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ACTION-MINIMIZING PERIODIC AND QUASI-PERIODIC SOLUTIONS IN THE n-BODY PROBLEM

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Abstract. Considering any set of n-positive masses, $n \geq 3$, moving in $\mathbb{R}^2$ under Newtonian gravitation, we prove that action-minimizing solutions in the class of paths with rotational and reflection symmetries are collision-free. For an open set of masses, the periodic and quasi-periodic solutions we obtained contain and extend the classical Euler–Moulton relative equilibria. We also show several numerical results on these action-minimizing solutions. Using a natural topological classification for collision-free paths via their braid types in a rotating frame, these action-minimizing solutions change from trivial to non-trivial braids as we vary masses and other parameters.

1. Introduction

Consider a system of n ($\geq$3) positive masses $m_1, m_2, \dots, m_n$ moving in the complex plane $\mathbb{C}$ under Newton's law of gravitation:

$$m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U(x), \quad k = 1, \dots, n, \tag{1.1}$$

where $x_k \in \mathbb{C}$ is the position of m_k and

$$U(x) = U(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|x_i - x_j|}$$

is the potential energy. The kinetic energy is given by

$$K(\dot{x}) = \frac{1}{2} \sum_{i=1}^{n} m_i |\dot{x}_i|^2.$$

Assume the mass center is at the origin and let V be the *configuration space*:

$$V := \left\{ x \in \mathbb{C}^n : \sum_{i=1}^{n} m_i x_i = 0 \right\}.$$

For any fixed positive constant T, equations (1.1) are the Euler–Lagrange equations for the action functional $\mathcal{A}_T : H^1_{\mathrm{loc}}(\mathbb{R}, V) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{A}_T(x) := \int_0^T K(\dot{x}) + U(x)\, dt.$$

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By rescaling properties of the n-body problem, there is no loss of generality by fixing $T = 1$.

In this paper, we are interested in searching for action-minimizing solutions that return to their original configurations up to rotations, after a period of time T. A path x in $H^1_{\mathrm{loc}}(\mathbb{R}, V)$ is said to has *ϕ-rotational symmetry*, $\phi \in (0, \pi]$, if there exists some $T > 0$ such that

$$x(t + T) = e^{\phi i} x(t)$$

for any $t \in \mathbb{R}$. The number T is called a *relative period* or simply a period of x. Let

$$H_{\phi,T} := \{x \in H^1_{\mathrm{loc}}(\mathbb{R}, V) : x(t + T) = e^{\phi i} x(t)\}.$$

The conventional definition of inner product on the Sobolev space $H^1([0, T], V)$ also defines an inner product on $H_{\phi,T}$:

$$\langle x, y\rangle_{\phi,T} = \int_0^T \langle x(t), y(t)\rangle + \langle \dot{x}(t), \dot{y}(t)\rangle \, dt.$$

Here $\langle \cdot, \cdot\rangle$ stands for the standard scalar product on $(\mathbb{R}^2)^n$. For any x in $H_{\phi,T}$, we have $\langle x(0), x(T)\rangle = |x(0)||x(T)| \cos \phi$. The assumption $\phi \in (0, \pi]$ ensures that $\mathcal{A}_T$ is coercive and attains its infimum on $H_{\phi,T}$ (see [2, Proposition 2], for instance). Critical points of $\mathcal{A}_T$ on $H_{\phi,T}$ are critical points of $\mathcal{A}_T$ on $H^1([0, T], V)$. One can easily verify that, for any $x \in H_{\phi,T}$ and $\tau \in \mathbb{R}$,

$$\begin{aligned} \mathcal{A}_T(x) &= \int_\tau^{T+\tau} K(\dot{x}) + U(x) \, dt, \\ \langle x, y\rangle_{\phi,T} &= \int_\tau^{T+\tau} \langle x(t), y(t)\rangle + \langle \dot{x}(t), \dot{y}(t)\rangle \, dt. \end{aligned}$$

Following these observations, any critical point x of $\mathcal{A}_T$ on $H_{\phi,T}$ is a solution of (1.1), except when there are collisions. If we can show that x has no collision on $[0, T)$, then there is no collision at all and x indeed solves (1.1) for all $t \in \mathbb{R}$. Moreover, x is periodic if $\phi/2\pi$ is rational, it is quasi-periodic if $\phi/2\pi$ is irrational, provided that x is not circular.

Fix $T > 0$ and $\phi \in (0, \pi]$. Imagine that we are standing on the plane of motion which rotates counterclockwise with angular velocity $\frac{\phi}{T}$. By adding the time axis, collision-free trajectories in $H_{\phi,T}$ draw out braids on n strands in the three-dimensional space-time. These braids are pure braids; that is, each strand ends at the same space coordinates as those at which it begins. We say two paths are having the same *braid type* if one can be continuously deformed to the other among pure braids. This defines an equivalence relation on collision-free paths in $H_{\phi,T}$ and the equivalence classes can be identified as the normal subgroup of pure braids for the classical Artin braid group on n strands. This point of view, introduced in [13] where the inertia plane is not rotating ($\phi = 0$), is an analogy of Poincaré's [17] classification using homology types. It provides a natural topological classification for collision-free planar motions.

The most well-known solutions with rotational symmetry are relative equilibria. These are the only solutions satisfying ϕ-rotational symmetry for every $\phi \in (0, \pi]$, and they are all topologically equivalent in the sense that they all draw out trivial braids in an appropriate rotating frame. For very small ϕ the classical Poincaré continuation method [16, 19] has been used to construct certain solutions with non-trivial braids. The perturbative nature of this approach requires either the presence of nearly zero masses or a very tight subsystem.

