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SHIMURA CORRESPONDENCE FOR FINITE GROUPS

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Abstract. Let $\mathbb{Q}_{2^s}$ be the unique unramifed extension of the two-adic field $\mathbb{Q}_2$ of the degree s. Let R be the ring of integers in $\mathbb{Q}_{2^s}$ Let G be a simply connected Chevalley group corresponding to an irreducible simply laced root system. Then the finite group $G(R/4R)$ has a two-fold central extension $G'(R/4R)$ constructed by means of the Hilbert symbol on $\mathbb{Q}_{2^s}$. In this paper, we construct a natural correspondence between genuine representations of $G'(R/4R)$ and representations of the Chevalley group $G(R/2R)$.

1. Introduction

Let Φ be an irreducible simply laced root system and let $G = G_{\rm sc}$ be the simply connected Chevalley group corresponding to Φ. Let F be a p-adic field and R its ring of integers. Then $G(F)$ has a unique non-trivial central extension $G'(F)$ by $\mu_2 = \{\pm 1\}$. If p is odd, then the central extension splits (uniquely) over $G(R)$. In particular, $G(R)$ can be viewed as a subgroup of $G'(F)$. On the other hand, if $F = \mathbb{Q}_{2^s}$ then for every $n > 1$ the Hilbert symbol $(u, v)_2$ defines a nontrivial central extension $G'(R/2^nR)$ of $G(R/2^nR)$ and the inverse image of $G(R)$ is a projective limit of $G'(R/2^nR)$, for $n > 1$. Thus we are led to study genuine representations of $G'(R/4R)$ in order to understand the simplest types of genuine representations of $G'(F)$.

We now describe our results in more details. The kernel of the natural projection from $G(R/4R)$ to $G(R/2R)$ can be identified with $\mathfrak{g}(R/2R)$, the Lie algebra of G over the residual field $R/2R$. Let $\mathfrak{g}'(R/2R)$ be the preimage of $\mathfrak{g}(R/2R)$ in $G'(R/4R)$. The group commutator of any two elements in $\mathfrak{g}'(R/2R)$ is an element in μ_2 and it depends only on the projection of the two elements onto $\mathfrak{g}(R/2R)$. Thus, the commutator defines a bilinear μ_2-valued form $\omega(x, y)$ on $\mathfrak{g}(R/2R)$. Our first result is the description of this form. Let κ be the Killing form on $\mathfrak{g}_{\mathbb{Z}}$, a Chevalley lattice in $\mathfrak{g}$. Then for all $x = X \otimes u$ and $y = Y \otimes v$ in $\mathfrak{g}_{\mathbb{Z}} \otimes (R/2R) = \mathfrak{g}(R/2R)$,

$$\omega(x, y) = (1 + 2u, 1 + 2v)_2^{\kappa(X,Y)}.$$

Let Z be the kernel of the form ω. Then Z', the inverse image of Z in $\mathfrak{g}'(R/2R)$, is the center of $\mathfrak{g}'(R/2R)$. Let χ be a genuine character of Z'. It is well known that there exists a unique irreducible representation ρ_χ of $\mathfrak{g}'(R/2R)$ with the central character χ. Our second result is that the representation ρ_χ extends to a representation of $G'(R/4R)$, denoted by ρ'_χ. This extension is unique unless $G'(R/4R) = \mathrm{SL}'(\mathbb{Z}/4\mathbb{Z})$. Now the classification of genuine representations of $G'(R/4R)$ is easy. Indeed, since Z' is contained in the center of $G'(R/4R)$ any irreducible representation π of $G'(R/4R)$,

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when restricted to $\mathfrak{g}'(R/2R)$, is a multiple of ρ_χ for some character χ of Z'. Thus one can canonically write

$$\pi = \mathrm{Hom}_{\mathfrak{g}'(R/2R)}(\rho'_\chi, \pi) \otimes \rho'_\chi,$$

where $G'(R/4R)$ acts on $T \in \mathrm{Hom}_{\mathfrak{g}'(R/2R)}(\rho'_\chi, \pi)$ by $\pi(g) \circ T \circ \rho'_\chi(g^{-1})$. Since T intertwines the action of $\mathfrak{g}'(R/2R)$, this action descends to $G(R/2R)$. In this way, we have constructed a correspondence (in fact a functor) between representations of $G'(R/4R)$ on which Z' acts by the genuine character χ and representations of $G(R/2R)$. This correspondence gives a bijection between equivalence classes of irreducible representations.

