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COUNTING FUNCTION OF THE EMBEDDED EIGENVALUES FOR SOME MANIFOLD WITH CUSPS, AND MAGNETIC LAPLACIAN

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Abstract. We consider a non-compact, complete manifold $\mathbf{M}$ of finite area with cuspidal ends. The generic cusp is isomorphic to $\mathbf{X} \times]1, +\infty[$ with metric $ds^2 = (h+dy^2)/y^{2\delta}$. $\mathbf{X}$ is a compact manifold equipped with the metric h. For a one-form A on $\mathbf{M}$ such that in each cusp A is a non-exact one-form on the boundary at infinity, we prove that the magnetic Laplacian $-\Delta_A = (id+A)^\star(id+A)$ satisfies the Weyl asymptotic formula with sharp remainder. We deduce an upper bound for the counting function of the embedded eigenvalues of the Laplace–Beltrami operator $-\Delta = -\Delta_0$.

1. Introduction

We consider a smooth, connected n-dimensional Riemannian manifold $(\mathbf{M}, \mathbf{g}), (n \geq 2)$, such that

$$\mathbf{M} = \bigcup_{j=0}^{J} \mathbf{M}_j \quad (J \geq 1), \tag{1.1}$$

where the $\mathbf{M}_j$ are open sets of $\mathbf{M}$. We assume that the closure of $\mathbf{M}_0$ is compact and that the other $\mathbf{M}_j$ are cuspidal ends of $\mathbf{M}$.

This means that $\mathbf{M}_j \cap \mathbf{M}_k = \emptyset$, if $1 \leq j < k$, and that there exists, for any j, $1 \leq j \leq J$, a closed compact $(n-1)$-dimensional Riemannian manifold $(\mathbf{X}_j, \mathbf{h}_j)$ such that $\mathbf{M}_j$ is isometric to $\mathbf{X}_j \times]a_j^2, +\infty[\,, (a_j > 0)$ equipped with the metric

$$ds_j^2 = y^{-2\delta_j}(\mathbf{h}_j \ + \ dy^2\ ); \quad (1/n < \delta_j \leq 1). \tag{1.2}$$

So there exists a smooth real one-form $A_j \in T^\star(\mathbf{X}_j)$, non-exact, such that

$$\begin{cases} \text{(i) } dA_j \neq 0 \\ \text{or} \\ \text{(ii) } dA_j = 0 \text{ and } [A_j] \text{ is not integer.} \end{cases} \tag{1.3}$$

In (ii), we mean that there exists a smooth closed curve γ in $\mathbf{X}_j$ such that

$$\int_\gamma A_j \ \notin \ 2\pi\mathbb{Z}.$$

Then one can always find a smooth real one-form $A \in T^\star(\mathbf{M})$ such that

$$\forall\, j,\ 1 \leq j \leq J, \quad A = A_j \quad \text{on } \ \mathbf{M}_j. \tag{1.4}$$

We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A = (i\ d + A)^\star (i\ d + A), \tag{1.5}$$

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$(i = \sqrt{-1}, (i\ d + A)u = i\ du + uA,\ \forall\ u\ \in\ C_0^\infty(\mathbf{M};\mathbb{C})$, the upper star, *, stands for the adjoint between the square-integrable one-forms and $L^2(\mathbf{M})$), so $d^\star(Z)$ is the usual Hodge–de Rham codifferential, and

$$A^\star(Z) =< A; Z >_{T^\star\mathbf{M}},\ \forall Z \in \Lambda_0^1(\mathbf{M}),$$

where $\Lambda_0^1(\mathbf{M})$ denotes the vector space of smooth one-forms with compact support.

As $\mathbf{M}$ is a complete metric space, by Hopf–Rinow theorem $\mathbf{M}$ is geodesically complete, so it is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M};\mathbb{C})$, the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant: for any $f \in C^1(\mathbf{M};\mathbb{R}), -\Delta_A$ and $-\Delta_{A+df}$ are unitary equivalent, hence they have the same spectrum.

For a self-adjoint operator P on a Hilbert space H,

$$\mathrm{sp}(P),\ \mathrm{sp}_{\mathrm{ess}}(P),\ \mathrm{sp}_{\mathrm{p}}(P),\ \mathrm{sp}_{\mathrm{d}}(P)$$

will denote respectively the spectrum, the essential spectrum, the point spectrum and the discrete spectrum of P. We recall that

$$\mathrm{sp}(P) = \mathrm{sp}_{\mathrm{ess}}(P) \cup \mathrm{sp}_{\mathrm{d}}(P),\ \mathrm{sp}_{\mathrm{d}}(P) \subset \mathrm{sp}_{\mathrm{p}}(P)\ \text{ and }\ \mathrm{sp}_{\mathrm{ess}}(P) \cap \mathrm{sp}_{\mathrm{d}}(P) = \emptyset.$$

Theorem 1.1. *Under the above assumptions on* $\mathbf{M}$*, the essential spectrum of the Laplace–Beltrami operator on* $\mathbf{M}$*,* $-\Delta = -\Delta_0$ *is given by*

$$\begin{cases} \mathrm{sp}_{\mathrm{ess}}(-\Delta) = [0, +\infty[, & \text{if} \quad 1/n < \delta < 1, \\ \mathrm{sp}_{\mathrm{ess}}(-\Delta) = [\frac{(n-1)^2}{4}, +\infty[, & \text{if} \quad \delta = 1. \end{cases} \tag{1.6}$$

($\delta = \min_{1\le j\le J} \delta_j$ *).*

When (1.3) and (1.4) are satisfied, the magnetic Laplacian $-\Delta_A$ *has a compact resolvent. The spectrum* $\mathrm{sp}(-\Delta_A) = \mathrm{sp}_{\mathrm{d}}(-\Delta_A)$ *is a sequence of non-decreasing eigenvalues* $(\lambda_j)_{j\in\mathbb{N}}, \lambda_j \le \lambda_{j+1}, \lim\limits_{j\to+\infty} \lambda_j = +\infty$*, such that the sequence of normalized eigenfunctions* $(\varphi_j)_{j\in\mathbb{N}}$ *is a Hilbert basis of* $L^2(\mathbf{M})$*. Moreover* $\lambda_0 > 0$. *(*$\mathbb{N}$ *denotes the set of natural numbers.)*

This theorem is not new. The case $A = 0$ was proved in [Don2], and the other case in [Go-Mo], but in the two cases, for a wider class of Riemann metrics. We will give a short proof for our simple class of Riemann metrics, by following the classical method used in [Don1, Don2, Do-Li].

