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THE MUCKENHOUPT A_∞ CLASS AS A METRIC SPACE AND CONTINUITY OF WEIGHTED ESTIMATES

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Abstract. We show how the A_∞ class of weights can be considered as a metric space. As far as we know this is the first time that a metric d_* is considered on this set. We use this metric to generalize the results obtained in [8]. Namely, we show that for any Calderón–Zygmund operator T and an A_p, $1 < p < \infty$, weight w_0, the numbers $\|T\|_{L^p(w)\to L^p(w)}$ converge to $\|T\|_{L^p(w_0)\to L^p(w_0)}$ as $d_*(w, w_0) \to 0$. We also find the rate of this convergence and prove that it is sharp.

1. Introduction and useful results

The main purpose of this paper is to define a natural metric structure on the classical Muckenhoupt A_p classes, and generalize a continuity result obtained in [8]. As far as we know, this is the first time that such metric has been studied in the context of continuity of norms of Calderón–Zygmund operators. Classically, the A_p spaces have only been treated as sets with no additional structure on them. Weighted inequalities have been studied extensively during the last 15 years and it is interesting that all $L^p(w)$ norms of Calderón–Zygmund operators turn out to be continuous with respect to the weight w, as we shall see in this paper. Moreover, we find the "rate" of this continuity with respect to the weight and prove that it is sharp (see Theorem 1.3). In addition, we can realize the completion of the A_p metric spaces as subspaces of the BMO space. Many properties of these new complete metric spaces are going to be considered in Section 2. In [8], the two authors showed that weighted estimates for classical operators are continuous at the constant weight 1. Thus, Theorem 1.3 is a generalization of these results, since we prove the whole continuity of the operator norm with respect to the weight w. At the time that [8] appeared, the authors did not guess that such a metric on the Muckenhoupt classes behaves in a regular manner, i.e., forcing all weighted estimates to be continuous.

Such continuity results have been coming up recently in connection with partial differential equations (PDE) with random coefficients and continuity of norms of Calderón–Zygmund operators. For example, the continuity at $w = 1$, was used in [2]. The continuity at any weight can also be important in various questions of PDE. This and the fact that the proofs here involve some subtle changes in the approach of [8], made us believe that the main results of the present paper can be of independent interest.

The metric A_p classes will be considered in Section 2, where we study many properties of these new spaces, and the main Theorem 1.3, in Section 3. Before we state and prove the main Theorems in Sections 2 and 3, we need some definitions and

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some already known results about the weighted theory and its relation with the BMO space.

We are going to work with functions $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ that are positive almost everywhere with respect to Lebesgue measure. Functions like these are known as weights. The celebrated A_p classes of weights are defined in the following way:

For $1 < p < \infty$, we say that $w \in A_p$ if for all cubes Q in $\mathbb{R}^n$ we have that ($[w]_{A_p}$ is called the A_p characteristic of the weight):

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} < \infty,$$

where p' is the conjugate exponent to p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

The class of A_1 weights consists of those w such that there is a positive constant c with the property:

$$Mw(x) \le cw(x),$$

for almost every x in $\mathbb{R}^n$, where M is the Hardy–Littlewood maximal function. The smallest such constant is denoted by $[w]_{A_1}$ and is called the A_1 characteristic.

We define the class of A_∞ weights as:

$$[w]_{A_\infty} := \sup_Q \left(\frac{\frac{1}{|Q|}\int_Q w}{\exp(\frac{1}{|Q|}\int_Q \log w)}\right) < \infty.$$

It is really easy to see that any A_p weight is actually an A_∞ weight, and that we have the estimate $[w]_{A_\infty} \le [w]_{A_p}$. It is also true that any A_∞ weight is an A_p weight for some $1 < p < \infty$. This means that we have the equality:

$$A_\infty = \bigcup_{1<p<\infty} A_p.$$

Another nice property is that for $1 \le p \le q \le \infty$ we have $A_1 \subset A_p \subset A_q \subset A_\infty$, where the inclusions here are strict. All of these sets are different for different values of p and q.

The space of BMO functions in $\mathbb{R}^n$, consists of locally integrable functions f such that the norm

$$\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q |f - f_Q|\, dx$$

is finite. The BMO space and the A_∞ space, have many nice properties. First of all, if f is a BMO function then for any number $\lambda \in (0, \frac{c}{\|f\|_*}]$, the function $\mathrm{e}^{\lambda f}$ is an A_p weight, $1 < p < \infty$, where the constant c depends on p and the dimension n. Secondly, for small BMO norm, the A_p norm of the weight $\mathrm{e}^{\lambda f}$ is bounded by the number 2 for example (see e.g., [3]).