In this paper, our major focus is on the existence and minimizing properties of solutions that are not included by these classical results. The angle ϕ is not assumed to be small and in most cases all masses are comparable in size. In particular, for an open set of masses including equal masses, we prove the existence of a class of periodic and quasi-periodic solutions that contain and extend the classical Euler–Moulton relative equilibria. Our main theorems will be stated in the next section and the proofs will be given in Sections 3 and 4. Section 5 contains several related numerical results. Minimizing arguments have been previously used to construct miscellaneous solutions, mostly with equal masses. We refer the readers to the excellent survey [7] and references therein for these relevant results.

2. Main theorems

Throughout this paper it is always assumed $\phi \in (0, \pi]$. Without any constraint on the space $H_{\phi,T}$, the action minimizers for $\mathcal{A}_T$ on $H_{\phi,T}$ are planar relative equilibria. This was proved in [8] for the case $\phi = \pi$, and general cases can be obtained by imitating the proof in there. For the three-body problem, these action-minimizers are Lagrange equilateral solutions; for general n-body problem, action-minimizers are planar relative equilibria with the smallest normalized potential.

In order to obtain solutions other than relative equilibria, we impose a reflection symmetry on the path space $H_{\phi,T}$. Fix ϕ and T, a path x in $H_{\phi,T}$ is said to has *reflection symmetry* if

$$x(t) = \overline{x(-t)},$$

for all $t \in \mathbb{R}$, where $\overline{x(t)}$ denotes the complex conjugate of $x(t)$.

Let $H_{\phi,T,R}$ be the set of paths in $H_{\phi,T}$ with the reflection symmetry. By Palais' principle of symmetric criticality [15], critical points of $\mathcal{A}_T$ on $H_{\phi,T,R}$ are also critical points on $H_{\phi,T}$. Due to the reflection symmetry, $[0, T/2]$ is a *fundamental domain* for the curves in $H_{\phi,T,R}$; that is, the canonical projection $H_{\phi,T,R} \to H^1([0, T/2], V)$ is injective and there is no proper closed subinterval of $[0, T/2]$ with this property. It is an easy exercise to check that for curves in $H_{\phi,T,R}$, the n-bodies are collinear at $t = 0$ and $t = T/2$. This greatly simplifies our minimization process. To simply put it, what we are looking for are the action-minimizers among the paths of the n-body problem that start from a collinear configuration and return to a collinear configuration at $t = T/2$.

Our first result is:

Theorem 2.1. *For any fixed set of n (≥ 3) positive masses $m_1, m_2, \dots, m_n$ and any fixed $\phi \in (0, \pi]$, the minimum of $\mathcal{A}_T$ on $H_{\phi,T,R}$ is attained and the minimizing trajectories are collision-free.*

Observe that the collinear relative equilibria, also called *Euler–Moulton relative equilibria* [9, 14], are critical points of $\mathcal{A}_T$ in $H_{\phi,T,R}$. This may suggest that the action-minimizer in the above theorem could very well be these known, hence trivial, solutions. Indeed, for ϕ small, the solutions given by the theorem are Euler–Moulton relative equilibria. However, for an open set of masses, for larger values of ϕ, it can be proved that they are not relative equilibria, and consequently what we obtained is a larger class of new solutions that contains the classical Euler–Moulton relative equilibria.

Theorem 2.2. *There is an open set $\mathcal{M}$ of masses containing equal masses with the following properties: for any $m = (m_1, m_2, \ldots, m_n) \in \mathcal{M}$ there corresponds a $\phi_c \in (0, \pi)$, called the critical angle of m, such that Euler–Moulton relative equilibria are absolute minimizers of $\mathcal{A}_T$ in $H_{\phi,T,R}$ provided $0 < \phi \leq \phi_c$, but cease to be absolute minimizers of $\mathcal{A}_T$ in $H_{\phi,T,R}$ when $\phi_c < \phi \leq \pi$.*

We have restricted ϕ to $(0, \pi]$ for simplicity. If ϕ is larger than π, then action-minimizers may be attained by a shorter path through a clockwise rotation. It may happen that, for some set of masses, the global action-minimizers are always Euler–Moulton solutions. However, numerical results suggest that for most, if not all, positive masses, the critical angle ϕ_c is less than π.

As discussed before, Euler–Moulton solutions all have the trivial braid type. To better illustrate some of the interesting braids, we take the simple case, $n = 3$. The first simple braid types are those of the so-called prograde and retrograde trajectories. The prograde (respectively, retrograde) trajectories are the ones such that two particles revolves around each other in one direction, while the center of their masses revolves around the third particle in the same direction (respectively, opposite direction). We can further classify the trajectories according to the number of rotations of the close binaries within one period. Let B_k, $k \in \mathbb{Z}$, be the braid type of the trajectories where the binaries complete k revolutions within $[0, T]$. We use positive k's for prograde and negative k's for retrograde. B_0 is the trivial braid type. The class of paths with trivial braid type for $n \geq 3$ will be also denoted by B_0.

Let B be a pure braid on n strands and $H_{\phi,T,R,B}$ be the set of all collision-free paths in $H_{\phi,T,R}$ whose braid type is B. It is natural to ask whether the minimum of $\mathcal{A}_T$ over $H_{\phi,T,R,B}$ can be attained in the interior of $H_{\phi,T,R,B}$. This is essentially the question raised by Poincaré [17] in 1896. These minimizers, if exist, can be different from the more global minimizers provided in Theorem 2.1. Obviously, collision trajectories are the only possible obstruction for us to obtain these braid-type minimizers. In [4] it was proved that, for $n = 3$ and ϕ away from zero, action-minimizers within the class of retrograde paths exist for most masses. On the other hand, the results in [21] suggest that action-minimizers do not exist in most braid classes.

3. Action-minimizers are collision-free

This section is devoted to proving Theorem 2.1, which states that that action-minimizers in $H_{\phi,T,R}$ are collision-free.

