu g(x)\left(f\left(\frac{u_1}{\mu}\right)+\varepsilon\right).
\end{aligned} \tag{2.1}$$

By Proposition 1 with $\delta := 2N-1 > 0$ and $0 < \lambda = \mu < 1$ there exist some $k > 0$ such that

$$f\left(\frac{u_1}{\mu}\right) \ge 2Nf(u_1) - k,$$

hence one can rewrite (2.1) as

$$(-\Delta)^{\frac{1}{2}} u_1 \ge \mu g(x)(2Nf(u_1) - k + \varepsilon).$$

We let ϕ be as in the proof of Theorem 1 and examine $u_1 + t\phi$ where $t > 0$ is to be picked later. Note that

$$\begin{aligned}
(-\Delta)^{\frac{1}{2}} (u_1 + t\phi) &= (-\Delta)^{\frac{1}{2}} u_1 + t g(x)\\
&\ge 2N\mu g(x)(f(u_1)+\varepsilon) + m g(x),
\end{aligned}$$

where $m := t - \mu k + \varepsilon\mu(1-2N)$ and we now pick $t > 0$ big enough such that $m = 0$. Therefore, from the definition of u_2 we have that

$$(-\Delta)^{\frac{1}{2}} (u_1 + t\phi) \ge 2N\ (-\Delta)^{\frac{1}{2}} u_2 \qquad \text{in } \Omega.$$

So, from the maximum principle we obtain

$$u_2 \le \frac{1}{2N}(u_1 + t\phi) \quad \text{in } \Omega.$$

Since f is log convex, there is some smooth, convex increasing function β with $\beta(0) = 0$ and $f(t) = e^{\beta(t)}$. By the convexity of β and since $\beta(0) = 0$, we have

$$\beta(u_2) \le \frac{1}{2N}\beta(u_1 + t\phi) \le \frac{1}{2N}\beta(\mu u_0 + t\phi),$$

but

$$\beta(\mu u_0 + t\phi) = \beta\left(\mu u_0 + (1-\mu)\frac{t\phi}{1-\mu}\right) \leq \mu\beta(u_0) + (1-\mu)\beta\left(\frac{t\phi}{1-\mu}\right).$$

From this, we can conclude

$$f(u_2)^{2N} \leq e^{\mu\beta(u_0)} e^{(1-\mu)\beta(\frac{t\phi}{1-\mu})} \leq f(u_0) f\left(\frac{t\phi}{1-\mu}\right)^{1-\mu}.$$

So, we see that $g(x)f(u_2)^{2N} \leq Cg(x)f(u_0) \in L^1(\Omega)$ for some large constant C.

Since $g(x)$ is bounded, we conclude that $g(x)f(u_2) \in L^{2N}(\Omega)$. But u_3 satisfies $(-\Delta)^{\frac{1}{2}} u_3 = \mu g(x)(f(u_2)+\varepsilon)$in Ω and so by elliptic regularity we have that u_3 is bounded (since the right-hand side is an element of $L^p(\Omega)$ for some $p > N$) and now we use the fact that $0 \leq u_3 \leq u_2$ and the monotone iteration argument to obtain a regular energy solution w to $(-\Delta)^{\frac{1}{2}} w = \mu g(x)(f(w)+\varepsilon)$ in Ω.

Now, set $\xi := \mu w$ and note that ξ is a regular energy solution of

$$(-\Delta)^{\frac{1}{2}} \xi = \mu^2 g(x)\left(f\left(\frac{\xi}{\mu}\right) + \varepsilon\right) \quad \text{in } \Omega,$$

and from Proposition 1, we have

$$(-\Delta)^{\frac{1}{2}} \xi \geq g(x)\left(f(\xi) + \frac{\varepsilon}{2}\right) \quad \text{in } \Omega,$$

and so by an iteration argument, we have the desired result.

Now, assume that f satisfies (S). In this case, the proof is much simpler. Define $w := \mu\tau$ where $0 < \mu < 1$ is from Proposition 1. Then note that $0 \leq w \leq \mu$ a.e. and

$$\begin{aligned}(-\Delta)^{\frac{1}{2}} w = \mu l(x) &\geq \mu g(x)\left(f\left(\frac{w}{\mu}\right) + \varepsilon\right)\\ &\geq g(x)\left(f(w) + \frac{\varepsilon}{2}\right).\end{aligned}$$

Hence, w is a regular energy supersolution of

$$(-\Delta)^{\frac{1}{2}} w \geq g(x)\left(f(w) + \frac{\varepsilon}{2}\right),$$

and we have the desired result after an application of the monotone iteration argument.