For any self-adjoint operator P with compact resolvent, and for any real $\lambda, N(\lambda, P)$ will denote the number of eigenvalues, (repeated according to their multiplicity), of P less then λ,

$$N(\lambda, P) = \mathrm{trace}\ (\chi_{]-\infty,\lambda[}(P)), \tag{1.7}$$

(for any $I \subset \mathbb{R},\ \chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ if $x \in \mathbb{R} \setminus I$).

The asymptotic behavior of $N(\lambda, -\Delta_A)$ satisfies the Weyl formula with the following sharp remainder.

Theorem 1.2. *Under the above assumptions on* $\mathbf{M}$ *and on* A*, we have the Weyl formula with remainder as* $\lambda \to +\infty$*,*

$$N(\lambda, -\Delta_A) = |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2}\ +\ \mathbf{O}(\mathbf{r}(\lambda)), \tag{1.8}$$

with

$$r(\lambda) = \begin{cases} \lambda^{(n-1)/2}\ln(\lambda), & \text{if} \quad 1/(n-1) \le \delta, \\ \lambda^{1/(2\delta)}, & \text{if} \quad 1/n < \delta < 1/(n-1), \end{cases} \tag{1.9}$$

$\delta = \min_{1\le j\le J}\delta_j$, $|\mathbf{M}|$ *is the Riemannian measure of* $\mathbf{M}$ *and* ω_d *is the euclidian volume of the unit ball of* $\mathbb{R}^d$, $\omega_d = \dfrac{\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}$.

The asymptotic formula (1.8) without remainder is given in [Go-Mo], and with remainder but only for $n=2$ (and $\delta_j = 1$ for any $1 \le j \le J$) in [Mo-Tr].

The Laplace–Beltrami operator $-\Delta = -\Delta_0$ may have embedded eigenvalues in its essential spectrum $\mathrm{sp}_{\mathrm{ess}}(-\Delta)$. Let $N_{\mathrm{ess}}(\lambda, -\Delta)$ denote the number of eigenvalues of $-\Delta$, (counted according to their multiplicity), less then λ.

Theorem 1.3. *There exists a constant* $C_{\mathbf{M}}$ *such that, for any* $\lambda \gg 1$,

$$N_{\mathrm{ess}}(\lambda, -\Delta) \;\le\; |\mathbf{M}|\frac{\omega_n}{(2\pi)^n}\lambda^{n/2} \;+\; C_{\mathbf{M}} r_0(\lambda)\,, \tag{1.10}$$

with $r_0(\lambda)$ *defined by*

$$r_0(\lambda) = \begin{cases} \lambda^{\frac{n-1}{2}}\ln(\lambda), & \text{if} \quad 2/n \le \delta \le 1, \\ \lambda^{\frac{n-(n\delta-1)}{2}}, & \text{if} \quad 1/n < \delta < 2/n, \end{cases} \tag{1.11}$$

δ *is the one defined in Theorem 1.2.*

The above upper bound proves that any eigenvalue of $-\Delta$ has finite multiplicity. There exist shorter proofs of the multiplicity, see for example [Don1] or Lemma B1 in [Go-Mo].

The estimate (1.10) is sharp when $n = 2$. There exist hyperbolic surfaces $\mathbf{M}$ of finite area so that

$$N_{\mathrm{ess}}(\lambda, -\Delta) \;=\; |\mathbf{M}|\frac{\omega_2}{(2\pi)^2}\lambda + \Gamma_{\mathbf{M}}\lambda^{1/2}\ln(\lambda) + \mathbf{O}(\lambda^{1/2}),$$

for some constant $\Gamma_{\mathbf{M}}$. See [Mul] for such examples.

Still in the case of surfaces, a compact perturbation of the metric of non-compact hyperbolic surface $\mathbf{M}$ of finite area can destroy all embedded eigenvalues, see [Col1].

For the proof of Theorem 1.2, we will follow the standard method of partitioning $\mathbf{M}$ and using min–max principle to estimate the number of eigenvalues by the sum of the ones of the Dirichlet operators and Neumann operators associated to the partition. In a cusp partition, we will diagonalize $-\Delta_A$ to an infinite sum of Schrödinger operators in a half-line, and then we can use standard estimates of the number of eigenvalues for those Schrödinger operators.

For the proof of Theorem 1.3, we will prove that Theorem 1.2 is still valid when one changes A into $\lambda^{-\rho}A$, for some one-form A. Then we will show that the number of embedded eigenvalues of $-\Delta$ less than λ is bounded above by the number of eigenvalues of $-\Delta_{(\lambda^{-\rho}A)}$ less than λ.

2. Proofs

Since by the Persson [Per] argument used in [Do-Li], the essential spectrum of an elliptic operator on a manifold is invariant by compact perturbation of the manifold, (see also Proposition C3 in [Go-Mo]), we can write

$$\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \bigcup_{j=1}^{J} \mathrm{sp}_{\mathrm{ess}}\left(-\Delta_A^{\mathbf{M}_j,D}\right), \tag{2.1}$$

where $-\Delta_A^{\mathbf{M}_j,D}$ denotes the self-adjoint operator on $L^2(\mathbf{M}_j)$ associated to $-\Delta_A$ with Dirichlet boundary conditions on the boundary $\partial\mathbf{M}_j$ of $\mathbf{M}_j$.

2.1. Diagonalization of the magnetic Laplacian. Let us consider a cusp $\mathbf{M}_j = \mathbf{X}_j \times]a_j^2, +\infty[$ equipped with the metric (1.2). Then for any $u \in C^2(\mathbf{M}_j)$,

$$-\Delta_A u = -y^{2\delta_j}\Delta_{A_j}^{\mathbf{X}_j} u \; - \; y^{n\delta_j}\partial_y(y^{(2-n)\delta_j}\partial_y u), \tag{2.2}$$

where $\Delta_{A_j}^{\mathbf{X}_j}$ is the magnetic Laplacian on $\mathbf{X}_j$: if for local coordinates $\mathbf{h}_j = \sum_{k,\ell} G_{k\ell}\ dx_k dx_\ell$ and $A_j = \sum_{k=1}^{n-1} a_{j,k}\ dx_k$, then

$$-\Delta_{A_j}^{\mathbf{X}_j} = \frac{1}{\sqrt{\det(G)}}\sum_{k,\ell}(i\partial_{x_k} + a_{j,k})\left(\sqrt{\det(G)}G^{k\ell}(i\partial_{x_\ell} + a_{j,\ell})\right).$$

We perform the change of variables $y = e^t$, and define the unitary operator $U : L^2(\mathbf{X}_j \times]2\ln(a_j), +\infty[) \to L^2(\mathbf{M}_j)$, where $]2\ln(a_j), +\infty[$ is equipped with the standard euclidian metric dt^2, by $U(f) = y^{(n\delta_j - 1)/2} f$. Thus $L^2(\mathbf{M}_j)$ is unitarily equivalent to $L^2(\mathbf{X}_j \times]2\ln(a_j), +\infty[)$, and

$$-U^\star\Delta_A U f = -\, e^{2\delta_j t}\Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n\delta_j - 1)[3 + \delta_j(n-4)]}{4} e^{2t(\delta_j - 1)} f - \partial_t(e^{2t(\delta_j-1)}\partial_t f). \tag{2.3}$$

Let us denote by $(\mu_\ell(j))_{\ell\in\mathbb{N}}$ the increasing sequence of eigenvalues of $-\Delta_{A_j}^{\mathbf{X}_j}$, each eigenvalue repeated according to its multiplicity.