First we consider the minimizing problem with fixed ends:

$$\inf\left\{\int_{\tau_1}^{\tau_2} K(\dot y)+U(y)\,dt : y\in H^1([\tau_1,\tau_2],\mathbb{C}^n),\ y(\tau_1)=\xi_1,\ y(\tau_2)=\xi_2\right\}. \tag{3.1}$$

A fundamental result by Marchal [12] (see Chenciner [6] for a more complete proof) states

Marchal's theorem. *Given any ξ_1, $\xi_2\in\mathbb{C}^n$. Minimizers of the fixed-ends problem (3.1) are collision-free on the interval (τ_1,τ_2).*

Note that the space $\mathbb{C}^n$ in (3.1) and Marchal's theorem can be replaced by the configuration space V. This is because any minimizer for the fixed-ends problem is a solution to (1.1), then the linear momentum is conserved, and therefore it has to stay on V at any instant if both ξ_1 and ξ_2 belong to V.

In this and the next section, the proofs of our main results require analysis on how the values of $\mathcal{A}_T$ vary as one collision path is deformed to a non-collision one. For the convenience of this deformation argument, we "enlarge" the space $H_{\phi,T}$ and $H_{\phi,T,R}$ by replacing V with $\mathbb{C}^n$. Actually, the rotational symmetry assumption automatically implies that action-minimizers stay entirely in V.

Near an isolated collision, it is well known that the bodies involved in this collision will approach the set of central configurations [18]. We can tell more if the solution under concern is action-minimizing:

Lemma 3.1. *(Venturelli [22, Theorem 4.1.18], Chenciner [6, Section 3.2.1]). If a minimizer x of the fixed-ends problem on time interval $[\tau_1,\tau_2]$ has an isolated collision of $k\le n$ bodies, then there is a parabolic homothetic collision–ejection solution $\bar x$ of the k-body problem, which is also a minimizer of the fixed-ends problem on $[\tau_1,\tau_2]$.*

Let $\gamma(t)\in H_{\phi,T,R}$ be a minimizer of $\mathcal{A}_T$. By Marchal's theorem, $\gamma(t)$ is collision-free for $t\in(0,T/2)$. Therefore, the only possible collisions occur at $t=0 \pmod T$ or $t=T/2 \pmod T$ or both. We will show first, as a simple case, that it is impossible to have binary collisions and then we will show that it is impossible to have any collisions.

Lemma 3.2. *Let $\gamma(t)\in H_{\phi,T,R}$ be a minimizer of $\mathcal{A}_T$. Then $\gamma(t)$ has no binary collision.*

Proof. As stated above, we only need to exclude binary collisions at $t=0$, $T/2$. By the minimizing properties of $\gamma(t)$, $\gamma(t)$ is a solution to the Euler–Lagrange equation for $t\in(0,T/2)$. Without loss of generality, assume m_1 and m_2 collide at $t=0$. The case a binary collision occurs at $t=T/2$ is similar. We will show that, by replacing $\gamma(t)$ with a different but collision-free path in $H_{\phi,T,R}$, the action can be reduced, hence contradicting that $\gamma(t)$ is an action-minimizer.

Let $q_1(t) \in \mathbb{R}^2$ and $q_2(t) \in \mathbb{R}^2$ be the position of m_1 and m_2 relative to their center of masses; i.e.,

$$q_i(t) = x_i(t) - (m_1 x_1(t) + m_2 x_2(t))/(m_1 + m_2), \quad i = 1, 2.$$

According to Lemma 3.1 and the fact that $\gamma(t)$ is an action-minimizer for fixed-ends problem on small time interval $[0, \tau]$, to prove by contradiction, we only need to handle the case $q = (q_1, q_2)$ has zero energy and approaches homothetically to zero as t approaches zero. By Sundman's estimate [20], there is a unit vector $v \in \mathbb{R}^2$ and a non-zero number a such that

$$q_1(t) = am_2\big(t^{2/3} + o(t^{2/3})\big)v$$

and

$$q_2(t) = -am_1\big(t^{2/3} + o(t^{2/3})\big)v.$$

Moreover,

$$\dot{q}_1(t) = \frac{2}{3}am_2\big(t^{-1/3} + o(t^{-1/3})\big)v$$

and

$$\dot{q}_2(t) = -\frac{2}{3}am_1\big(t^{-1/3} + o(t^{-1/3})\big)v.$$

By switching m_1 and m_2 if necessary, we may assume that $a > 0$. The motion of the binary is parabolic, this implies that $a^3 = \frac{9}{2(m_1+m_2)^2}$. Let $\mathcal{A}_\tau^{12}$ be the action of the binary m_1 and m_2 with respect to their center of masses for $t \in [0, \tau]$. Then

$$\begin{aligned} \mathcal{A}_\tau^{12}(q_1, q_2) &= \int_0^\tau \frac{1}{2}m_1|\dot{q}_1(t)|^2 + \frac{1}{2}m_2|\dot{q}_2(t)|^2 + \frac{m_1 m_2}{|q_1(t) - q_2(t)|}\, dt \\ &= \frac{4}{3}a^2 m_1 m_2 (m_1 + m_2)\tau^{1/3} + o(\tau^{1/3}). \end{aligned}$$