2. Finite Chevalley groups

Let $(\alpha|\beta)$ denote the inner product on Φ normalized such that $(\alpha|\alpha) = 2$ for long roots. Co-roots can be identified with $\alpha^\vee := \frac{2\alpha}{(\alpha|\alpha)}$. Since Φ is simply laced, $\alpha^\vee = \alpha$. In particular, we can identify the root and the co-root lattices.

The root system Φ defines a split, simple Lie algebra $\mathfrak{g}$ over $\mathbb{Z}$. More precisely, we have a Chevalley lattice

$$\mathfrak{g}_{\mathbb{Z}} = X \oplus_{\alpha\in\Phi} \mathbb{Z} \cdot E_\alpha,$$

where X is the co-root lattice. The co-roots, considered as elements in the Chevalley lattice, will be denoted by H_α.

We can define an invariant (Killing) form on $\mathfrak{g}$ by

$$\begin{cases} \kappa(H_\alpha, H_\beta) = (\alpha^\vee|\beta^\vee), \\ \kappa(E_\alpha, E_{-\alpha}) = 1, \end{cases}$$

and 0 for any other combinations of Chevalley generators as entries of κ. Let $\mathfrak{g}(\mathbb{Z}/2\mathbb{Z})$ denote the Lie algebra over the finite field $\mathbb{Z}/2\mathbb{Z}$. Note that $\mathfrak{g}(\mathbb{Z}/2\mathbb{Z})$ is simply obtained by reducing the Chevalley lattice modulo 2. The Killing form κ can now be viewed as an invariant form on $\mathfrak{g}(\mathbb{Z}/2\mathbb{Z})$ with values in $\mathbb{Z}/2\mathbb{Z}$. Note that the kernel of κ is equal to the kernel of the restriction of κ to $X/2X$. This kernel is trivial if and only if the determinant of the Cartan matrix of the root system is odd.

Let $G = G_{\mathrm{sc}}$ be the simply connected Chevalley group corresponding to the root system Φ. By fixing the Chevalley lattice, we have also fixed a structure of G as a group scheme over $\mathbb{Z}$. Recall that there is a maximal, split torus T in G preserving root spaces in $\mathfrak{g}$ under the adjoint action. If A is a ring, then $T(A) \cong X \otimes_{\mathbb{Z}} A^\times$. We shall also need the adjoint group G_{ad}. Let T_{ad} be the maximal split torus in G_{ad}. Then $T_{\mathrm{ad}}(A) \cong Y \otimes_{\mathbb{Z}} A^\times$, where Y is the co-character lattice of T_{ad}. In the simply laced case Y is the dual lattice to X with respect to the product $(\alpha|\beta)$.

We shall be mostly interested in the case $A = R/4R$, where R is the ring of integers in $\mathbb{Q}_{2^s}$. Since $R/4R$ is a local ring the group $G(R/4R)$ is generated by one-parameter subgroups $U_\alpha \simeq R/4R$ for all α in Φ (see [1], Proposition 1.6). The choice of Chevalley basis fixes an isomorphism of $R/4R$ and U_α, $u \mapsto e_\alpha(u)$ for every $\alpha \in \Phi$. For example, if $G = \mathrm{SL}_2$ then $e_\alpha(u)$ and $e_{-\alpha}(u)$ are

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

For every v in $(R/4R)^\times$ define elements

$$\begin{cases} w_\alpha(v) = e_\alpha(v)\, e_{-\alpha}(-v^{-1})\, e_\alpha(v), \\ h_\alpha(v) = w_\alpha(v)\, w_\alpha(-1). \end{cases}$$

If $G = \mathrm{SL}_2$ then $w_\alpha(v)$ and $h_\alpha(v)$ are

$$\begin{pmatrix} 0 & v \\ -v^{-1} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}.$$