A subset of BMO that appears in many applications is BLO. It stands for the functions of bounded lower oscillation. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to belong in BLO if there is a positive constant c such that:

$$\frac{1}{|Q|}\int_Q f - \inf_{x\in Q} f(x) \le c,$$

for all cubes Q, where the infimum is understand as the essential infimum. It can be proved that for any $w \in A_1$, the function $\log w$ is in BLO. Also if a function $f \in \text{BLO}$

then for sufficiently small $\lambda > 0$ the function $e^{\lambda f} \in A_1$. The reference for all these results is [3].

For the proofs of our theorems interpolation with change of measure is going to play an important role and for this reason we need some preliminary results on this subject as well. In the following $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ will denote measure spaces. Suppose T is an operator of a class of functions on X into a class of functions on Y. T is called a sub-linear operator, if it satisfies the following properties:

(i) If $f = f_1 + f_2$ and Tf_1, Tf_2 are defined then Tf is defined,
(ii) $|T(f_1 + f_2)| \leq |Tf_1| + |Tf_2|$, μ almost everywhere,
(iii) For any scalar k, we have $|T(kf)| = |k||Tf|$, μ almost everywhere.
Let $p, q \geq 1$ be two real numbers. We say that T is of type (p, q), if T is defined for all functions f in $L^p(X, \mathcal{M}, \mu)$ and there exists a positive number, K, independent of f, such that

$$\|Tf\|_{q,\nu} \leq K\|f\|_{p,\mu},$$

where

$$\|Tf\|_{q,\nu} = \left(\int_Y |Tf|^q d\nu\right)^{\frac{1}{q}}$$

and

$$\|f\|_{p,\mu} = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}.$$

Let μ_0, μ_1 be two measures for $(X, \mathcal{M})$. If we define the measure $\mu = \mu_0 + \mu_1$, then μ_0, μ_1 are each absolutely continuous with respect to μ. Thus, by the Radon–Nikodym theorem, there exists two functions, α_0, α_1 such that for any $E \in \mathcal{M}$,

$$\mu_j(E) = \int_E \alpha_j \, d\mu,$$

where $j = 0, 1$. In the following we will assume that α_0, α_1 are never zero. This is equivalent to asserting that the sets of measure zero with respect to μ_j, $j = 0, 1$, are the same as the sets of measure zero with respect to μ. Thus, in the various measure spaces that we will consider, the equivalence classes of functions will be the same.

Let $0 \leq s \leq 1$, and define the measure μ_s on X by

$$\mu_s(E) = \int_E \alpha_0^{1-s} \alpha_1^s \, d\mu,$$

for each $E \in \mathcal{M}$. Also assume, that we have two measures ν_0, ν_1 on $\mathcal{N}$, and define the measures ν_r, for $0 \leq r \leq 1$, just as we did for μ_s above.

Given any real numbers $1 \leq p_0, p_1, q_0, q_1$ and any $0 \leq t \leq 1$, we define $p_t, q_t, s(t), r(t)$ as follows:

$$\frac{(1-t)p_t}{p_0} + \frac{tp_t}{p_1} = 1, \quad \frac{(1-t)q_t}{q_0} + \frac{tq_t}{q_1} = 1,$$
$$s(t) = \frac{(tp_t)}{p_1}, \quad r(t) = \frac{(tq_t)}{q_1}.$$

We have the following theorem by [9]:

Theorem 1.1. *Suppose that T is a sub-linear operator satisfying*

$$\|Tf\|_{q_j,\nu_j} \leq K_j \|f\|_{p_j,\mu_j}$$

for all $f \in L^{p_j}(X, \mathcal{M}, \mu_j)$, $j = 0, 1$. Then, for $0 \leq t \leq 1$, we have

$$\|Tf\|_{q_t,\nu_{r(t)}} \leq K_0^{1-t} K_1^t \|f\|_{p_t,\mu_{s(t)}},$$

for all $f \in L^{p_t}(X, \mathcal{M}, \mu_{s(t)})$.

In addition to the previous theorem, we need also the following proved in [7]:

Theorem 1.2. *If the A_∞ norm of a weight w is small, i.e., $[w]_{A_\infty} \leq 1 + \delta < 2$, then the function $f = \log w$, and any cube Q satisfy*

$$\frac{1}{|Q|} \int_Q |f - f_Q| \, dx \leq 32\sqrt{\delta}.$$

Our purpose is to generalize the main Theorem proved in [8]. This generalization is the following:

Theorem 1.3. *Let T be a linear operator such that for some $1 < p < \infty$,*

$$\|T\|_{L^p(w) \to L^p(w)} \leq F([w]_{A_p}),$$

for any A_p weight w in $\mathbb{R}^n$, where F is an increasing, real valued function. Fix an A_p weight w_0. Then:

$$\lim_{d_*(w,w_0) \to 0} \|T\|_{L^p(w) \to L^p(w)} = \|T\|_{L^p(w_0) \to L^p(w_0)},$$

and in addition for any sub-linear operator satisfying the hypothesis of the theorem we have the estimate:

$$\|T\|_{L^p(w) \to L^p(w)} \leq \|T\|_{L^p(w_0) \to L^p(w_0)} (1 + c d_*(w, w_0))$$

for all weights $w \in A_p$ with sufficiently small $d_*(w, w_0)$, where c is a positive constant that depends on p, on the function F, on the dimension n and on $[w_0]_{A_p}$.