We now change the path within our path space so as to reduce the binary action, thereby contradicting the assumption that $\gamma(t)$ is an action-minimizer. First, we remark that near collisions, the collision particles contribute most to the action, of the order of $O(\tau^{1/3})$, whereas the other particles only contribute of the order of $O(\tau)$. When the direction of the binary collision is parallel to the x-axis, $v = (1, 0)$, we fix τ small and choose the new path $(\tilde{q}_1, \tilde{q}_2)$ that is stationary: $\tilde{q}_1(t) = q_1(\tau) = (am_2\tau^{2/3}, 0) + o(\tau^{2/3})$ and $\tilde{q}_2(t) = q_2(\tau) = (-am_1\tau^{2/3}, 0) + o(\tau^{2/3})$, for all $t \in (0, \tau)$. The new action

$$\mathcal{A}_\tau^{12}(\tilde{q}_1, \tilde{q}_2) = \frac{2}{9}a^2 m_1 m_2 (m_1 + m_2)\tau^{1/3} + o(\tau^{1/3})$$

is approximately one sixth of the original action. More generally, when the collision direction $v = (\xi, \eta)$ is not perpendicular to the x-axis, consider the new path $(\tilde{q}_1, \tilde{q}_2)$

$$\begin{aligned} \tilde{q}_1(t) &= (\operatorname{Re} q_1(\tau), \operatorname{Im} q_1(t)) = (am_2\xi\tau^{2/3}, am_2\eta t^{2/3}) + o(\tau^{2/3}, t^{2/3}), \\ \tilde{q}_2(t) &= (\operatorname{Re} q_2(\tau), \operatorname{Im} q_2(t)) = -(am_1\xi\tau^{2/3}, am_1\eta t^{2/3}) + o(\tau^{2/3}, t^{2/3}), \end{aligned}$$

$t \in [0, \tau]$. For small τ, it clearly reduces the original action because both kinetic and potential energy are reduced.

When the collision direction is perpendicular to the x-axis, $v = (0, 1)$, it is harder in this case to reduce the action, for we need to bring the particles to the x-axis at $t = 0$. We may naïvely take a circular trajectory starting at $t = \tau$. It turns out that the action is not reduced by this path. Instead, we take a straight line with constant speed:

$$\tilde{q}_1(t) = \operatorname{Im} q_1(\tau)(1 - t/\tau,\ t/\tau) = am_2\tau^{2/3}(1 - t/\tau,\ t/\tau) + o(\tau^{2/3}),$$
$$\tilde{q}_2(t) = \operatorname{Im} q_2(\tau)(1 - t/\tau,\ t/\tau) = am_1\tau^{2/3}(-1 + t/\tau,\ -t/\tau) + o(\tau^{2/3}),$$

for $t \in (0, \tau)$. Again, we compute the action of the binary under this new path.

$$\mathcal{A}_\tau^{12}(\tilde{q}_1, \tilde{q}_2) < \left(\frac{3\sqrt{2}}{8} + \frac{\sqrt{2}}{6}\right)\frac{4}{3}a^2 m_1 m_2 (m_1 + m_2)\tau^{1/3} + o(\tau^{1/3}),$$

which is very close to, but smaller than the original action. In the original binary action, the kinetic part and the potential part in the Lagrangian contributes equally to the action. In this new path, the kinetic part of the action is about $3\sqrt{2}/4$ of the original action and the contribution from the potential part is less than $\sqrt{2}/3$ of the original potential contribution. Here we used the shortest distance in $(0, \tau)$ for the upper bound of the potential function. There are other paths that reduce both kinetic and potential actions.

When the collision direction is other than x- or y-axis, we choose the shorter path moving the particles to the x-axis, then the action is smaller than when the collision direction is in the y-axis.

This proves that no binary collision is possible for the action-minimizer. □

We now suppose that there is a k-tuple collision, involving $m_1, m_2, \ldots, m_k$, at $t = 0$ for the action-minimizer $\gamma(t)$. Again, according to Lemma 3.1, we only need to rule out the case that $m_1, m_2, \ldots, m_k$ forms a central configuration and approaches homothetically to the k-tuple collision as $t \to 0^+$ with zero total energy. Let $q_1(t)$, $q_2(t), \ldots,$ $q_k(t)$ be the positions of the k bodies relative to their center of masses. Then there exist vectors a_1, a_2, $\ldots$, a_k in $\mathbb{R}^2$ such that $q_i(t) = a_i\big(t^{2/3} + o(t^{2/3})\big)$ and $\dot{q}_i(t) = \frac{2}{3}a_i\big(t^{-1/3} + o(t^{-1/3})\big)$, for $i = 1, 2, \ldots, k$. The vectors a_1, $a_2, \ldots, a_k$ form a central configuration for the k particles with the energy constraint:

$$\sum_{i=1}^{k} \frac{1}{2} m_i \left|\frac{2}{3}a_i\right|^2 - \sum_{1 \le i < j \le k} \frac{m_i m_j}{|a_i - a_j|} = 0$$

If a_1, a_2, $\ldots$, a_k form a collinear central configuration, then we can do the same as we did for the binary collisions to reduce the action through a collision-free path. Therefore, we assume that a_1, $a_2, \ldots, a_k$ are not collinear. Let $a_i = (\xi_i, \eta_i)$, $i = 1, 2, \ldots, k$. We take the following new path, for $t \in (0, \tau)$:

$$\tilde{q}_i(t) = (\operatorname{Re} q_i(\tau), \operatorname{Im} q_i(t)) = (\xi_i \tau^{2/3}, \eta_i t^{2/3}) + o(\tau^{2/3}, t^{2/3}), \quad i = 1, 2, \ldots, k.$$

Obviously, both the kinetic energy and the potential energy are reduced under the new path and therefore the action is reduced for $t \in (0, \tau)$. Since $a_1, a_2, \ldots, a_k$ are not collinear, the number of the particles involved in a possible collision at $t = 0$ is less than k.

By induction, there cannot be any collision for the action-minimizer at $t = 0$. This proves Theorem 2.1.

4. Bifurcation from Euler–Moulton relative equilibria

Action-minimizers described in Theorem 2.1 could be the Euler–Moulton relative equilibria, and this is indeed the case for small ϕ. In this section, we will show that, by treating ϕ as a parameter, for an open set of masses the action-minimizers undergo a bifurcation from Euler–Moulton relative equilibria to other classes of solutions. This will imply Theorem 2.2 and the existence of different classes of solutions.