3. Uniqueness of solutions for small λ

In this section, we prove uniqueness theorems for equation $(P)_\lambda$ for small enough λ. Throughout this section, we assume that $g = 0$ on $\partial\Omega$. We need the following regularity result.

Proposition 2. [6] *Let $\alpha \in (0,1)$, Ω be a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^N$ and suppose that u is a weak solution of $(-\Delta)^{\frac{1}{2}} u = h(x)$ in Ω with $u = 0$on $\partial\Omega$.*

1. *Suppose that $h \in L^\infty(\Omega)$. Then $u_e \in C^{0,\alpha}(\overline{\mathcal{C}})$ hence $u \in C^{0,\alpha}(\overline{\Omega})$.*
2. *Suppose that $h \in C^{k,\alpha}(\overline{\Omega})$ where $k = 0$ or $k = 1$ and $h = 0$on $\partial\Omega$. Then $u_e \in C^{k+1,\alpha}(\overline{\mathcal{C}})$ hence $u \in C^{k+1,\alpha}(\overline{\Omega})$.*

Using this one easily obtains the following:

Corollary 1. *For each* $0 < \lambda < \lambda^*$ *the minimal solution of* $(P)_\lambda$, u_λ, *belongs to* $C^{2,\alpha}(\overline{\Omega})$. *In addition* $u_\lambda \to 0$ *in* $C^1(\overline{\Omega})$ *as* $\lambda \to 0$.

We now come to our main theorem of this section.

Theorem 2. *Suppose that* Ω *is a star-shaped domain with respect to the origin and set* $\gamma := \sup_\Omega \frac{x\cdot\nabla g(x)}{g(x)}$.

(1) *Suppose that* f *satisfies (R) and that*

$$\limsup_{t\to\infty} \frac{F(t)}{f(t)t} < \frac{N-1}{2(N+\gamma)}. \tag{3.1}$$

Then for sufficiently small λ, u_λ *is the unique regular energy solution of* $(P)_\lambda$.

(2) *Suppose that* f *satisfies (S). Then for sufficiently small* λ, u_λ *is the unique regular energy solution* $(P)_\lambda$.

Proof: Let f satisfy (R) and (3.1) or let f satisfy (S) and suppose that u is a second regular energy solution of $(P)_\lambda$ which is different from the minimal solution u_λ. Set $v := u - u_\lambda$ and note that $v \geq 0$ by the minimality of u_λ and $v \neq 0$ since u is different from the minimal solution.

A computation shows that v satisfies the equation

$$(-\Delta)^{\frac{1}{2}} v = \lambda g(x)\{f(u_\lambda + v) - f(u_\lambda)\}. \tag{3.2}$$

Applying Proposition 2 to u and u_λ separately shows that $v_e \in C^{2,\alpha}(\overline{\mathcal{C}})$.

A computation shows the following identity holds:

$$\operatorname{div}\left\{(z,\nabla v_e)\nabla v_e - z\frac{|\nabla v_e|^2}{2}\right\} + \frac{N-1}{2}|\nabla v_e|^2 = (z,\nabla v_e)\Delta v_e,$$

where $z = (x, y)$. Integrating this identity over $\Omega \times (0, R)$ we end up with

$$\frac{1}{2}\int_{\partial\Omega\times(0,R)} |\nabla v_e|^2\ x\cdot\nu + \int_\Omega x\cdot\nabla_x v_e\ \partial_\nu v_e + \frac{N-1}{2}\int_{\Omega\times(0,R)} |\nabla v_e|^2 + \varepsilon(R) = 0, \tag{3.3}$$

where

$$\varepsilon(R) := \int_{\Omega\times\{y=R\}} (x\cdot\nabla_x v_e + R\ \partial_y v_e)\,\partial_y v_e - \frac{R}{2}|\nabla v_e|^2.$$

One can show that $\varepsilon(R) \to 0$ as $R \to \infty$, for details on this and the above calculations see [24]. Sending $R \to \infty$ and since Ω is star-shaped with respect to the origin, we have

$$\frac{N-1}{2}\int_{\mathcal{C}} |\nabla v_e|^2 \leq -\int_\Omega x\cdot\nabla_x v\ \partial_\nu v_e,$$

and after using (3.2) one obtains

$$\frac{N-1}{2}\int_{\mathcal{C}} |\nabla v_e|^2 \leq -\lambda\int_\Omega x\cdot\nabla_x v\ g(x)\{f(u_\lambda + v) - f(u_\lambda)\}. \tag{3.4}$$