In order to do that we will define a metric in the A_∞ space and this is going to generalize the convergence $[w]_{A_p} \to 1$ in the sense that this will be equivalent to the convergence $d_*(w, 1) \to 0$ in the metric d_*.

2. The $(\mathcal{A}_\infty, d_*)$ metric space

Let us observe that if we have any weight w, any positive constant $c > 0$ and any $1 \leq p \leq \infty$, then $[w]_{A_p} = [cw]_{A_p}$. We define an equivalence relation in A_∞ in the following way: for $u, v \in A_\infty$ we will write $u \sim v$ if and only if there is a positive constant c such that $u = cv$ almost everywhere in $\mathbb{R}^n$. This allows us to define the quotient space:

$$\mathcal{A}_\infty = A_\infty \big/ \sim .$$

In the same way, we define for $1 \leq p < \infty$:

$$\mathcal{A}_p = A_p \big/ \sim .$$

For two elements $u, v \in \mathcal{A}_\infty$, we define the distance function d_* as:

$$d_*(u,v) = \|\log u - \log v\|_*.$$

Again it is obvious that all the requirements of a metric are satisfied and the reason for defining the equivalence relation is exactly because we need to have:

$$d_*(u,v) = 0 \Leftrightarrow u \sim v.$$

So we define a metric in $\mathcal{A}_\infty$, going through the BMO space. We can check that for an A_p weight w, $[w]_{A_p} \to 1$ is equivalent to $d_*(w,1) \to 0$ and since $\mathcal{A}_p \subset \mathcal{A}_\infty$, the restriction of the d_* metric to $\mathcal{A}_p$, makes the class a metric space. The drawback of these "new" metric spaces is that none of them is complete.

However, the following is an obvious remark that gives more informations about this "new" spaces. It states that small balls around the constant weight 1, are complete in the d_* metric.

Theorem 2.1. *Consider a closed ball $\bar{B}(1,r)$ of sufficiently small radius $r > 0$ and center at the weight 1, in the metric space $(\mathcal{A}_\infty, d_*)$, i.e., $\bar{B}(1,r) = \{w \in \mathcal{A}_\infty : d_*(w,1) \le r\}$. Then $\bar{B}(1,r)$ is a complete metric space with respect to the metric d_*.*

Proof. Consider a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ in $(\bar{B}(1,r), d_*)$. This means that the sequence $\{\log w_n\}_{n\in\mathbb{N}}$ is Cauchy in the BMO space. But BMO is a Banach space and so there is a function $f \in$ BMO such that $\log w_n \to f$ in BMO as $n \to \infty$. By the John–Nirenberg inequality we know that there is a dimensional constant $c > 0$ such that for all $\lambda \in (0, \frac{c}{\|f\|_*}]$ the function $\mathrm{e}^{\lambda f} \in A_2$. But $|\|\log w_n\|_* - \|f\|_*| \le \|\log w_n - f\|_* \to 0$ as $n \to \infty$. Here, we use the fact that $w_n \in \bar{B}(1,r)$. This means that $\|\log w_n\|_* = \|\log w_n - \log 1\|_* \le r$ and r is sufficiently small. Therefore, the number $\|f\|_*$ is small and so the number $\frac{c}{\|f\|_*}$ is really big. We are now allowed to choose for $\lambda = 1$ and we get that $\mathrm{e}^f \in A_2$ or equivalently there is a weight $w \in A_2 \subset A_\infty$ with $f = \log w$. It is trivial now to see that $d_*(w_n, w) \to 0$ as $n \to \infty$. □

Of course in the previous Theorem, we can replace the $\mathcal{A}_\infty$ space by any of the other $\mathcal{A}_p$ spaces. We already mentioned that none of the $\mathcal{A}_p$ spaces is complete. The proof of this fact is very simple. Let us prove that $\mathcal{A}_1$ is not complete by finding a Cauchy sequence in the space that has no limit inside $\mathcal{A}_1$. It will follow that this example works for anyone of the $\mathcal{A}_p$ spaces. Consider a decreasing sequence $-1 < r_n < 0$ with $\lim_{n\to\infty} r_n = -1$. Define the A_1 weights $w_n = |x|^{r_n}$. Then:

$$d_*(w_{r_n}, w_{r_m}) = \|r_n \log|x| - r_m \log|x|\|_* = |r_n - r_m| \|\log|x|\|_*$$

and since $r_n \to 1$ we see that $\{w_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{A}_1$, or equivalently the sequence $\{\log w_n\}_{n\in\mathbb{N}}$ is Cauchy in BMO. Its limit in the BMO space is obviously the function $f(x) = -\log|x|$. This means that for $w(x) = \frac{1}{|x|}$ we have $d_*(w_n, w) \to 0$ as $n \to \infty$, but since w is not in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ it cannot be an A_1 weight. So the space $(\mathcal{A}_1, d_*)$ is not complete.