Then $-\Delta_A^{\mathbf{M}_j,D}$ is unitarily equivalent to $\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D$,

$$\mathrm{sp}(-\Delta_A^{\mathbf{M}_j,D}) = \mathrm{sp}\left(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D\right), \tag{2.4}$$

where $L_{j,\ell}^D$ is the Dirichlet operator on $L^2(]2\ln(a_j), +\infty[)$ associated to

$$L_{j,\ell} = e^{2\delta_j t}\mu_\ell(j) \; + \; \frac{(n\delta_j - 1)}{4}[3 + \delta_j(n-4)]e^{2t(\delta_j - 1)} \; - \; \partial_t(e^{2t(\delta_j-1)}\partial_t). \tag{2.5}$$

The operator $L_{j,\ell}^D$ depends on A_j since $\mu_\ell(j)$ depends on A_j but we skip this dependence in notations for the sake of simplicity,

$$0 \le \mu_\ell(j) \le \mu_{\ell+1}(j) \quad \text{and} \quad \lim_{\ell\to\infty}\mu_\ell(j) = +\infty.$$

It is well known that assumption (1.3) implies that

$$0 \ < \ \mu_0(j).$$

As a matter of fact, if $\mu_0(j) = 0$ and u_0 is an associated eigenfunction, then $idu_0 = -u_0 A_j$, so $Re(\overline{u_0} du_0) = 0$, and then $|u_0|$ is constant. We can assume that $u_0 = e^{-i\varphi}$ with φ a real function. Then locally $d\varphi = A_j$, which yields $dA_j = 0$, so for any $x_0 \in \mathbf{X}_j$, and for any regular curve $\Gamma_{x_0,x}$ joining x_0 to x, we have $\varphi(x) = \oint_{\Gamma_{x_0,x}} A_j$. Therefore $e^{i\varphi}$ will be a well-defined function on $\mathbf{X}_j$ iff part (ii) of (1.3) is not satisfied, (see for example [Hel]).

When $1/n < \delta_j < 1$, another change of variables can be done. Precisely, we set $y = [(1-\delta_j)t]^{1/(1-\delta_j)}$, and define the unitary operator

$$U : L^2\left(\mathbf{X}_j \times \left]\frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, +\infty\right[\right) \ \to \ L^2(\mathbf{M}_j), \text{ by } U(f) = y^{(n-1)\delta_j/2} f.$$

Then we compute

$$-U^\star y^{n\delta_j} \partial_y [y^{(2-n)\delta_j} \partial_y U(f)] = -y^{(n+1)\delta_j/2} \partial_y [y^{(3-n)\delta_j/2} \partial_y f] - \frac{(n-1)\delta_j}{2} y^{2\delta_j - 1} \partial_y f + \frac{(n-1)\delta_j[(n-3)\delta_j + 2]}{4} y^{-2(1-\delta_j)} f,$$

so using that $y^{\delta_j}\partial_y = \partial_t$ and that $t^\rho \partial_t = \partial_t(t^\rho.) - \rho t^{\rho-1}$, we get easily that

$$-U^\star \Delta_A U f = -[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} \Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2 t^2} f - \partial_t^2 f. \tag{2.6}$$

Thus, in the case $1/n < \delta_j < 1$, equality (2.4) holds also when $L_{j,\ell}^D$ is the Dirichlet operator on $L^2\left(\left]\frac{a^{2(1-\delta_j)}}{1-\delta_j}, +\infty\right[\right)$ associated to

$$L_{j,\ell} = \mu_\ell(j)[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} \ + \ \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2 t^2} \ - \ \partial_t^2. \tag{2.7}$$

2.2. Proof of Theorem 1.1. To study the spectrum, we use the first diagonalization given by (2.4) and (2.5).

If $\mu_\ell(j) > 0$ then $\mathrm{sp}(L_{j,\ell}^D) = \mathrm{sp}_d(L_{j,\ell}^D) = \{\mu_{\ell,k}(j);\ k \in \mathbb{N}\}$, where $(\mu_{\ell,k}(j))_{k\in\mathbb{N}}$ is the increasing sequence of eigenvalues of $L_{j,\ell}^D$, $\lim_{k\to+\infty} \mu_{\ell,k}(j) = +\infty$.

If $\mu_\ell(j) = 0$ then $\mathrm{sp}(L_{j,\ell}^D) = \mathrm{sp}_{\mathrm{ess}}(L_{j,\ell}^D) = [\alpha_n, +\infty[$, with $\alpha_n = 0$ if $\delta_j < 1$, and $\alpha_n = (n-1)^2/4$ if $\delta_j = 1$, (by (2.5), if $\delta_j = 1$, $L_{j,\ell}^D u = -\partial_t^2 u + (n-1)^2/4u$, and by (2.5), if $1/n < \delta_j < 1, L_{j,\ell}^D u = -\partial_t^2 u + V(t)u$ with $\lim_{t\to\infty} V(t) = 0$).

Since we have $\mu_0(j) = 0$ when $A = 0$, we get that $\mathrm{sp}_{\mathrm{ess}}(-\Delta_0) = [\alpha_n, +\infty[$.