If $\Phi \neq A_1$, by a result of Stein ([8], Corollary 2.14), the group $G(R/4R)$ is abstractly generated by the one-parameter groups U_α modulo the relations

$$[e_\alpha(u), e_\beta(v)] = \begin{cases} e_{\alpha+\beta}(\pm uv), & \text{if } \alpha+\beta \text{ is a root,} \\ 1, & \text{if not, and } -\alpha \neq \beta. \end{cases} \tag{2.1}$$

and

$$h_\alpha(u)h_\alpha(v) = h_\alpha(uv). \tag{2.2}$$

The group $G(R/4R)$ has a two-step filtration with $G(R/2R)$ as a quotient and a subgroup isomorphic to $\mathfrak{g}(R/2R) = \mathfrak{g}_{\mathbb{Z}} \otimes R/2R$. This isomorphism is explicitly given by

$$\begin{cases} h_\alpha(1+2u) \mapsto H_\alpha \otimes u, \\ \quad e_\alpha(2u) \mapsto E_\alpha \otimes u. \end{cases}$$

Note that the relation (2.1) implies that the groups $G(R/4R)$ and $G(R/2R)$ are perfect if $\Phi \neq A_1$. The relation $[h_\alpha(v), e_\alpha(u)] = e_\alpha((v^2-1)u)$ implies that $\mathrm{SL}_2(R/4R)$ and $\mathrm{SL}_2(R/2R)$ are also perfect if $|R/2R| > 2$.

3. Central extensions

Assume that $\Phi \neq A_1$. Since the group $G(R/4R)$ is perfect, it has a universal central extension. The universal central extension (with some low rank exceptions) is given by the Steinberg group $G''(R/4R)$. The group $G''(R/4R)$ is generated by elements $e''_\alpha(u)$, for all $u \in R/4R$ and $\alpha \in \Phi$, satisfying $e''_\alpha(u)e''_\alpha(v) = e''_\alpha(u+v)$ and the relation (2.1). Define $h''_\alpha(v)$ in $G''(R/4R)$ in the same way as $h_\alpha(v)$ was defined in $G(R/4R)$. Then $h''_\alpha(v)$ do not necessarily satisfy the relation (2.2). Thus the Steinberg symbol $(u,v)_S$ is defined as the obstruction to the relation (2.2):

$$(u,v)_S = h''_\alpha(u)h''_\alpha(v)h''_\alpha(uv)^{-1}.$$

The symbol does not depend on the choice of the root α. It is a central element in $G''(R/4R)$. The elements $(1+2v, 1+2u)_S$ are of order at most 2 and generate the kernel of the projection of $G''(R/4R)$ onto $G(R/4R)$ ([8], Theorem 3.10).

Let $(u,v)_2$ be the Hilbert symbol on $\mathbb{Q}_{2^s}$. When restricted to $(1+2R)\times(1+2R)$, the kernel of the Hilbert symbol is $1+4R$ and, by passing to the quotient $1+2R/1+4R \cong R/2R$, the symbol induces a non-degenerate bilinear form on $R/2R$. The group $G(R/4R)$ has a non-trivial central extension by $\mu_2 = \{\pm 1\}$, denoted by $G'(R/4R)$ obtained by specializing the Steinberg symbol to the Hilbert symbol:

$$(u,v)_S \mapsto (u,v)_2.$$

Let $e'_\alpha(u)$ and $h'_\alpha(v)$ in $G'(R/4R)$ be the projections of $e''_\alpha(u)$ and $h''_\alpha(v)$ in $G''(R/4R)$, respectively. The elements $e'_\alpha(u)$ satisfy the relation (2.1). However, the relation (2.2) is replaced by

$$h'_\alpha(u)h'_\alpha(v) = h'_\alpha(uv) \cdot (u,v)_2.$$

The elements $h'_\alpha(u)$ and $h'_\beta(v)$ generally do not commute. Their commutator is

$$[h'_\alpha(u), h'_\beta(v)] = (u,v)_2^{(\alpha^\vee|\beta^\vee)}.$$

Define $\mathrm{SL}'_2(R/4R)$ as a subgroup of $G'(R/4R)$ generated by elements $e'_\alpha(u)$ and $e'_{-\alpha}(u)$, for all $u \in R/4R$ and α one fixed root. This definition does not depend on the choice of the root system $\Phi \neq A_1$ and the root α in Φ. In this way, we have defined $G'(R/4R)$ for all simply laced root systems including A_1.