Let us also mention the following result in [4], by Garnett and Jones, that helps to understand better when a ball in $(\mathcal{A}_p, d_*)$ is complete. It states that for a function $f \in \mathrm{BMO}$,

$$\mathrm{dist}_{\mathrm{BMO}}(f, L^\infty) := \inf\{\|f-g\|_* : g \in L^\infty\} \sim \frac{1}{\sup\{\lambda > 0 : \mathrm{e}^{\lambda f} \in A_2\}}.$$

This means that if we have a Cauchy sequence in $\mathcal{A}_p$, the closer the sequence is to the L^∞ space, the more chances it has to have a limit in $\mathcal{A}_p$.

So now we can try and find the completion of these spaces under the metric d_*. By definition the completion of $(\mathcal{A}_p, d_*)$ is the space $\bar{\mathcal{A}}_p$ that consists of the equivalence classes of all Cauchy sequences of $\mathcal{A}_p$. We can identify this space as a subspace of BMO. Indeed:

$$\bar{\mathcal{A}}_p = \{f \in \mathrm{BMO} : \exists \{w_n\}_{n\in\mathbb{N}} \subset A_p : \lim_{n\to\infty} \|\log w_n - f\|_* = 0\},$$

and we can think of the $\mathcal{A}_p$ class as a subset of $\bar{\mathcal{A}}_p$, by identifying every weight w with its logarithm, $\log w$, in BMO. Since the classical A_p spaces form an increasing "sequence" of the variable p (and of course the same is true for the $\mathcal{A}_p$ spaces), the same is true for this new subspaces of BMO, $\bar{\mathcal{A}}_1 \subset \bar{\mathcal{A}}_p \subset \bar{\mathcal{A}}_q \subset \bar{\mathcal{A}}_\infty \subset \mathrm{BMO}$, for $1 \leq p \leq q \leq \infty$.

They are also *convex* subsets of BMO. Indeed, consider $1 < p < \infty$, and $f, g \in \bar{\mathcal{A}}_p$. This means that there are sequences $\{w_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}} \subset A_p$ such that: $f = \lim_{n\to\infty} w_n, g = \lim_{n\to\infty} v_n$, in BMO. Let $0 < t < 1$ be fixed. We will show that $tf+(1-t)g \in \bar{\mathcal{A}}_p$. For this, we only need to see that $tf+(1-t)g = \lim_{n\to\infty}\log(w_n^t v_n^{1-t})$, in BMO, and check using Hölder that the weight $w_n^t v_n^{1-t} \in A_p$, for all n, since:

$$[w^t v^{1-t}]_{A_p} \leq [w]_{A_p}^t [v]_{A_p}^{1-t},$$

for all $w, v \in A_p$. Thus, $tf + (1-t)g \in \bar{\mathcal{A}}_p$. It is trivial to see now that $\bar{\mathcal{A}}_\infty$ is also a convex subset of BMO. For $\bar{\mathcal{A}}_1$ the same holds, since if we have two A_1 weights, w, v, it is trivial to see that $w^t v^{1-t} \in A_1$ and actually that $[w^t v^{1-t}]_{A_1} \leq [w]_{A_1}^t [v]_{A_1}^{1-t}$.

Here, let us observe that for any $1 < p < \infty$, we have that $L^\infty \subset \bar{\mathcal{A}}_p$. There is a nice result of weighted theory (see [3]) that states the following (we will present the statement only for A_2): there are dimensional constants $c_1, c_2 > 0$, such that for a function ϕ in $\mathbb{R}^n$ we have:

(a) $\mathrm{e}^\phi \in A_2$ provided $\inf\{\|\phi - g\|_* : g \in L^\infty\} \leq c_1$ and
(b) $\inf\{\|\phi - g\|_* : g \in L^\infty\} \leq c_2$ provided $\mathrm{e}^\phi \in A_2$. This means that all functions $f \in \mathrm{BMO}$ that satisfy the assumption (a), belong to the $\bar{\mathcal{A}}_2$ space. Equivalently, there is a small neighborhood of L^∞ inside BMO, that lies inside the $\bar{\mathcal{A}}_2$ space.