The first claim in Theorem 2.2 follows from a classical Weierstrass theorem, which states that for positive-definite Lagrangian systems, all solutions are action-minimizers for small time intervals. Even though we fixed the period T, which is not necessarily small, we can rescale the n-body system so that the Euler–Moulton solution, with small angle ϕ and large size, can be regarded as Euler–Moulton solution with fixed size, small T. Therefore, for ϕ small enough, they are action-minimizers.

For the rest of the theorem, we first take $n = 3$ and let $m_1 = m_2 = m_3 = 1$, the Euler circular solution is given by

$$\begin{aligned} x_1(t) &= (-a\cos\omega t, -a\sin\omega t), \\ x_2(t) &= (0,0), \\ x_3(t) &= (a\cos\omega t, a\sin\omega t), \end{aligned}$$

where

$$a^3\omega^2 = 5/4.$$

If we take $\phi = \pi$ and $T = 1$, for the solution to be in $H_{\phi,T,R}$, we have $\omega = \pi$ and therefore $a^3 = 5/(4\pi^2)$. Calculating the action of this trajectory is easy. It is approximately 7.467.

Now we replace it with a different path,

$$\begin{aligned} x_1(t) &= (-0.5\cos\omega t, -0.5\sin\omega t), \\ x_2(t) &= (0, 0.5\sin\omega t), \\ x_3(t) &= (0.5\cos\omega t, 0), \end{aligned}$$

This new path and the Euler solution belong to different braid classes. It is a retrograde path; that is, its braid type is B_{-1}. The action of this path is approximately 7.213, which is smaller than the action of the Euler solution. By continuity, this implies that the Euler solution is not an action-minimizer in $H_{\phi,T}$ for an open set of masses and an open set of ϕ containing π. This proves Theorem 2.2 for the case $n = 3$.

Now for arbitrary $n \geq 4$, we again take $m_1 = m_2 = \cdots = m_n = 1$ and $T = 1$. Let $b_1 < b_2 < \cdots < b_n$ be a collinear central configuration. The collinear central configurations are symmetric for equal masses and therefore $b_i = -b_{n+1-i}$ for $i = 1, 2, \dots, n$. The Euler–Moulton relative equilibrium solution is given by

$$x_i(t) = (ab_i \cos \omega t, ab_i \sin \omega t), \quad 1 \le i \le n,$$

where again $a^3\omega^2$ is equal to a fixed constant depending on the chosen central configuration, so as to balance the centrifugal force with the gravitational force.

To reduce the action, we keep all the paths for $m_2, m_3, \dots, m_{n-1}$ and change the path for m_1 and m_n to:

$$x_{1,n}(t) = (ab_{1,n} \cos \omega t, -ab_{1,n} \sin \omega t).$$

The pair m_1 and m_n rotate clockwise as oppose to counterclockwise rotations for all other bodies:

$$x_i(t) = (ab_i \cos \omega t, ab_i \sin \omega t), \quad 2 \le i \le n-1.$$

The kinetic energy for the new pair stay the same. However, the potential energy is reduced, for one easily verifies that under the new path

$$\frac{a}{|x_1 - x_i|} + \frac{a}{|x_1 - x_{n+1-i}|} \le \frac{1}{|b_1| - |b_i|} + \frac{1}{|b_1| + |b_i|},$$

for $2 \le i \le n/2$.

Therefore, for equal masses or close to equal masses, the critical angle $\phi_c < \pi$. This completes the proof of Theorem 2.2.

Same arguments can be applied to many other set of masses. Numerical evidences suggest that Theorem 2.2 holds for all masses. See Section 5 for examples.

Remark 4.1. The idea of proof for Theorem 2.2 comes from the observation that retrograde paths can have lower action than Euler–Moulton solutions when ϕ is close to π. However, determining the braid types of the solutions (other than relative equilibria) obtained in Theorem 2.2 is a non-trivial task. For most choices of masses in the three-body problem, existence of solutions in $H_{\phi,T,R,B_{-1}}$ (that is, the space of retrograde paths in $H_{\phi,T,R}$) with ϕ away from zero was proved in [4] (see also [5]). We conjecture that, for any n, the action-minimizing solutions obtained in Theorem 2.2 for $\phi \in (\phi_c, \pi]$ have non-trivial braid types.

5. Some numerical results

This section includes some numerical solutions for the three- and four-body problems in the function space $H_{\pi,T,R}$. For convenience we set $T = 1$. As we shall see, for various choices of masses, the critical angles ϕ_c described in Theorem 2.2 fall inside the interval $(\pi/2, \pi)$.

Figure 1 shows some action-minimizing retrograde orbits with action lower than all Euler solutions. See [4,5] for a rigorous existence proof for such orbits. The upper left orbit was first numerically obtained by Hénon [11]. Below the other three orbits, there are values of $\mathcal{A}_1$ for the Euler solutions with the same initial ordering as the designated retrograde orbits. For instance, below the upper right orbit, the value of $\mathcal{A}_1$ for the Euler solution $x \in H_{\pi,1,R}$ with $(m_1, m_2, m_3) = (1, 1, 10)$ and $x_1(0) > x_2(0) > x_3(0)$ is approximately 31.6355. However, this is not the Euler solution with minimum possible action. The one with minimum action is initially ordered $x_1(0) > x_3(0) > x_2(0)$, for