We now compute the right-hand side of (3.4). Set $h(x,\tau) := f(u_\lambda(x)+\tau) - f(u_\lambda(x))$ and let $H(x,t) = \int_0^t h(x,\tau)d\tau$. For this portion of the proof, we are working on Ω and hence all gradients are with respect to the x-variable. To clarify our notation note that the chain rule can be written as

$$\nabla H(x,v) = \nabla_x H(x,v) + h(x,v)\nabla v,$$

where we recall $v = v(x)$. Some computations now show that

$$H(x,t) = F(u_\lambda + t) - F(u_\lambda) - f(u_\lambda)t,$$

and

$$\nabla_x H(x,t) = \{f(u_\lambda + t) - f(u_\lambda) - f'(u_\lambda)t\}\nabla u_\lambda,$$

and so the right-hand side of (3.4) can be written as

$$\begin{aligned} -\lambda \int_\Omega g(x)\{f(u_\lambda + v) - f(u_\lambda)\}x \cdot \nabla v &= -\lambda \int_\Omega g(x)h(x,v)x \cdot \nabla v \\ &= -\lambda \int_\Omega g(x)x \cdot \{\nabla H(x,v) - \nabla_x H(x,v)\} \\ &= \lambda \int_\Omega g(x)x \cdot \nabla_x H(x,v) + \lambda N \int H(x,v)g(x) \\ &\quad + \lambda \int_\Omega H(x,v)x \cdot \nabla g(x). \end{aligned}$$

Therefore, (3.4) can be written as

$$\begin{aligned} \frac{N-1}{2} \int_{\mathcal{C}} |\nabla v_e|^2 \leq \lambda \int_\Omega & x \cdot \nabla u_\lambda g(x)\{f(u_\lambda + v) - f(u_\lambda) - f'(u_\lambda)v\} \\ & + N\lambda \int_\Omega g(x)\{F(u_\lambda + v) - F(u_\lambda) - f(u_\lambda)v\} \\ & + \lambda \int_\Omega x \cdot \nabla g(x)\{F(u_\lambda + v) - F(u_\lambda) - f(u_\lambda)v\}. \end{aligned} \tag{3.5}$$

We now assume that we are in case (1). Let α be such that

$$\limsup_{\tau \to \infty} \frac{F(\tau)}{\tau f(\tau)} < \alpha < \frac{N-1}{2(N+\gamma)},$$

so there exist some $\tau_0 > 0$ such that $F(\tau) < \alpha \tau f(\tau)$for all $\tau \geq \tau_0$. Let $0 < \theta < 1$ be such that $\frac{\theta(N-1)}{2} - \alpha(N+\gamma) > 0$ and we now decompose the left-hand side of (3.5) into the convex combination

$$\frac{\theta(N-1)}{2} \int_{\mathcal{C}} |\nabla v_e|^2 + \frac{(N-1)(1-\theta)}{2} \int_{\mathcal{C}} |\nabla v_e|^2. \tag{3.6}$$

Using the following trace theorem: there exist some $\tilde{C} > 0$ such that

$$\int_{\mathcal{C}} |\nabla w|^2 \geq \tilde{C} \int_\Omega w^2, \quad \forall w \in H^1_{0,L}(\mathcal{C}), \tag{3.7}$$

one sees that (3.6) is bounded below by

$$\frac{\theta(N-1)}{2} \int_{\mathcal{C}} |\nabla v_e|^2 + C \int_\Omega v^2.$$

By taking $C > 0$ smaller if necessary one can bound this from below by

$$\frac{\theta(N-1)}{2} \int_{\mathcal{C}} |\nabla v_e|^2 + C \int_\Omega g(x) v^2,$$

and after using (3.2), this last quantity is equal to

$$\frac{\lambda\theta(N-1)}{2} \int_\Omega g(x)\{f(u_\lambda + v) - f(u_\lambda)\}v + C \int_\Omega g(x) v^2. \tag{3.8}$$

Substituting (3.8) into (3.4) and rearranging one arrives at an inequality of the form

$$\int_\Omega g(x) T_\lambda(x, v) \le 0,$$

where

$$\begin{aligned} T_\lambda(x,\tau) = {} & \frac{\theta(N-1)}{2}\{f(u_\lambda+\tau) - f(u_\lambda)\}\tau + \frac{C}{\lambda}\tau^2 \\ & - N\{F(u_\lambda+\tau) - F(u_\lambda) - f(u_\lambda)\tau\} \\ & - \frac{x\cdot\nabla g}{g}\{F(u_\lambda+\tau) - F(u_\lambda) - f(u_\lambda)\tau\} \\ & - x\cdot\nabla u_\lambda\{f(u_\lambda+\tau) - f(u_\lambda) - f'(u_\lambda)\tau\}. \end{aligned}$$