We should also mention that since:

$$\mathrm{BLO} = \{\alpha \log w : \alpha \geq 0, w \in A_1\},$$

we can ask the question if the spaces $\bar{\mathcal{A}}_1$, BLO are equal. Let us assume that they are. A classical result of weighted theory is that BMO = BLO − BLO. By our assumption we have that $\mathrm{BMO} = \bar{\mathcal{A}}_1 - \bar{\mathcal{A}}_1$. Now consider a function $f \in \mathrm{BMO}$. There are functions $\phi, \psi \in \bar{\mathcal{A}}_1$ such that $f = \phi - \psi$. We know that there are sequences of A_1 weights $\{\phi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}}$ such that $f = \lim_{n\to\infty}\log\phi_n - \lim_{n\to\infty}$

$\log \psi_n = \lim_{n\to\infty} \log \phi_n \psi_n^{-1}$, where the limit is in BMO. But $\phi_n \psi_n^{-1}$ is an A_2 weight for all n. So we get that $\bar{\mathcal{A}}_2 = \text{BMO}$. But this is obviously false.

Note that from the argument follows the inclusion, $\bar{\mathcal{A}}_1 - \bar{\mathcal{A}}_1 \subset \bar{\mathcal{A}}_2$. Trivially, we have the more general fact, that for any $1 < p < \infty$, $\bar{\mathcal{A}}_1 + (1-p)\bar{\mathcal{A}}_1 \subset \bar{\mathcal{A}}_p$. Also, since we have that $w \in A_p \Leftrightarrow w^{1-p'} \in A_{p'}$, we get the equivalence $f \in \bar{\mathcal{A}}_p \Leftrightarrow (1-p')f \in \bar{\mathcal{A}}_{p'}$. For $p = 2$ we have $f \in \bar{\mathcal{A}}_2 \Leftrightarrow -f \in \bar{\mathcal{A}}_2$, which means that the $\bar{\mathcal{A}}_2$ class is symmetric with respect to the origin in the BMO space. No other $\bar{\mathcal{A}}_p$ class has this property. Here we should remember the following about power weights. A function of the form $|x|^\alpha$ is an A_p weight in $\mathbb{R}^n$, if and only if $-n < \alpha < n(p-1)$. The interval for α is symmetric with respect to the origin, if and only if $p = 2$. Now we can see that there is a "correspondence" between the $\bar{\mathcal{A}}_2$ space and the interval $(-n, n)$.

3. The continuity in the weight

Our goal in this section is to prove the main Theorem:

Theorem 3.1. *Let T be a linear operator such that for some $1 < p < \infty$,*

$$\|T\|_{L^p(w)\to L^p(w)} \leq F([w]_{A_p}),$$

for any A_p weight w in $\mathbb{R}^n$, where F is an increasing, real valued function. Fix an A_p weight w_0. Then:

$$\lim_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)} = \|T\|_{L^p(w_0)\to L^p(w_0)},$$

and in addition for any sub-linear operator satisfying the hypothesis of the theorem we have the estimate:

$$\|T\|_{L^p(w)\to L^p(w)} \leq \|T\|_{L^p(w_0)\to L^p(w_0)}(1 + c d_*(w, w_0))$$

for all weights $w \in A_p$ with sufficiently small $d_*(w, w_0)$, where c is a positive constant that depends on p, on the function F, on the dimension n and on $[w_0]_{A_p}$.

Here let us mention something that is important. Say that our A_∞ weight w is of the "order" $[w]_{A_\infty} < 1 + \delta$. Then by Theorem 1.2 we get that $d_*(w, 1) \leq c\sqrt{\delta}$. Since Theorem 1.3 is going to be a generalization of the main Theorem in [8], the rate of convergence that we have in both theorems should agree. The above observation explains exactly this.

Remark 3.1. Notice that the second half of the previous theorem is true for the Maximal function (since it is true for all sub-linear operators that are bounded in the way described in the theorem), i.e., for all weights $w \in A_p$ that are sufficiently close to $w_0 \in A_p$, with $d_*(w, w_0) \leq \delta$:

$$\|M\|_{L^p(w)\to L^p(w)} \leq \|M\|_{L^p(w_0)\to L^p(w_0)}(1 + c\delta).$$

It is well known (see [1]) that $\|M\|_{L^p(w)\to L^p(w)} \leq c[w]_{A_p}^{\frac{1}{p-1}}$, which can be used here.

The argument is similar to the one given in [8], but it uses some small new ideas in order to overcome the problem that we have to work with two weights w and w_0, instead of "one" weight w, as was done in [8]. We are going to present it because we have to point out these nice differences.

Proof. First we will show that for any *sub-linear* operator that satisfies the assumptions of our theorem we have:

$$\|T\|_{L^p(w)\to L^p(w)} \leq \|T\|_{L^p(w_0)\to L^p(w_0)}(1+c\delta)$$

for all weights $w \in A_p$ with $d_*(w,w_0) \leq \delta$. Let $0 < \delta$ be a small number that we consider to be fixed. Fix also an A_p weight w, with $d_*(w,w_0) < \delta$. This means that $\|\log \frac{w}{w_0}\|_* \leq \delta$. We would like to write our weight w as $w = w_0^{1-t}W^t$, for some small and positive number t (which is going to be about δ), and some weight $W \in A_p$. At exactly this point, the justification of this fact is diffirent than the one given in [8]. The argument used there can not be used here. Fortunately, we are able to continue as follows. From the expression we can see that $W = \frac{w^{\frac{1}{t}}}{w_0^{\frac{1}{t}}}w_0$. For this, let us consider only the case $p = 2$, but the general case is identical to this one. Since $w_0 \in A_2$ we know that there is a small $\epsilon > 0$ such that $w_1 := w_0^{1+\epsilon} \in A_2$. Then obviously $w_0 = w_1^{1-s}$ for small $s > 0$. To continue, consider the function $f = \log\left(\frac{w}{w_0}\right)^{\frac{1}{s}}$. The BMO norm of f is really small since:

$$\|f\|_* = \frac{1}{s}d_*(w,w_0) \leq \frac{1}{s}\delta,$$

and so by the John–Nirenberg inequality we have that for all $\lambda \in (0, \frac{c}{\|f\|_*}]$ the function $e^{\lambda f} = \left(\frac{w}{w_0}\right)^{\frac{\lambda}{s}} \in A_2$, where c is a positive constant that depends only on the dimension. If we choose $\lambda = \frac{c_0}{\delta}$, $c_0 > 0$ is any constant less than or equal to sc, we see that $w_2 := \left(\frac{w}{w_0}\right)^{\frac{c_0}{\delta s}} \in A_2$, which implies that the function $w_1^{1-s}w_2^s \in A_2$. Then:

$$W := \frac{w^{\frac{1}{t}}}{w_0^{\frac{1}{t}}}w_0 = w_1^{1-s}w_2^s \in A_2,$$

where we put $t = \frac{\delta}{c_0}$. Here we should mention that the A_2 norm of W can be chosen to be bounded above by a constant that depends only on the A_2 norm of w_1. On the other hand, $[w_1]_{A_2}$ depends only on the A_2 norm of w_0, and this is fixed. With this in mind, let us assume that the A_2 characteristic of W is bounded above by c. The important thing here is that it does not depend on δ.

Now the proof continues as in [8], namely: Write $\gamma = \|T\|_{L^p(w_0)\to L^p(w_0)}$. By the interpolation result of Stein and Weiss, Theorem 1.1, for $X = Y = \mathbb{R}^n$, $\mathcal{M} = \mathcal{N} = \mathcal{L}$ and $\mu_0 = \nu_0 = w_0dx, \mu_1 = \nu_1 = Wdx$, where by $\mathcal{L}$ we denote the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^n$, we get

$$\begin{aligned}\|T\|_{L^p(w)\to L^p(w)} &\leq \gamma^{1-t}\|T\|^t_{L^p(W)\to L^p(W)} \\ &\leq \gamma^{1-t}c^tF\Big([W]_{A_p}\Big)^t \\ &\leq \gamma^{1-t}c^tF(c)^t,\end{aligned}$$

and the right-hand side goes to γ as $t \to 0^+$ or equivalently as $\delta \to 0^+$. In other words,

$$\limsup_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)} \leq \|T\|_{L^p(w_0)\to L^p(w_0)},$$

and in addition we have the desired estimate:

$$\|T\|_{L^p(w)\to L^p(w)} \le \|T\|_{L^p(w_0)\to L^p(w_0)}(1+c\delta),$$

where c is a constant depending on n, p and $[w_0]_{A_p}$, for all weights w in A_p that are δ close to w_0 in the d_* metric.

We can also conclude the following:

Proposition 3.1. *The set*

$$\{\log w : w \in A_p\}$$

is open in BMO for all $1 < p < +\infty$.

Proof. To see this fix $w_0 \in A_p$ and choose sufficiently small $\delta > 0$. For $f \in$ BMO with $\|f - \log w_0\|_* \le \delta$, write $f = \log u$, where u is a positive function. Then follow the previous reasoning in the beginning of the proof, with $w = u$ and write $u = w_0^{1-t}W^t$, for $0 < t < 1$. It follows that $W \in A_p$, if $\delta > 0$ is small depending only on the A_p norm of w_0, and so $u = w_0^{1-t}W^t$ is an A_p weight, by Hölder's inequality. As we can see, this is exactly the same argument as before. This result is new and could not be obtained from the techniques used in [8]. □

Now we show that for a *linear* operator we have the estimate:

$$\|T\|_{L^p(w_0)\to L^p(w_0)} \le \liminf_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)}.$$

Here again the reasoning requires subtle, but not difficult modification from the one given in [8]. Let us assume for simplicity that $p = 2$ and that $\|T\|_{L^2(w_0)\to L^2(w_0)} = 1$. Note that other p's can be treated similarly. So far, we have proved that:

$$\limsup_{d_*(w,w_0)\to 0} \|T\|_{L^2(w)\to L^2(w)} \le 1$$

and:

$$d_*(w, w_0) \le \delta < 1 \Rightarrow \|T\|_{L^2(w)\to L^2(w)} \le 1 + c\delta.$$