If A satisfies assumptions (1.3) and (1.4), we have seen that $0 < \mu_0(j)$, then $0 < \mu_\ell(j)$ for all j and ℓ, and then

$$\mathrm{sp}(-\Delta_A^{\mathbf{M}_j,D}) = \{\mu_{\ell,k}(j);\ (\ell,k) \in \mathbb{N}^2\}.$$

As $\mu_\ell(j) \le \mu_{\ell,k}(j) < \mu_{\ell,k+1}(j)$ with $\lim_{\ell\to+\infty} \mu_\ell(j) = +\infty$ and $\lim_{k\to+\infty} \mu_{\ell,k}(j) = +\infty$, each $\mu_{\ell,k}(j)$ is an eigenvalue of $-\Delta_A^{\mathbf{M}_j,D}$ of finite multiplicity, so $\mathrm{sp}\left(-\Delta_A^{\mathbf{M}_j,D}\right) = \mathrm{sp}_d\left(-\Delta_A^{\mathbf{M}_j,D}\right)$. Therefore, we get that $\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \emptyset$. $\square$

2.3. Proof of Theorem 1.2. We proceed as in [Mo-Tr].

We begin by establishing for $\mathbf{M}_j$, $1 \le j \le J$, formula (1.8) with $-\Delta_A^{\mathbf{M}_j,D}$ defined in (2.1) instead of $-\Delta_A$. When $\delta_j = 1$ we use the decomposition given by (2.4) and (2.5), but when $1/n < \delta_j < 1$, we use the decomposition given by (2.4) and (2.7).

From now on, any constant depending only on δ_j and on $\min_j \mu_0(j)$ will be invariably denoted by C.

As in [Mo-Tr], we will follow Titchmarsh's method. Using Theorem 7.4 in [Tit, page 146], we prove the following lemma.

Lemma 2.1. *There exists $C > 1$ so that for any $\lambda \gg 1$ and any $\ell \in K_\lambda = \left\{ l \in \mathbb{N};\ \mu_\ell(j) \in \left[0, \lambda / \min_j a_j^{4\delta_j}\right[\right\}$,*

$$\left| N(\lambda, L_{j,\ell}^D) \;-\; \frac{1}{\pi} w_{j,\ell}(\lambda) \right| \;\le\; C \ln(\lambda), \tag{2.8}$$

with $w_{j,\ell}(\mu) = \displaystyle\int_{\alpha_j}^{+\infty} [\mu - V_{j,\ell}(t)]_+^{1/2}\, dt = \int_{\alpha_j}^{T_j(\mu)} [\mu - V_{j,\ell}(t)]_+^{1/2}\, dt.$

The potential $V_{j,\ell}$ is defined as follows:

$$\begin{cases} \text{if } \delta_j = 1 \\ V_{j,\ell}(t) = \mu_\ell(j) e^{2t} + \frac{(n-1)^2}{4}, \\ \text{if } 1/n < \delta_j < 1 \\ V_{j,\ell}(t) = \mu_\ell(j) [(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} \;+\; \frac{(n-1)\delta_j [(n-3)\delta_j + 2]}{4(1-\delta_j)^2} t^{-2}, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \text{if } \delta_j = 1 \\ \alpha_j = 2\ln(a_j), \quad T_j(\mu) = \dfrac{1}{2} \ln(\mu/\mu_0(j)), \\ \text{if } 1/n < \delta_j < 1 \\ \alpha_j = \frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, \quad T_j(\mu) = \dfrac{1}{1-\delta_j} \left(\dfrac{\mu}{\mu_0(j)} \right)^{\frac{1-\delta_j}{2\delta_j}} . \end{cases} \tag{2.10}$$

Proof. When $1/n < \delta_j < 1$, by enlarging $\mathbf{M}_0$ and reducing $\mathbf{M}_j$, we can take α_j large enough so that $V_{j,\ell}(t)$ is an increasing function on $[\alpha_j, +\infty[$ and $\lambda/\mu_\ell(j) \gg 1$ when $\ell \in K_\lambda$. Then, if $\alpha_j \le Y < X(\lambda) = V_{j,\ell}^{-1}(\lambda)$, following the proof of Theorem 7.4 in [Tit, pages 146–147], we get that

$$\begin{aligned} &\left| N(\lambda, L_{j,\ell}^D) - \frac{1}{\pi} w_{j,\ell}(\lambda) \right| \\ &\quad \le C[\ln(\lambda - V_{j,\ell}(\alpha_j)) - \ln(\lambda - V_{j,\ell}(Y)) + (X(\lambda) - Y)(\lambda - V_{j,\ell}(Y)) + 1]. \end{aligned} \tag{2.11}$$

When $\delta_j = 1$, we choose $Y = X(\lambda) - \frac{\sqrt{\ln \lambda}}{\sqrt{\lambda}}$.

When $1/n < \delta_j < 1$, we choose $Y = X(\lambda) - \dfrac{\sqrt{\ln \lambda}}{\sqrt{\lambda}} \left(\dfrac{\lambda}{\mu_\ell(j)} \right)^{\frac{1-\delta_j}{4\delta_j}}$;

$\left(X(\lambda) \sim \frac{1}{1-\delta_j} \left(\frac{\lambda}{\mu_\ell(j)} \right)^{\frac{1-\delta_j}{2\delta_j}} \right).$ □

Let us apply to $-\Delta_{A_j}^{\mathbf{X}_j}$, the magnetic Laplacian which lies on $\mathbf{X}_j$, on a "boundary at infinity", the sharp asymptotic Weyl formula of Hörmander [Hor1] (see also [Hor2]),

Theorem 2.1. *There exists $C > 0$ so that for any $\mu \gg 1$*

$$\left| N(\mu, -\Delta_{A_j}^{\mathbf{X}_j}) \; - \; \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\mathbf{X}_j| \mu^{(n-1)/2} \right| \; \leq \; C\mu^{(n-2)/2}. \tag{2.12}$$

Lemma 2.2. *There exists $C > 0$ such that for any $\lambda \gg 1$*

$$\left| N(\lambda, -\Delta_A^{\mathbf{M}_j,D}) - \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_j| \lambda^{n/2} \right| \leq C \begin{cases} \lambda^{(n-1)/2} \ln(\lambda), & \text{if} \quad 1/(n-1) \leq \delta_j \leq 1, \\ \lambda^{1/(2\delta_j)}, & \text{if} \quad 1/n < \delta_j < 1/(n-1). \end{cases} \tag{2.13}$$

Proof. By formula (2.4),

$$N\left(\lambda, -\Delta_A^{\mathbf{M}_j,D}\right) = \sum_{\ell=0}^{+\infty} N(\lambda, L_{j,\ell}^D). \tag{2.14}$$

When $\ell \notin K_\lambda$, (K_λ is defined in Lemma 2.1), and thanks to formula (2.9) we have $V_{j,\ell} \geq \mu_\ell(j) a_j^{4\delta_j} \geq \lambda$ so $N(\lambda, L_{j,\ell}^D) = 0$. Therefore the estimates (2.8), (2.12) and formula (2.14) prove that

$$\left| N\left(\lambda, -\Delta_A^{\mathbf{M}_j,D}\right) \; - \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda) \right| \; \leq \; C\lambda^{(n-1)/2} \ln(\lambda). \tag{2.15}$$

Let us denote

$$\Theta_j(\lambda) = \sum_{\ell=0}^{+\infty} \frac{1}{\pi} w_{j,\ell}(\lambda) \quad \text{and} \quad R_j(\mu) = \sum_{\ell=0}^{+\infty} [\mu - \mu_\ell(j)]_+^{1/2}. \tag{2.16}$$

As $R_j(\mu) = \dfrac{1}{2} \displaystyle\int_0^{+\infty} [\mu - s]_+^{-1/2} N(s, -\Delta_{A_j}^{\mathbf{X}_j}) ds$, the Hörmander estimate (2.12) entails the following one.