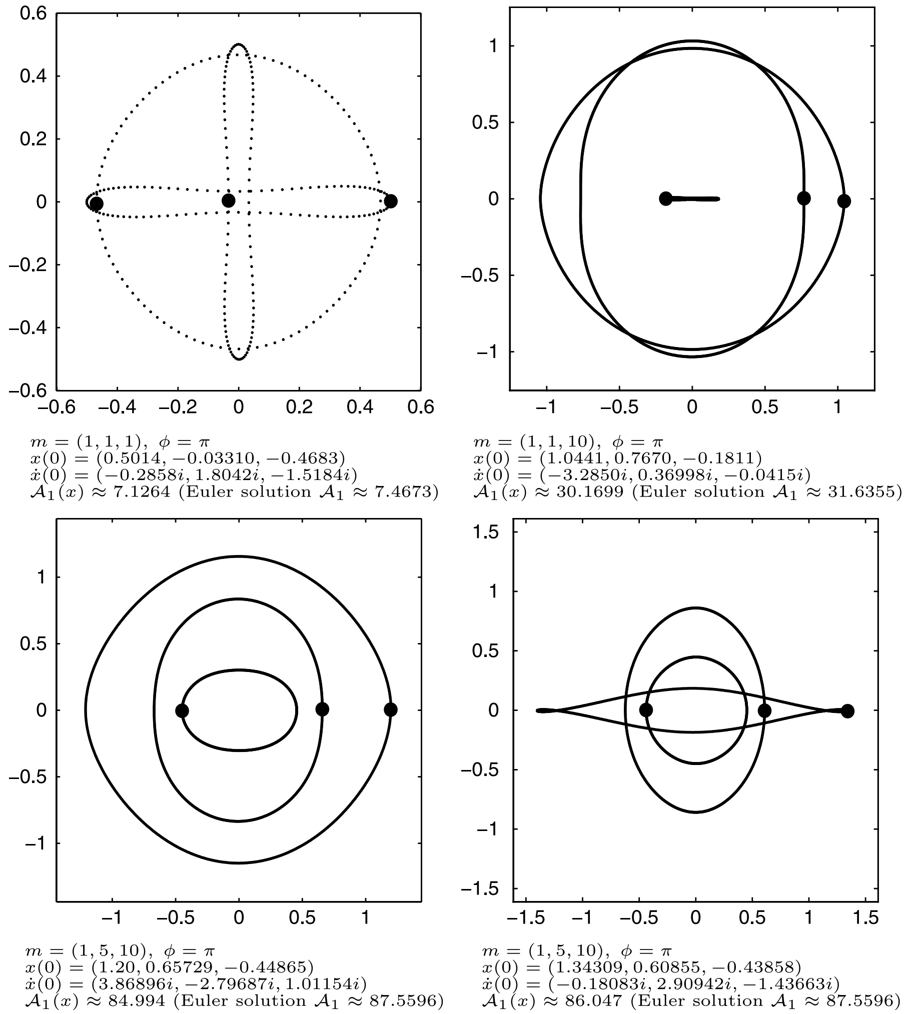


FIGURE 1. Action-minimizing retrograde orbits with $\phi = \pi$.

which we have $\mathcal{A}_1 \approx 30.3648$. This is still greater than the action of the retrograde path we obtained. Therefore, the set $\mathcal{M}$ described in Theorem 2.2 may include a neighborhood of $m = (1, 1, 10)$ (and any other ordered triple consisting of 1,1,10).

The two orbits in Figure 1 with $(m_1, m_2, m_3) = (1, 5, 10)$ are both retrograde but in $H_{\pi,1,R}$ they are topologically distinct. When viewed from a rotation frame, as described in Section 1, the one on the left has m_2, m_3 wind around each other in clockwise direction, whereas the one on the right has m_1 in place of m_3. The value of $\mathcal{A}_1 \approx 87.5596$ for the Euler solution is for the case $x_1(0) > x_2(0) > x_3(0)$. The Euler solution with minimum action is ordered $x_1(0) > x_3(0) > x_2(0)$. In this case, the value of $\mathcal{A}_1$ is approximately 86.3204, which is still greater than the action of both retrograde orbits. Therefore, the set $\mathcal{M}$ described in Theorem 2.2 may also contain neighborhoods for ordered triples consisting of $\{1, 5, 10\}$.

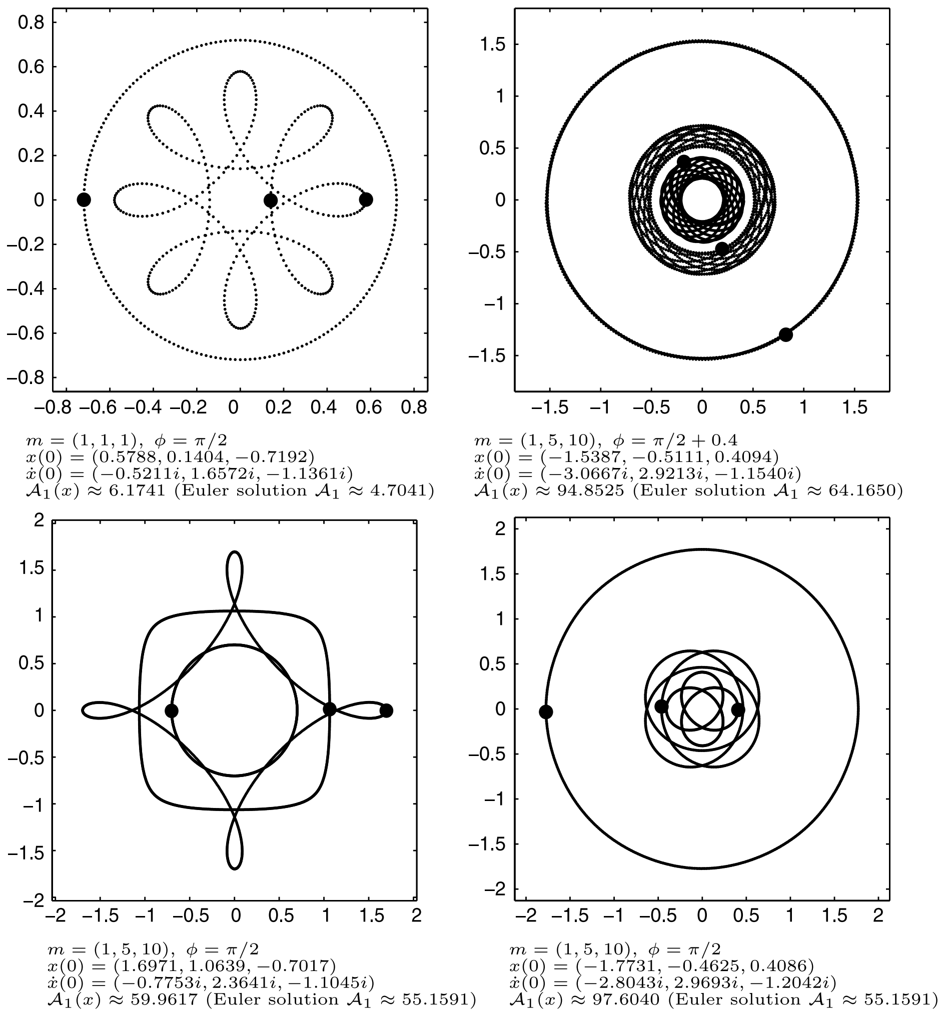
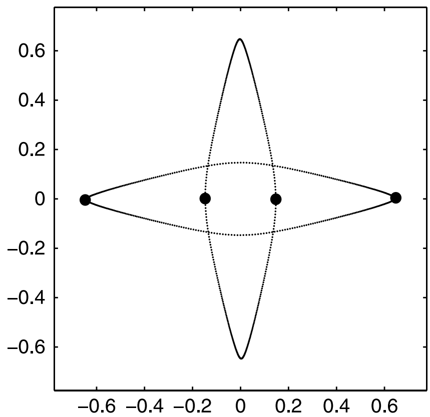