Let M_ϕ denote the operation of multiplication by ϕ. To finish the proof of the continuity at $w = w_0$ we are going to assume that:

$$\liminf_{d_*(w,w_0)\to 0} \|T\|_{L^2(w)\to L^2(w)} = \liminf_{d_*(w,w_0)\to 0} \left\| M_{w_0^{-\frac12} w^{\frac12}} T M_{w_0^{\frac12} w^{-\frac12}} \right\|_{L^2(w_0)\to L^2(w_0)} < 1$$

and get a contradiction. This means that there is $\tau > 0$ small, and a sequence of A_2 weights w_n such that $d_*(w_n, w_0) \to 0$ as $n \to \infty$ and in addition:

$$\|w_0^{-\frac12} w_n^{\frac12} T w_0^{\frac12} w_n^{-\frac12} g\|_{L^2(w_0)} \le (1-\tau)\|g\|_{L^2(w_0)} \tag{3.1}$$

for all functions $g \in L^2(w_0)$.

Fix now any cube Q in $\mathbb{R}^n$. Here we can make the normalization assumption $\frac{1}{|Q|}\int_Q \frac{w_n}{w_0}dx = 1$ for all $n \in \mathbb{N}$. We claim two things:

- (1*) $\|w_n^{-\frac12} - w_0^{-\frac12}\|_{L^2(w_0,Q)} \to 0$ as $n \to \infty$ where by $L^2(w_0, Q)$ we mean the $L^2(w_0)$ norm over Q, and
- (2*) there exists a subsequence k_n such that $w_{k_n} \to w_0$ almost everywhere in the cube Q.

Obviously (2^*) follows from (1^*). For a proof of 1^*, see lemma after the end of this proof. Now without loss of generality we can assume that the subsequence is the original sequence w_n. Note that (1^*) implies $\|w_n^{-\frac{1}{2}}f - w_0^{-\frac{1}{2}}f\|_{L^2(w_0,Q)} \to 0$ as $n \to \infty$ for all bounded f, and so for $g = fw_0^{-\frac{1}{2}}$, we get $\|T(w_0^{\frac{1}{2}}w_n^{-\frac{1}{2}}g) - Tg\|_{L^2(w_0,Q)} \to 0$ as $n \to \infty$ and this implies that for a subsequence of w_n (which again we assume that is the whole sequence), $w_0^{-\frac{1}{2}}w_n^{\frac{1}{2}}Tw_0^{\frac{1}{2}}w_n^{-\frac{1}{2}}g \to Tg$ almost everywhere in the cube Q. It is time to apply Fatou's lemma in inequality (3.1) and get:

$$\Big\|\liminf_{n\to\infty} w_0^{-\frac{1}{2}}w_n^{\frac{1}{2}}Tw_0^{\frac{1}{2}}w_n^{-\frac{1}{2}}g\Big\|_{L^2(w_0,Q)} \le \liminf_{n\to\infty}\Big\|w_0^{-\frac{1}{2}}w_n^{\frac{1}{2}}Tw_0^{\frac{1}{2}}w_n^{-\frac{1}{2}}g\Big\|_{L^2(w_0,Q)} \le (1-\tau)\|g\|_{L^2(w_0,Q)}.$$

Here $g = fw_0^{-\frac{1}{2}}$ with bounded f form a dense family in $L^2(w_0,Q)$. For g from this dense family it follows:

$$\|Tg\|_{L^2(w_0)} \le (1-\tau)\|g\|_{L^2(w_0)}$$

by letting the cube Q expand to infinity, for g in some dense subclass of $L^2(w_0)$. By assumption $\|T\|_{L^2(w_0)\to L^2(w_0)} = 1$ and this is how we have our contradiction. □

All that remains is the following lemma:

Lemma 3.1. *Let $w_0, w \in A_2$ such that $d_*(w,w_0) \le \epsilon$, where ϵ is sufficiently small. Let us have a normalization assumption $\frac{1}{|Q|}\int_Q \frac{w}{w_0}dx = 1$. Then $\|w_n^{-\frac{1}{2}} - w_0^{-\frac{1}{2}}\|_{L^2(w_0,Q)} \le |Q|^{\frac{1}{2}}c(\epsilon)^{\frac{1}{2}}$, where $c(\epsilon)$ goes to 0 as ϵ goes to 0.*

Proof. We want to estimate the expression:

$$\frac{1}{|Q|}\Big\|w^{-\frac{1}{2}} - w_0^{-\frac{1}{2}}\Big\|^2_{L^2(w_0,Q)} = \frac{1}{|Q|}\int_Q \frac{w_0}{w} + 1 - \frac{2}{|Q|}\int_Q\Big(\frac{w_0}{w}\Big)^{\frac{1}{2}}.$$