The group $G'(R/4R)$ is perfect unless $G'(R/4R) = \mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$, for the same reason as $G(R/4R)$. The conjugation action of $G'(R/4R)$ on $G'(R/4R)$ descends down to an action of $G(R/4R)$ on $G'(R/4R)$. In fact, an element $t = \lambda \otimes v$ in $T(R/4R) = X \otimes (\mathbb{Z}/4\mathbb{Z})^\times$ acts on the generating elements of $G'(R/4R)$ by

$$te'_\alpha(u)t^{-1} = e'_\alpha(v^{(\lambda|\alpha)}u). \tag{3.1}$$

Moreover, since the formula (3.1) makes sense for any $t = \lambda \otimes s$ in $T_{\mathrm{ad}}(4) = Y \otimes (\mathbb{Z}/4\mathbb{Z})^\times$, the adjoint group $G_{\mathrm{ad}}(R/4R)$ acts on $G'(R/4R)$.

We have the following diagram of groups:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & G(R/2R) & & G(R/2R) & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \to & \mu_2 & \to & G'(R/4R) & \to & G(R/4R) & \to & 1 \\
 & & & & \uparrow & & \uparrow & & \\
1 & \to & \mu_2 & \to & \mathfrak{g}'(R/2R) & \to & \mathfrak{g}(R/2R) & \to & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

We now describe the central extension $\mathfrak{g}'(R/2R)$ of $\mathfrak{g}(R/2R)$ appearing in the diagram. (See [3] and [6] for more on the subject of extensions of elementary two-groups.) We can define a symplectic form ω on $\mathfrak{g}(R/2R)$ with values in μ_2 by

$$\omega(x,y) = [x', y'],$$

where x' and y' are any two elements in $\mathfrak{g}'(R/2R)$ that project to x and y, respectively, and $[x', y']$ denotes the group commutator.

Proposition 3.1. *Let κ be the Killing form on $\mathfrak{g}_\mathbb{Z}$. Then, for any two elements $X \otimes u$ and $Y \otimes v$ in $\mathfrak{g}_\mathbb{Z} \otimes R/2R = \mathfrak{g}(R/2R)$,*

$$\omega(X \otimes u, Y \otimes v) = (1+2u, 1+2v)_2^{\kappa(X,Y)}.$$

Proof. There are several cases to consider. Assume first that $x = H_\alpha \otimes u$ and $y = H_\beta \otimes v$. We can take $x' = h'_\alpha(1+2u)$ and $y' = h'_\beta(1+2v)$. Since

$$[h'_\alpha(1+2u), h'_\beta(1+2v)] = (1+2u, 1+2v)_2^{(\alpha^\vee|\beta^\vee)},$$

this case has been checked. Next, assume that $x = E_\alpha \otimes u$ and $y = E_\beta \otimes v$ where $\alpha \neq -\beta$. Then $\kappa(E_\alpha, E_\beta) = 0$. We can take $x' = e'_\alpha(2u)$ and $y' = e'_\beta(2v)$. Since

$$[e'_\alpha(2u), e'_\beta(2v)] = e'_{\alpha+\beta}(\pm 4uv) = 1$$

in $G'(R/4R)$, this case has been also checked. If $\beta = -\alpha$, then this is Corollary 2.9 in [8]. The remaining cases are trivial. □