To obtain a contradiction, we show that for sufficiently small $\lambda > 0$ that $T_\lambda(x,\tau) > 0$ on $(x,\tau) \in \Omega \times (0,\infty)$ and hence we must have that $v = 0$. Define

$$\begin{aligned} S_\lambda(x,\tau) = {} & \frac{\theta(N-1)}{2}\{f(u_\lambda+\tau) - f(u_\lambda)\}\tau + \frac{C}{\lambda}\tau^2 \\ & - (N+\gamma)\{F(u_\lambda+\tau) - F(u_\lambda) - f(u_\lambda)\tau\} \\ & - \varepsilon_\lambda\{f(u_\lambda+\tau) - f(u_\lambda) - f'(u_\lambda)\tau\}. \end{aligned}$$

where $\varepsilon_\lambda := \|\nabla u_\lambda \cdot x\|_{L^\infty}$. Note that since f is increasing and convex that $T_\lambda(x,\tau) \ge S_\lambda(x,\tau)$ for all $\tau \ge 0$. We now show the desired positivity for S_λ and to do this we examine large and small τ separately.

Large τ: Take $\tau \ge \tau_0$ and $0 < \lambda \le \frac{\lambda^*}{2}$. Since f is convex and increasing

$$\begin{aligned} S_\lambda(x,\tau) \ge {} & \frac{\theta(N-1)}{2} f(u_\lambda+\tau)\tau - (N+\gamma)F(u_\lambda+\tau) \\ & - \varepsilon_\lambda f(u_\lambda+\tau) + \frac{C}{\lambda}\tau^2 \\ & - \frac{\theta(N-1)}{2} f(u_\lambda)\tau, \end{aligned} \tag{3.9}$$

but $F(u_\lambda+\tau) < \alpha(u_\lambda+\tau)f(u_\lambda+\tau)$ for all $\tau \ge \tau_0$ and so the right-hand side of (3.9) is bounded below by

$$\begin{aligned} & f(u_\lambda+\tau)\left[\tau\left\{\frac{\theta(N-1)}{2} - (N+\gamma)\alpha\right\} - \varepsilon_\lambda - (N+\gamma)\alpha u_\lambda\right] \\ & - \frac{\theta(N-1)}{2} f(u_\lambda)\tau + \frac{C}{\lambda}\tau^2. \end{aligned}$$

Using the fact that f is superlinear at ∞ there exist some $\tau_1 \geq \tau_0$ such that $S_\lambda(x,\tau) > 0$ for all $\tau \geq \tau_1$ and $0 < \lambda \leq \frac{\lambda^*}{2}$.

Small τ: Let $0 < \lambda_0 < \frac{\lambda^*}{2}$ be such that $\|u_\lambda\|_{L^\infty} \leq 1$. Using the convexity and monotonicity of f and Taylor's Theorem there exist some $C_1 > 0$ such that

$$F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda)\tau \leq C_1\tau^2, \qquad f(u_\lambda + \tau) - f(u_\lambda) - f'(u_\lambda)\tau \leq C_1\tau^2,$$

for all $0 \leq \tau \leq \tau_0$, $0 < \lambda \leq \lambda_0$ and $x \in \Omega$. Noting that the first term of $S_\lambda(x,\tau)$ is positive for $\tau > 0$ one sees that for all $0 < \tau \leq \tau_0$, $x \in \Omega$ and $0 < \lambda < \lambda_0$ one has the lower bound

$$S_\lambda(x,\tau) \geq \frac{C}{\lambda}\tau^2 - (N + \gamma + \varepsilon_\lambda)C_1\tau^2,$$

and hence by taking λsmaller if necessary we have the desired result.

(2) We now assume that f satisfies (S). One uses a similar approach to arrive at an inequality of the form

$$\int_\Omega T_\lambda(x,v) \leq 0,$$

where as before $v = u - u_\lambda \geq 0$ and where we assume that $v \neq 0$. To arrive at a contradiction we show that for sufficiently small λ that $T_\lambda(x,\tau) > 0$ for all $x \in \Omega$ and for all $0 < \tau < 1 - u_\lambda(x)$. Again the idea is to break the interval into two regions. For τ such that $\tau + u_\lambda(x)$ close to 1, we use Proposition 1, 2 (ii) to see the desired positivity. For the remainder of the interval, we again use Taylor's Theorem.

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