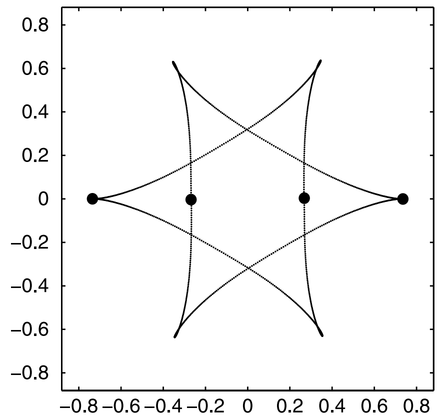
FIGURE 2. Some retrograde orbits with $\phi < \pi$.

Figure 2 shows some retrograde orbits that are local action-minimizers with $\pi/2 \leq \phi < \pi$. The upper left orbit is also included in Hénon's family [11]. The upper right orbit is quasi-periodic since the angle ϕ is not commensurable with π. None of these orbits is absolute minimizer since their actions are all greater than the actions of the Euler solutions with the same initial ordering. This suggests that the critical angle ϕ_c described in Theorem 2.2 are most likely to fall inside $(\pi/2, \pi)$.

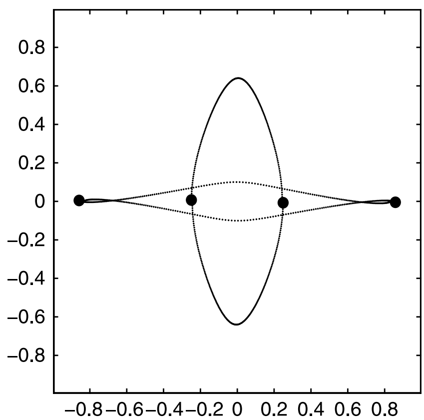
In Figure 3, except the lower right orbit, the other three orbits are "double retrograde" in the sense that their braids are constituted by two separate pairs of retrograde braids. To be more precise, their braid type is the one on the left of Figure 4. The lower right orbit, on the other hand, has the more complicated braid depicted on the right of Figure 4. This orbit periodically changes its shape between collinear and square configurations. The proof for its existence can be found in [1, 2]. The upper two orbits can be found in [3, 10]. The value of $\mathcal{A}_1$ for the Euler–Moulton solution



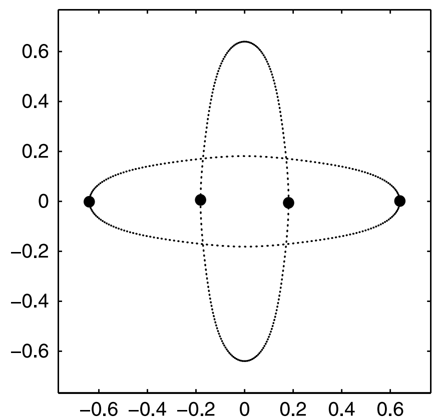
$m = (1, 1, 1, 1),\ \phi = \pi$
$x(0) = (0.64698, 0.14665, -0.14665, -0.64698)$
$\dot{x}(0) = (0.17385i, 2.4798i, -2.4798i, -0.17385i)$
$\mathcal{A}_1(x) \approx 13.560$ (Euler-Moulton $\mathcal{A}_1 \approx 14.6132$)



$m = (1, 1, 1, 1),\ \phi = 2\pi/3$
$x(0) = (0.7343, 0.26644, -0.26644, -0.7343)$
$\dot{x}(0) = (-0.0172i, 2.16063i, -2.16063i, 0.0172i)$
$\mathcal{A}_1(x) \approx 12.50$ (Euler-Moulton $\mathcal{A}_1 \approx 11.1519$)



$m = (1, 2, 2, 1),\ \phi = \pi$
$x(0) = (0.82965, 0.24415, -0.24415, -0.82965)$
$\dot{x}(0) = (-0.1286i, 2.32943i, -2.32943i, 0.1286i)$
$\mathcal{A}_1(x) \approx 25.461$ (Euler-Moulton $\mathcal{A}_1 \approx 26.9129$)



$m = (1, 1, 1, 1),\ \phi = \pi$
$x(0) = (0.63953, 0.18136, -0.18136, 0.63953)$
$\dot{x}(0) = (-0.6618i, 2.33371i, -2.33371i, 0.6618i)$
$\mathcal{A}_1(x) \approx 13.365$ (Euler-Moulton $\mathcal{A}_1 \approx 14.6132$)

FIGURE 3. Action-minimizing orbits with four bodies.