Let Z be the kernel of the form ω. In order to describe Z, it suffices to describe the kernel of the Killing form considered modulo 2. Recall that the kernel of the Killing form on $\mathfrak{g}(\mathbb{Z}/2\mathbb{Z})$ is equal to the kernel of the Killing form restricted on $X/2X$. Let $\hat{X}$ be a lattice, $X \subseteq \hat{X} \subseteq Y$, such that $\hat{X}/X$ is the two-torsion in Y/X. Since Y is dual to X, it follows that $2\hat{X}/2X$ is the kernel of the Killing form. It follows that

$$Z \cong (2\hat{X}/2X) \otimes (1 + 2R/1 + 4R) \cong (2\hat{X}/X) \otimes (R/2R).$$

Proposition 3.2. *Let Z' be the center of the nilpotent group $\mathfrak{g}'(R/2R)$.*

(1) *The group Z' is the preimage of Z, the kernel of ω.*
(2) *The group Z' is contained in the center of $G'(R/4R)$.*
(3) *The adjoint action of $T_{\mathrm{ad}}(R/4R)$ on Z' induces a transitive action on the set of genuine characters of Z'.*

Proof. The first statement is obvious. Let t in $T'(R/4R)$ such that its image in $T(R/4R)$ is $t = \lambda \otimes v$. Then the conjugation action of t on a generating element $e'_\alpha(u)$ is given by

$$te'_\alpha(u)t^{-1} = e'_\alpha(v^{(\lambda|\alpha)}u).$$

If t is in Z' then $\lambda \in 2\hat{X}$ and $v \in 1 + 2R$. But then $v^{(\lambda|\alpha^\vee)} = 1$ and Z' is in the center of $G'(R/4R)$. This proves the second statement. Finally, the conjugation action of an element $t = \lambda \otimes v$ in $T_{\mathrm{ad}}(R/4R) \cong Y \otimes (R/4R)^\times$ on $h'_\alpha(u)$ is given by

$$th'_\alpha(v)t^{-1} = h'_\alpha(v) \cdot (v, u)_2^{(\lambda|\alpha^\vee)}.$$

Since the lattice Y is dual to X this formula shows that the group $T_{\mathrm{ad}}(R/4R)$ acts transitively on the set of genuine characters of Z'. □

4. Main results

Fix a genuine character χ of Z', the center of $\mathfrak{g}'(R/2R)$. Let $S \subseteq \mathfrak{g}(R/2R)$ be a maximal subspace such that the bilinear form ω is trivial on S. Let $S' \subseteq \mathfrak{g}'(R/2R)$ be the inverse image of S. Then S' is a maximal abelian subgroup of $\mathfrak{g}'(R/2R)$ and χ extends (in more than one way) to a character of S'. Let χ_S be one extension. Define

$$\rho_\chi = \mathrm{Ind}_{S'}^{\mathfrak{g}'(R/2R)}(\chi_S).$$

It is not difficult to see that ρ_χ does not depend on the choice of χ_S and that it is the unique irreducible representation of $\mathfrak{g}'(R/2R)$ with the central character χ. Indeed, the restriction of ρ_χ to S' is the sum of all characters of S' extending χ, thus the claim follows from the Frobenius reciprocity and Mackey's irreducibility criterion. We note that the square of the dimension of ρ_χ is

$$\dim(\rho_\chi)^2 = \frac{|\mathfrak{g}'(R/2R)|}{|Z'|} = \frac{|\mathfrak{g}(R/2R)|}{|Z|},$$

where, we remind the reader, Z is the kernel of the pairing ω on $\mathfrak{g}(R/2R)$. In the following table we give the size of Z:

Φ	A_{2n-1}	A_{2n}	D_{2n-1}	D_{2n}	E_6	E_7	E_8
$\lvert Z\rvert$	1	2^s	2^s	2^{2s}	1	2^s	1

Proposition 4.1. *The representation ρ_χ of $\mathfrak{g}'(R/2R)$ extends to a representation of $G'(R/4R)$. This extension is denoted by ρ'_χ. The extension is unique unless $G(R/4R) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$.*