There exists a constant $C > 0$ such that, for any $\mu \gg 1$,

$$|R_j(\mu) \; - \; \frac{\omega_{n-1}}{2(2\pi)^{n-1}} |\mathbf{X}_j| \int_0^{+\infty} [\mu - s]_+^{-1/2} s^{(n-1)/2} ds| \; \leq \; C\mu^{(n-1)/2}. \tag{2.17}$$

Writing in (2.9)

$$V_{j,\ell}(t) = \mu_\ell(j) V_j(t) \; + \; W_j(t), \tag{2.18}$$

we get that $\Theta_j(\lambda) = \dfrac{1}{\pi} \displaystyle\int_{\alpha_j}^{T_j(\lambda)} V_j^{1/2}(t) R_j \left(\frac{\lambda - W_j(t)}{V_j(t)} \right) dt.$

So according to (2.17)

$$\left| \Theta_j(\lambda) \; - \; \frac{\omega_{n-1} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{(2\pi)^n \Gamma\left(1 + \frac{n}{2}\right)} |\mathbf{X}_j| \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)} dt \right| \leq C \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)} dt. \tag{2.19}$$

From Definitions (2.9) and (2.18), we get that

$$\left| \int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)} dt \; - \; \lambda^{n/2} \frac{1}{(\delta_j n - 1) a_j^{2(\delta_j n - 1)}} \right| \; \leq \; C\lambda^{(n-1)/2}, \tag{2.20}$$

and

$$\begin{aligned} &\int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)} dt \\ &\quad \leq C \begin{cases} \lambda^{(n-1)/2}, & \text{if} \quad 1/(n-1) < \delta_j \leq 1, \\ \lambda^{(n-1)/2} \ln \lambda, & \text{if} \quad 1/(n-1) = \delta_j, \\ \lambda^{1/(2\delta_j)}, & \text{if} \quad 1/n < \delta \leq 1/(n-1). \end{cases} \end{aligned} \tag{2.21}$$

As $|\mathbf{M}_j| = \dfrac{|\mathbf{X}_j|}{(\delta_j n - 1) a_j^{2(\delta_j n - 1)}}$, we get (2.13) from (2.15), (2.16) and (2.19)–(2.21). $\square$

To achieve the proof of Theorem 1.2, we proceed as in [Mo-Tr].

We denote $\mathbf{M}_0^0 = \mathbf{M} \setminus \left(\bigcup_{j=1}^J \overline{\mathbf{M}_j} \right)$, then

$$\mathbf{M} = \overline{\mathbf{M}_0^0} \bigcup \left(\bigcup_{j=1}^{J} \overline{\mathbf{M}_j} \right). \tag{2.22}$$

Let us denote respectively by $-\Delta_A^{\Omega,D}$ and $-\Delta_A^{\Omega,N}$ the Dirichlet operator and the Neumann-like operator on an open set Ω of $\mathbf{M}$ associated to $-\Delta_A$.
$-\Delta_A^{\Omega,N}$ is the Friedrichs extension defined by the associated quadratic form $q_A^{\Omega}(u) = \int_{\Omega} |idu + Au|^2 d\mathbf{m}, \; u \in C^{\infty}(\overline{\Omega}; \mathbb{C}), \; u$ with compact support in $\overline{\Omega}$. ($d\mathbf{m}$ is the n-form volume of $\mathbf{M}$ and $|Z|^2 = < Z; Z >_{T^\star(M)}$ for any complex one-form Z on $\mathbf{M}$.)

The min–max principle and (2.22) imply that

$$\begin{aligned} N\left(\lambda, -\Delta_A^{\mathbf{M}_0^0, D}\right) + \sum_{1 \leq j \leq J} N\left(\lambda, -\Delta_A^{\mathbf{M}_j, D}\right) &\leq N(\lambda, -\Delta_A) \\ \leq N\left(\lambda, -\Delta_A^{\mathbf{M}_0^0, N}\right) + \sum_{1 \leq j \leq J} N\left(\lambda, -\Delta_A^{\mathbf{M}_j, N}\right)&. \end{aligned} \tag{2.23}$$

The Weyl formula with remainder, (see [Hor2] for Dirichlet boundary condition and [Sa-Va, page 9] for Neumann-like boundary condition), gives that

(2.24)

$$N\left(\lambda, -\Delta_A^{\mathbf{M}_0^0, Z}\right) = \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_0^0| \lambda^{n/2} + \mathbf{O}(\lambda^{(n-1)/2}) \quad (\text{for } Z = D \quad \text{and} \quad \text{for } Z = N).$$

For $1 \leq j \leq J$, the asymptotic formula for $N\left(\lambda, -\Delta_A^{\mathbf{M}_j, N}\right)$,

$$N\left(\lambda, -\Delta_A^{\mathbf{M}_j, N}\right) = \frac{\omega_n}{(2\pi)^n} |\mathbf{M}_j| \lambda^{n/2} + \mathbf{O}(r(\lambda)), \tag{2.25}$$

is obtained as for the Dirichlet case (2.13) by noticing that

$$N(\lambda, L_{j,\ell}^D) \leq N(\lambda, L_{j,\ell}^N) \leq N(\lambda, L_{j,\ell}^D) + 1,$$

where $L^D_{j,\ell}$ and $L^N_{j,\ell}$ are Dirichlet and Neumann-like operators on a half-line $I =]\alpha_j, +\infty[$, associated to the same differential Schrödinger operator $L_{j,\ell}$ defined by (2.5) when $\delta_j = 1$, and by (2.7) otherwise.

(The Neumann-like boundary condition is of the form $\partial_t u(\alpha_j) + \beta_j u(\alpha_j) = 0$ because of the change of functions performed by $U^\star$).