The last integral can be taken care of really easy, since by our normalization assumption and Cauchy–Schwartz we get the following:

$$\frac{1}{|Q|}\int_Q\Big(\frac{w_0}{w}\Big)^{\frac{1}{2}} = \frac{1}{|Q|}\int_Q\Big(\frac{w}{w_0}\Big)^{-\frac{1}{2}} \ge \Bigg(\frac{1}{|Q|}\int_Q\Big(\frac{w}{w_0}\Big)^{\frac{1}{2}}\Bigg)^{-1} \ge \Big(\frac{1}{|Q|}\int_Q\frac{w}{w_0}\Big)^{-\frac{1}{2}} = 1.$$

Therefore, the quantity that we need to estimate is bounded above by:

$$\frac{1}{|Q|}\Big\|w^{-\frac{1}{2}} - w_0^{-\frac{1}{2}}\Big\|^2_{L^2(w_0,Q)} \le \frac{1}{|Q|}\int_Q\frac{w_0}{w} - 1.$$

It is time to use the fact that $d_*(w,w_0) \le \epsilon$. We get that the weight $\frac{w}{w_0}$ is in the A_2 class and actually because the BMO norm of $\log\Big(\frac{w}{w_0}\Big)$ is really small, the A_2 characteristic is bounded by $1 + c(\epsilon)$, where $c(\epsilon)$ is a constant that goes to 0 as ϵ goes to 0. So:

$$\frac{1}{|Q|}\Big\|w^{-\frac{1}{2}} - w_0^{-\frac{1}{2}}\Big\|^2_{L^2(w_0,Q)} \le \Big[\frac{w}{w_0}\Big]_{A_2} - 1 \le c(\epsilon).$$

□

4. Comments and observations

Let us have a closer look to what the previous Theorem tells us. Consider any linear operator T that satisfies the assumptions of Theorem 1.3. This means that for any $w \in A_p$ we have a number $\|T\|_{L^p(w)\to L^p(w)}$. So we have a map $F_T : \mathcal{A}_p \to \mathbb{R}$ defined by the formula:

$$F_T(w) = \|T\|_{L^p(w)\to L^p(w)}.$$

First, we should observe that for $w \in \mathcal{A}_p$ this is well defined since $\|T\|_{L^p(w)\to L^p(w)} = \|T\|_{L^p(cw)\to L^p(cw)}$ for all $w \in A_p$ and all positive constants $c > 0$. By Theorem 1.3 we have that this map is continuous, since:

$$\lim_{d_*(w,w_0)\to 0} |F_T(w) - F_T(w_0)| = 0$$

for all weights $w, w_0 \in \mathcal{A}_p$.

In [8] the authors showed that for the Hilbert transform, H, in $\mathbb{S}^1$ we have that for all sufficiently small δ's, there is a weight $w \in A_2$, with the properties that $[w]_{A_2} \leq 1 + \delta < 2$ and:

$$c\sqrt{\delta} \leq \|H\|_{L^2(w)\to L^2(w)} - 1.$$

This means that Theorem 1.3 is sharp for $p = 2$ at the constant weight $w_0 = 1$. This is true also for the Hilbert transform in the line and for the Martingale transform. It is a good point to mention that there are singular operators like the Riesz projection P_+ in $\mathbb{S}^1$, that converge faster to their $L^2(dx)$ norm than the previous mentioned operators (see [8]). Namely, there is universal constant $c > 0$ such that for all weights $[w]_{A_2} \leq 1 + \delta < 2$, we have:

$$\|P_+\|_{L^2(w)\to L^2(w)} - 1 \leq c\delta.$$

In addition, there is a universal constant $c_1 > 0$, such that for all sufficiently small δ's, there is weight $w \in A_2$ with the properties $[w]_{A_2} \leq 1 + \delta$ and:

$$c_1\delta \leq \|P_+\|_{L^2(w)\to L^2(w)} - 1.$$

We should also mention that in the proof of Theorem 1.3, there is only one time (namely in the second step) that we really need to use the fact that our operator is linear in order to get that:

$$\|T\|_{L^p(w_0)\to L^p(w_0)} \leq \liminf_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)}.$$

It is used when we claim that the convergence $\|w_n^{-\frac{1}{2}} f - w_0^{-\frac{1}{2}} f\|_{L^2(w_0,Q)} \to 0$ as $n \to \infty$, implies the convergence $\|T(w_0^{\frac{1}{2}} w_n^{-\frac{1}{2}} f) - Tf\|_{L^2(w_0,Q)} \to 0$ as $n \to \infty$.

By [5, 6], we know that for any Calderón–Zygmund operator T, any $1 < p < \infty$, and any A_p weight w, we have the estimate:

$$\|T\|_{L^p(w)\to L^p(w)} \leq c[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}},$$

where c is a universal constant that does not depend on the weight, and so we see that Theorem 1.3 can be applied for this class of operators, since the function F that appears in the statement of this Theorem can be chosen to be equal to $F(x) = cx^{\max\{1,\frac{1}{p-1}\}}$.

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