Proof. We note that $G(R/4R)$ acts by conjugation on irreducible representations of $\mathfrak{g}'(R/2R)$. Since the isomorphism class of ρ_χ depends on the central character χ, and Z' is central in $G'(R/4R)$, we see that the conjugation by $G(R/4R)$ does not change the isomorphism class of ρ_χ. In particular, ρ_χ gives rise to a projective representation of $G(R/4R)$. By Theorem 2.13 in [8], the Steinberg group $G''(R/4R)$ is universal if the rank of Φ is at least 5. Assume this. Then the projective representation lifts to a representation of $G''(R/4R)$. Let ρ'_χ denote this representation. We claim that this representation descends to $G'(R/4R)$. Indeed, for every $u \in R$, $e''_\alpha(2u) \in G''(R/4R)$ and $e'_\alpha(2u) \in \mathfrak{g}'(R/2R)$ are a scalar multiple of each other (when acting by ρ'_χ and ρ_χ respectively). In particular, their commutators coincide. Since

$$(1+2u, 1+2v)_S = [e''_\alpha(2u), e''_{-\alpha}(2v)] = [e'_\alpha(2u), e'_{-\alpha}(2v)] = (1+2u, 1+2v)_2,$$

we see that the Steinberg symbol acts through its Hilbert specialization, i.e., $G''(R/4R)$ acts through its quotient $G'(R/4R)$. Note, however, that the restriction of ρ'_χ to $\mathfrak{g}'(R/2R)$ is not necessarily isomorphic to ρ_χ. It may be isomorphic to a twist of ρ_χ by a character of $\mathfrak{g}(R/2R)$. Any such twist is isomorphic to $\rho_{\chi'}$ for a (possibly) different genuine character χ' of Z'. Since the maximal torus $T_{\mathrm{ad}}(R/4R)$ of the adjoint group acts transitively on the set of all genuine characters of Z', we can conjugate by an element in $T_{\mathrm{ad}}(R/4R)$, if necessary, to construct an extension of ρ_χ to $G'(R/4R)$. The uniqueness of extension is clear since $G'(R/4R)$ is perfect unless $G'(R/4R) = \mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$.

To deal with low rank groups of type A_m and D_m we proceed as follows. Let $\Phi_0 \subseteq \Phi$ be a root subsystem of the same type as Φ but of the rank $m-1$. Fix Φ^+, a set of positive roots. Let $P = MU$ be a maximal parabolic subgroup of G such that U is generated by the root groups U_α for $\alpha \in \Phi^+ \setminus \Phi_0$. Then $G_0 = [M, M]$ is a simply connected Chevalley group corresponding to Φ_0.

Let Z'_0 be the center of $\mathfrak{g}'_0(R/2R)$. If m is odd then $Z' \subseteq Z'_0$ and every genuine character χ of Z' is the restriction of 2^s genuine characters $\chi_1, \dots, \chi_{2^s}$ of Z'_0. Let U_2 be the subgroup of $G'(R/4R)$ generated $e'_\alpha(2u)$ for $\alpha \in \Phi^+ \setminus \Phi_0$. Let $\rho_\chi^{U_2}$ be the subspace of U_2-fixed vectors in ρ_χ. Then, as representations of $\mathfrak{g}'_0(R/2R)$,

$$\rho_\chi^{U_2} \cong \rho_{\chi_1} \oplus \cdots \oplus \rho_{\chi_{2^s}}.$$

Thus, if ρ_χ extends to a representation of $G'(R/4R)$ then, by taking U_2-fixed vectors, $G'_0(R/4R)$ acts naturally on $\rho_{\chi_1}, \dots, \rho_{\chi_{2^s}}$. Similarly, if m is even then $Z'_0 \subseteq Z'$ and every genuine character χ of Z' is the restriction of 2^s genuine characters $\chi_1, \dots, \chi_{2^s}$

of Z'_0. Then, as representations of $\mathfrak{g}'_0(R/2R)$,

$$\rho_{\chi_1}^{U_2} \cong \cdots \cong \rho_{\chi_{2^s}}^{U_2} \cong \rho_\chi,$$

and $G'_0(R/4R)$ acts naturally on ρ_χ. □

If σ is a representation of $G(R/2R)$ then, after inflating σ to $G'(R/4R)$, $\sigma \otimes \rho'_\chi$ is a genuine representation of $G'(R/4R)$. Clearly, $\sigma \otimes \rho'_\chi$ is an irreducible representation of $G'(R/4R)$ if and only if σ is an irreducible representation of $G(R/2R)$.