The above inequality is well known. It comes from the fact that the eigenvalues of $L^D_{j,\ell}$ and $L^N_{j,\ell}$ are of multiplicity one and there is no common eigenvalue, (we have used Theorem 2.1 of [Co-Le, page 225]). If $(\mu^Z_{\ell,k}(j))_{k\in\mathbb{N}}$ is the sequence of non-decreasing eigenvalues of $L^{\overline{Z}}_{j,\ell}$, $(Z = D$ or $D = N)$, and $(\varphi^Z_{\ell,k})_{k\in\mathbb{N}}$ an associated orthonormal basis of eigenfunctions, then $\mu^N_{\ell,0}(j) < \mu^D_{\ell,0}(j)$. As in $E_{k+1}(Z)$, the subspace of dimension $k+1$ spanned by $\varphi^Z_{\ell,0}$, $\varphi^Z_{\ell,1}, \ldots, \varphi^Z_{\ell,k}$, there exists, in $E_{k+1}(Z)$, a subspace of dimension k included in the domain of $L^{\overline{Z}}_{j,\ell}$, for $(Z, \overline{Z}) = (N, D)$ and for $(Z, \overline{Z}) = (D, N)$, the min–max principle involves $\mu^{\overline{Z}}_{\ell,k-1}(j) < \mu^Z_{\ell,k}(j)$. (For any k, $\varphi^N_{\ell,k+1} - \dfrac{\varphi^N_{\ell,k+1}(\alpha_j)}{\varphi^N_{\ell,k}(\alpha_j)}\varphi^N_{\ell,k}$ is in the domain of $L^D_{j,\ell}$ and $\varphi^D_{\ell,k+1} - \dfrac{\partial_t\varphi^D_{\ell,k+1}(\alpha_j)}{\partial_t\varphi^D_{\ell,k}(\alpha_j)}\varphi^D_{\ell,k}$ is in the domain of $L^N_{j,\ell}$.)

We get (1.8) from (2.13) and (2.23)–(2.25). □

2.4. Proof of Theorem 1.3.

Lemma 2.3. *For any $j \in \{1, \ldots, J\}$, there exists a one-form A_j satisfying (1.3) and the following property.*

There exists $\tau_0 = \tau_0(A_j) > 0$ and $C = C(A_j) > 0$ such that, if $\mu_0(j,\tau) = \inf_{u\in C_0^\infty(\mathbf{X}_j),\ \|u\|_{L^2(\mathbf{X}_j)}=1} \|idu + \tau u A_j\|^2_{L^2(\mathbf{X}_j)}$ denotes the first eigenvalue of $-\Delta^{\mathbf{X}_j}_{\tau A_j}$, then

$$\mu_0(j,\tau) \ \geq\ C\tau^2, \quad \forall\, \tau \in]0, \tau_0]. \tag{2.26}$$

($\|idu + \tau u A_j\|^2_{L^2(\mathbf{X}_j)} = \int_{\mathbf{X}_j} \langle idu + \tau u A_j; idu + \tau u A_j\rangle_{T^\star(\mathbf{X}_j)}\, d\mathbf{x}_j$).

Proof. When $n = 2$, we can take $A_j = \omega_j d\mathbf{x}_j$, ($d\mathbf{x}_j$ is the $(n-1)$-form volume of $\mathbf{X}_j$), for some constant $\omega_j \in \mathbb{R} \setminus \dfrac{2\pi}{|\mathbf{X}_j|}\mathbb{Z}$, then $\mu_0(j,\tau) = \tau^2\omega_j^2$ for small $|\tau|$.

When $n \geq 3$, we have $\mu_0(j, 0) = 0$, $\partial_\tau\mu_0(j, 0) = 0$ and

$$\partial^2_\tau\mu_0(j,0) = \frac{2}{|\mathbf{X}_j|}\int_{\mathbf{X}_j}\left[\,|A_j|^2 - \left(-\Delta_0^{\mathbf{X}_j}\right)^{-1}(d^\star A_j).(d^\star A_j)\,\right] d\mathbf{x}_j.$$

($d^\star$ is the Hodge–de Rham codifferential on $\mathbf{X}_j$, and $(-\Delta_0^{\mathbf{X}_j})^{-1}$ is the inverse of the Laplace–Beltrami operator on functions, which is well defined on the orthogonal of the first eigenspace, on the space $\{f \in L^2(\mathbf{X}_j);\ \int_{\mathbf{X}_j} f d\mathbf{x}_j = 0\}$).

The proof is standard. One writes $-\Delta^{\mathbf{X}_j}_{\tau A} = P_0 + \tau P_1 + \tau^2 P_2$, $P_0 = -\Delta_0$ and for all $u \in C^1(\mathbf{X}_j)$, $P_1(u) = i\,\langle du; A_j\rangle_{T^\star\mathbf{X}_j} - id^\star(uA_j)$ and $P_2(u) = u|A_j|^2 = u\,\langle A_j; A_j\rangle_{T^\star\mathbf{X}_j}$. The first eigenvalue of P_0, $\mu_0(j, 0) = 0$ is of multiplicity one. The associated normalized eigenfunction is $u_0 = 1/\sqrt{|\mathbf{X}_j|}$. Then $\tau \to \mu_0(j,\tau)$ is an analytic function, and there exists an associated eigenfunction $u_{0,\tau}$ analytic in τ. Then, as $\tau \to 0$, $\mu_0(j,\tau) = \tau c_1 + \tau^2 c_2 + \mathbf{O}(\tau^3)$ and $u_{0,\tau} = u_0 + \tau v_1 + \tau^2 v_2 + \mathbf{O}(\tau^3)$, with

$$\begin{cases} c_1 = \int_{\mathbf{X}_j} P_1(u_0).\overline{u_0} d\mathbf{x}_j, \\ v_1 = -P_0^{-1}[P_1(u_0) - c_1 u_0], \\ c_2 = \int_{\mathbf{X}_j} [P_2(u_0) + P_1(v_1)].\overline{u_0} d\mathbf{x}_j. \end{cases}$$

The operator P_1 is formally self-adjoint and P_2 is self-adjoint. We have $P_1(u_0) = -\frac{i}{\sqrt{|\mathbf{X}_j|}} d^\star(A_j)$ so $P_1(u_0)$ is orthogonal to the constant function u_0 and then $c_1 = 0$.