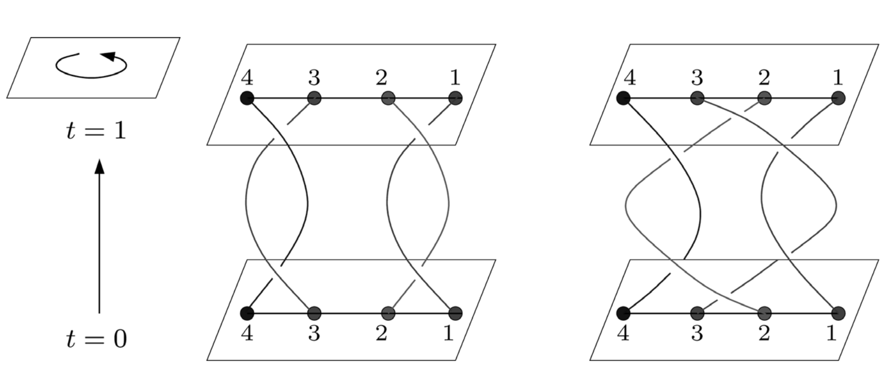


FIGURE 4. Braid types of the solutions in Figure 3.

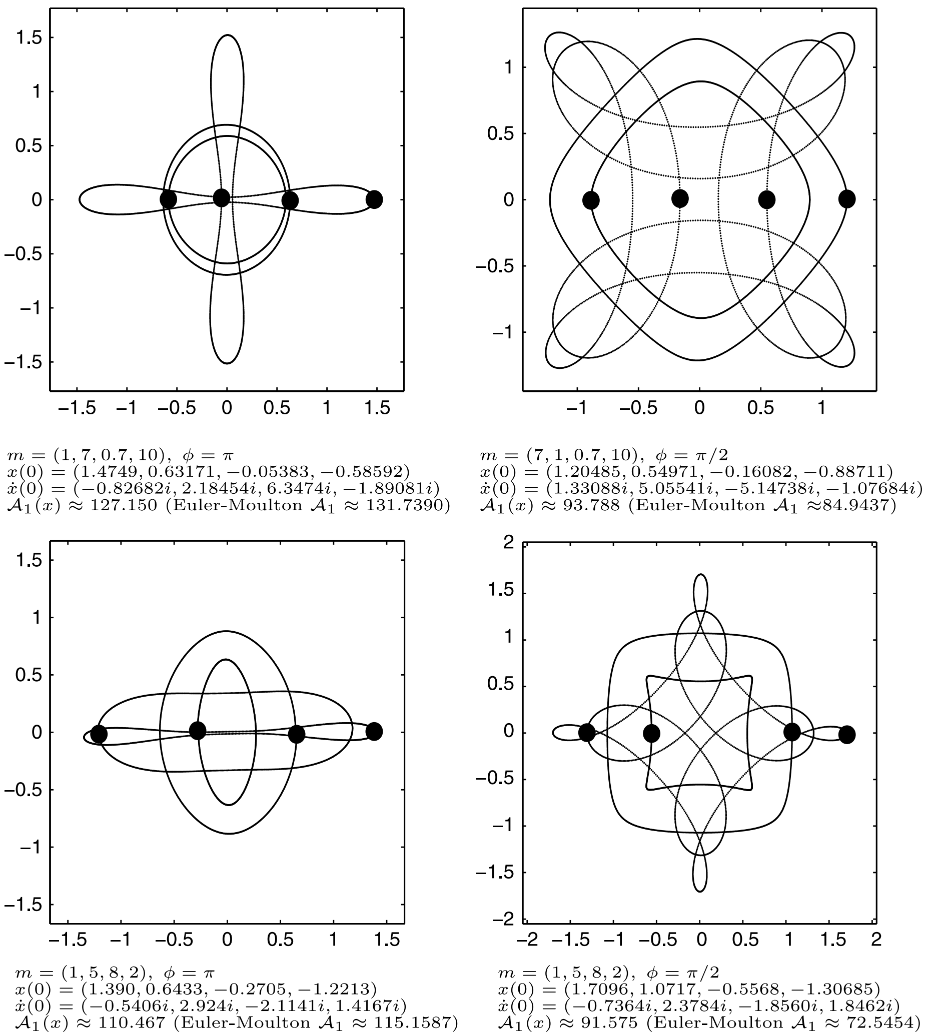


FIGURE 5. Double retrograde orbits with four distinct masses.

below the lower left orbit is already the minimum among all Euler–Moulton solutions with masses $\{1, 1, 2, 2\}$.

Figure 5 illustrates again some admissible masses described in Theorem 2.2. Assume $\phi = \pi$. For the case $\{m_1, m_2, m_3, m_4\} = \{0.7, 1, 7, 10\}$, the minimum possible action among all Euler–Moulton solutions is approximately 129.0106, which is greater than the action of the upper left orbit. As for the case $\{m_1, m_2, m_3, m_4\} = \{1, 2, 5, 8\}$, the value of $\mathcal{A}_1$ for the Euler–Moulton solution below the lower left orbit is already the minimum among Euler–Moulton solutions. Apart from $(1, 1, 1, 1)$ and all ordered quadruples for $\{1, 1, 2, 2\}$, as we saw from Figure 3, the set $\mathcal{M}$ in Theorem 2.2 may also include a neighborhood of any ordered quadruple from $\{0.7, 1, 7, 10\}$ or $\{1, 2, 5, 8\}$.

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