Theorem 4.1. *Let χ be a genuine character of Z', the center of $\mathfrak{g}'(R/2R) \subseteq G'(R/4R)$. The map $\sigma \mapsto \sigma \otimes \rho'_\chi$ gives a one to one correspondence between isomorphism classes of irreducible representations of $G(R/2R)$ and irreducible representations of $G'(R/4R)$ such that Z' acts by the character χ.*

Proof. Let π be an irreducible representation of $G'(R/4R)$ such that Z' acts by the character χ. Clearly, the restriction of π to $\mathfrak{g}'(R/2R)$ is a multiple of ρ_χ. Let

$$\sigma = \mathrm{Hom}_{\mathfrak{g}'(R/2R)}(\rho'_\chi, \pi).$$

Note that σ is naturally a $G'(R/4R)$-module with the action of g in $G'(R/4R)$ given by

$$\sigma(g)(T) = \pi(g) \circ T \circ \rho'_\chi(g^{-1}),$$

for every T in σ. Since T intertwines the action of $\mathfrak{g}'(R/2R)$, σ descends to a representation of $G(R/2R)$. Moreover, the natural map $T \otimes w \mapsto T(w)$ gives an isomorphism of $\sigma \otimes \rho'_\chi$ and π. The theorem is proved. □

We finish this paper with a remark on the relevance of our results to the local Shimura correspondence for two-adic groups. Let Z_2 be the algebraic subgroup of G defined as the two-torsion of the center of G. Let $\hat{G}$ be the algebraic group defined as the quotient of G by Z_2. The co-character lattice of $\hat{G}$ is $\hat{X}$. We remind the reader that $X \subseteq \hat{X} \subseteq Y$ and $\hat{X}/X$ is the two-torsion in Y/X. In particular, if $\hat{T}$ is a maximal torus of $\hat{G}$ then $\hat{T}(F) \cong \hat{X} \otimes F^\times$. One expects that there is a Shimura correspondence between genuine representations of $G'(\mathbb{Q}_{2^s})$ and representations of the linear group $\hat{G}(\mathbb{Q}_{2^s})$, by analogy with real groups [2] and p-adic groups with p odd [7]. Under this correspondence, once we have fixed the genuine character χ of Z', irreducible generic representations of $G'(\mathbb{Q}_{2^s})$ containing the type ρ'_χ should correspond to irreducible unramifed representations of $\hat{G}(\mathbb{Q}_2)$. More generally, irreducible genuine representations of $G'(\mathbb{Q}_{2^s})$ containing the type $\rho'_{\chi'}$ (now χ' is any genuine character of Z') should correspond to irreducible unramifed representations of $\hat{G}(\mathbb{Q}_2)$ containing a one-dimensional type of conductor 4 depending on χ'. More precisely, let $\hat{G}(R)^{\mathrm{der}}$ be the derived subgroup of $\hat{G}(R)$. Then

$$\hat{G}(R)/\hat{G}(R)^{\mathrm{der}} \cong (\hat{X}/X) \otimes R^\times.$$

Since $\hat{X}/X$ is two-torsion, any character μ of $\hat{G}(R)$ is necessarily quadratic. Since $(R/2R)^\times$ is odd, μ is determined by its restriction on $(\hat{X}/X) \otimes (1 + 2R)$. We say that

μ is of conductor 4 if it is trivial on $(\hat{X}/X) \otimes (1+4R)$. Thus quadratic characters of $\hat{G}(R)$ of conductor 4 correspond to characters of

$$(\hat{X}/X) \otimes (1+2R/1+4R) \cong (\hat{X}/X) \otimes (R/2R) \cong Z.$$

Thus, irreducible genuine representations of $G'(\mathbb{Q}_{2^s})$ containing the type $\rho'_{\chi'}$ for should correspond to irreducible unramifed representations of $\hat{G}(\mathbb{Q}_2)$ containing a one-dimensional type μ where, abusing the notation, $\chi' = \chi \cdot \mu$.

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