To the non-negative quadratic form $A_j \to \partial_\tau^2 \mu_0(j,0)$, we associate a self-adjoint operator P on $\Lambda^1(\mathbf{X}_j), \partial_\tau^2\mu_0(j,0) = \displaystyle\int_{\mathbf{X}_j} \langle P(A_j); A_j\rangle_{T^\star \mathbf{X}_j} d\mathbf{x}_j$, which is a pseudodifferential operator of order 0 with principal symbol, the square matrix $p_0(x,\xi) = (p_0^{ik}(x,\xi))_{1\le i,k\le n-1}$ defined as follows. In local coordinates, if $\mathbf{h}_j = \sum_{i,k} G_{ik}(x) dx_i dx_k$, then

$$\frac{|\mathbf{X}_j|}{2} p_0^{ik}(x,\xi) = G^{ik}(x) \; - \; \sum_{\ell,m} G^{im}(x) G^{\ell k}(x) \frac{\xi_m}{|\xi|}\frac{\xi_\ell}{|\xi|}; \quad \left(|\xi|^2 = \sum_{\ell,m} G^{m\ell}(x)\xi_m \xi_\ell \right),$$

so for any $\zeta \in \mathbb{R}^{n-1}$,

$$\sum_{i,k} \frac{|\mathbf{X}_j|}{2} p_0^{ik}(x,\xi)\zeta_i\zeta_k = \frac{2}{|\mathbf{X}_j|}\left[|\zeta|^2 - \frac{\langle \xi;\zeta\rangle^2}{|\xi|^2} \right] \ge 0; \quad \left(\langle \xi;\zeta\rangle = \sum_{i,k} G^{ik}\xi_i\zeta_k \right).$$

Thus we get $\partial_\tau^2\mu_0(j,0) = \int_{\mathbf{X}_j} \langle P(A_j); A_j\rangle_{T^\star \mathbf{X}_j} d\mathbf{x}_j > 0.$ □

Lemma 2.4. *For a one-form A satisfying (1.4), there exists a constant $C_A > 0$ such that, if u is a function in $L^2(\mathbf{M})$ such that $du \in L^2(\mathbf{M})$ and*

$$\forall j = 1,\dots,J, \quad \int_{\mathbf{X}_j} u(x_j,y) d\mathbf{x}_j = 0, \quad \forall y \in]a_j^2, +\infty[, \tag{2.27}$$

then $\forall \tau \in]0,1]$,

$$\|idu + \tau uA\|^2_{L^2(\mathbf{M})} \; \le \; (1+\tau C_A)\|idu\|^2_{L^2(\mathbf{M})} + C_A\|u\|^2_{L^2(\mathbf{M})}. \tag{2.28}$$

Proof. First we remark that the inequality

$$|idu + \tau uA|^2 \le (1+\rho)|du|^2 + (1+\rho^{-1})|\tau uA|^2 \tag{2.29}$$

is satisfied for any $\rho > 0$.

For $\rho = \tau$, we get that there exists a constant $C_A^0 > 0$, depending only on $A/\mathbf{M}_0$, such that

$$\|idu + \tau uA\|^2_{L^2(\mathbf{M}_0)} \; \le \; (1+\tau)\|idu\|^2_{L^2(\mathbf{M}_0)} + \tau C_A^0\|u\|^2_{L^2(\mathbf{M}_0)}. \tag{2.30}$$

We get also for $\rho = \tau$ that for any $j \in \{1, \dots, J\}$,

$$(2.31)\qquad \int_{a_j^2}^{+\infty} \|idu + \tau uA\|^2_{L^2(\mathbf{X}_j)} y^{(2-n)\delta_j} dy \leq \int_{a_j^2}^{+\infty} \left((1+\tau)\|idu\|^2_{L^2(\mathbf{X}_j)} + \tau C_A^j \|u\|^2_{L^2(\mathbf{X}_j)}\right) y^{(2-n)\delta_j} dy,$$

for some constant C_A^j depending only on A/X_j.

But (2.27) implies that

$$(2.32)\qquad \|u\|^2_{L^2(\mathbf{X}_j)} \leq \frac{1}{\mu_1(j,0)} \|idu\|^2_{L^2(\mathbf{X}_j)},$$

with $(\mu_\ell(j,0))_{\ell\in\mathbb{N}}$ the sequence of eigenvalues of Laplace–Beltrami operator on $\mathbf{X}_j$,

$$\mu_0(j,0) = 0 < \mu_1(j,0) \leq \mu_2(j,0) \leq \cdots .$$

So if (2.27) is satisfied then (2.31) and (2.32) imply that

$$(2.33)\qquad \|idu + \tau uA\|^2_{L^2(\mathbf{M}_j)} \leq \left(1 + \tau c_A^j\right) \|idu\|^2_{L^2(\mathbf{M}_j)},$$

for some constant c_A^j depending only on A/X_j.

The existence of a constant $C_A > 0$ satisfying the inequality (2.28) follows from (2.30) and (2.33) for $j = 1, \dots, J$. □

Lemma 2.5. *When A satisfies (1.3), (1.4) and Lemma 2.3 , then as $\lambda \to +\infty$, the following Weyl formula is satisfied.*

$$(2.34)\qquad N(\lambda, -\Delta_{(\lambda^{-\rho}A)}) = |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} \; + \; \mathbf{O}(\mathbf{r_0}(\lambda)),$$

with

$$(2.35)\qquad \rho = \begin{cases} 1/2, & \text{if } \; 2/n \leq \delta \leq 1, \\ (n\delta - 1)/2, & \text{if } \; 1/n < \delta < 2/n, \end{cases}$$

δ and ω_d are as in Theorem 1.2, and the function $r_0(\lambda)$ is the one defined by (1.11).

Proof. We follow the proof of Theorem 1.2.

Since A satisfies Lemma 2.3, we have for $\lambda \gg 1$ large enough that $-\Delta_{(\lambda^{-\rho}A)} - (-\Delta_0)$ is in $\mathbf{M}_0$ a partial differential operator of order 1 with bounded coefficients, so the part of the proof of Theorem 1.2 in $\mathbf{M}_0$ remains valid for the estimate of $N(\lambda, -\Delta^{\mathbf{M}_0,Z}_{(\lambda^{-\rho}A)})$, $(Z = D$ or $Z = N)$, because for any $\Lambda \gg 1, N(\Lambda, -\Delta_0^{\mathbf{M}_0,Z} + C(-\Delta_0^{\mathbf{M}_0,Z})^{1/2} + C) \leq N(\Lambda, -\Delta^{\mathbf{M}_0,Z}_{(\lambda^{-\rho}A)}) \leq N(\Lambda, -\Delta_0^{\mathbf{M}_0,Z} - C(-\Delta_0^{\mathbf{M}_0,Z})^{1/2} - C)$ and $|N(\Lambda, -\Delta_0^{\mathbf{M}_0,Z} \pm C(-\Delta_0^{\mathbf{M}_0,Z})^{1/2} \pm C) - |\mathbf{M}_0| \frac{\omega_n}{(2\pi)^n} \Lambda^{n/2}| \leq C\Lambda^{(n-1)/2}$.

For the part of the proof of Theorem 1.2 in $\mathbf{M}_j$, $1 \leq j$, we have also for any $\Lambda \gg 1, N(\Lambda, -\Delta_0^{\mathbf{X}_j} + C(-\Delta_0^{\mathbf{X}_j})^{1/2} + C) \leq N(\Lambda, -\Delta^{\mathbf{X}_j}_{(\lambda^{-\rho}A_j)}) \leq N(\Lambda, -\Delta_0^{\mathbf{X}_j} - C (-\Delta_0^{\mathbf{X}_j})^{1/2} - C)$ and $|N(\Lambda, -\Delta_0^{\mathbf{X}_j} \pm C(-\Delta_0^{\mathbf{X}_j})^{1/2} \pm C) - |\mathbf{X}_j| \frac{\omega_{n-1}}{(2\pi)^{n-1}} \Lambda^{(n-1)/2}|$

$\leq C\Lambda^{(n-2)/2}$. But the crucial step of the proof of Theorem 1.2 is Lemma 2.1, where we used, (with $\mu_\ell(j)$ to be replaced by $\mu_\ell(j,1)$ in our new notations), that

$$0 < C \leq \mu_0(j) \leq \mu_\ell(j) \leq \mu_{\ell+1}(j) \quad \text{and} \quad \lim_{\ell\to+\infty} \mu_\ell(j) = +\infty.$$

Here in $\mathbf{M}_j, (1 \leq j)$, if $(\mu_\ell(j,\lambda^{-\rho}))_{\ell\in\mathbb{N}}$ denotes the increasing sequence of eigenvalues of $-\Delta^{\mathbf{X}_j}_{\lambda^{-\rho}A_j}$, we have

$$C/\lambda^{2\rho} \leq \mu_0(j,\lambda^{-\rho}) \quad \text{and} \quad C \leq \mu_1(j,\lambda^{-\rho}) \leq \mu_{1+\ell}(j,\lambda^{-\rho}) \leq \mu_{2+\ell}(j,\lambda^{-\rho})$$

with $\lim_{\ell\to+\infty} \mu_\ell(j,\lambda^{-\rho}) = +\infty$.

More precisely, $\lim_{\lambda\to+\infty} \mu_\ell(j,\lambda^{-\rho}) = \mu_\ell(j,0)$ and $0 = \mu_0(j,0) < \mu_{1+\ell}(j,0)$ for any $\ell \in \mathbb{N}$. It follows that Lemma 2.1 holds for any $\ell \in K_\lambda, \ell \neq 0$. So taking (2.14) into account, the proof of Theorem 1.2 will remain valid if we can prove, (for $L^D_{j,0}$ as in Lemma 2.1, excepted that $\mu_0(j)$ is replaced by $\mu_0(j,\lambda^{-\rho})$), that

$$N\left(\lambda, L^D_{j,0}\right) = \mathbf{O}(r_0(\lambda)).$$

This can easily be done as follows.

When $\delta_j = 1$, $(\rho = 1/2)$, it is easy to see that

$$N\left(\lambda, L^D_{j,0}\right) \;\leq\; N\left(\lambda + C, L^{D,\lambda}\right) \;\leq\; C\lambda^{1/2}\ln(\lambda),$$

where $L^{D,\lambda}$ is the Dirichlet operator on $]0,+\infty[$ associated to $\dfrac{C}{\lambda}e^{2t} - \partial_t^2$.

When $0 < \delta_j < 1$, by scaling we have that

$$N\left(\lambda, L^D_{j,0}\right) \;\leq\; N\left((\lambda + C)^{1+2\rho(1-\delta_j)}, L^D\right) \;\leq\; C\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)},$$

where L^D is the Dirichlet operator on $]0,+\infty[$ associated to $\dfrac{1}{C^2}t^{\frac{2\delta_j}{1-\delta_j}} - \partial_t^2$.

When $2/n \leq \delta < 1$, as $2/n \leq \delta \leq \delta_j$, then

$$\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} = \lambda^{(2-\delta_j)/(2\delta_j)} \leq \lambda^{(2-\delta)/(2\delta)} \leq \lambda^{(n-1)/2} = \mathbf{O}(r_0(\lambda)).$$

When $1/n < \delta < 2/n$, as $\delta \leq \delta_j$, then $\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} \leq \lambda^{(1+2\rho(1-\delta))/(2\delta)} = \lambda^{(n-(n\delta-1))/2} = \mathbf{O}(r_0(\lambda))$. □

To achieve the proof of Theorem 1.3, we take a one-form A satisfying the assumptions of Lemma 2.5.

We remark that any eigenfunction u of the Laplace–Beltrami operator $-\Delta$ on $\mathbf{M}$ associated to an eigenvalue in $]\inf \mathrm{sp}_{\mathrm{ess}}(-\Delta), +\infty[$, satisfies (2.27). So if H_λ is the subspace of $L^2(\mathbf{M})$ spanned by eigenfunctions of $-\Delta$ associated to eigenvalues in $]0,\lambda[$, then, by (2.28) of Lemma 2.4 with $\tau = 1/\lambda^\rho$, with ρ defined by (2.35), we have

$$\begin{aligned}\forall u \in H_\lambda, \quad \|idu + \frac{1}{\lambda^\rho}uA\|^2_{L^2(\mathbf{M})} &\leq \left(1 + \frac{C_A}{\lambda^\rho}\right)\|du\|^2_{L^2(\mathbf{M})} + C_A\|u\|^2_{L^2(\mathbf{M})}\\ &\leq \left(\left(1 + \frac{C_A}{\lambda^\rho}\right)\lambda + C_A\right)\|u\|^2_{L^2(\mathbf{M})}.\end{aligned}$$

But if $(\lambda_j)_{j\in\mathbb{N}}$ is the non-decreasing sequence of eigenvalues of $-\Delta_{(\lambda^{-\rho}A)}$, then by max–min principle one must have

$$k < \dim(H_\lambda) \;\Rightarrow\; \lambda_k < \left(1 + \frac{C_A}{\lambda^\rho}\right)\lambda + C_A;$$

so

$$\dim(H_\lambda) \leq N\left(\left(1 + \frac{C_A}{\lambda^\rho}\right)\lambda + C_A, -\Delta_{(\lambda^{-\rho}A)}\right) + 1. \tag{2.36}$$

The estimates (2.34) and (2.36) prove (1.10), by noticing that $\lambda^{n/2}/\lambda^\rho = \mathbf{O}(r_0(\lambda))$. □

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