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DISCREPANCIES OF PRODUCTS OF ZETA-REGULARIZED PRODUCTS

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Abstract. Zeta-regularized products $\widehat{\prod}_m a_m$ are known not to commute with finite products, so one studies the discrepancy F_n given by

$$\exp(F_n) := \frac{\widehat{\prod}_m \left(\prod_{j=1}^n a_{m,j}\right)}{\prod_{j=1}^n \left(\widehat{\prod}_m a_{m,j}\right)}.$$

For a rather general class of products, associated to polynomials P_j in several variables, we show that the discrepancy $F_n(P_1,\ldots,P_n)$ of n products is a sum of pairwise contributions $F_2(P_i,P_j)$. Namely,

$$\left(\sum_{j=1}^n \deg P_j\right) F_n(P_1,\ldots,P_n) = \sum_{1\le i<j\le n} (\deg P_i + \deg P_j) F_2(P_i,P_j).$$

Thus, there are no higher interactions behind the non-commutativity.

1. Introduction

The zeta-regularized product $\widehat{\prod}_m a_m$ is an often useful substitute for the divergent product $\prod_m a_m$ [JL]. If the Dirichlet series $f(s) := \sum_m a_m^{-s}$ converges for $\mathrm{Re}(s) \gg 0$ and has an analytic continuation to $s = 0$, one defines $\widehat{\prod}_m a_m := \exp(-f'(0))$. It has been known since at least the work of Shintani [Sh] in the 1970's that taking regularized products does not commute with finite products, i.e., in general

$$\widehat{\prod}_m (a_m \cdot b_m) \neq \left(\widehat{\prod}_m a_m\right) \cdot \left(\widehat{\prod}_m b_m\right). \tag{1.1}$$

Nonetheless, both sides of this non-equality seem related, since in all known examples their ratio is far simpler than either side [Sh, KW, FR, Mi, DF]. For example, when a_m and b_m are given by positive polynomials of degree one in two variables, Shintani [Sh] and Mizuno [Mi] showed that this ratio is the exponential $\exp(F)$ of a rational function F in the coefficients of the polynomials and in the logarithms of these coefficients. In contrast, the right-hand side of (1.1) is a product of two Barnes double Γ-functions [Ba].

Given n zeta-regularized products $\widehat{\prod}_m a_{m,j}$ $(1 \le j \le n)$ we can define the discrepancy F_n measuring the non-commutativity of the process of regularization with that

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of taking finite products, namely

$$\exp(F_n) := \frac{\widehat{\prod}_m \left(\prod_{j=1}^n a_{m,j}\right)}{\prod_{j=1}^n \left(\widehat{\prod}_m a_{m,j}\right)}.$$

To prove properties of F_n we have to make some assumptions on the $a_{m,j}$ ensuring the meromorphic continuation of $\sum_m a_{m,j}^{-s}$ and regularity at $s = 0$.

The first study of continuations of rather general Dirichlet series seems to be Mellin's [Me]. Its rather informal style was put on a firm basis by Mahler [Ma], who assumed that $a_m = P(m)$ is the value of a polynomial P in r positive integer variables $m = (m_1, m_2, \ldots, m_r) \in \mathbb{N}^r$ satisfying.

Mahler's hypothesis on P [Ma, p. 385, Kl. A]. The polynomial $P(x) = P(x_1, \ldots, x_r) \in \mathbb{C}[x]$ does not vanish for any $x \in \mathbb{R}^r_{\geq 0} := [0, \infty)^r$. Its homogeneous part of highest degree $P_{\text{top}}(x)$ is not constant and vanishes nowhere in $\mathbb{R}^r_{\geq 0}$, except for $P_{\text{top}}(0) = 0$.

Later authors [Sa, Li, Es, deC] have weakened this assumption, but we shall stay with Mahler's hypothesis for simplicity. Since $P_{\text{top}} Q_{\text{top}} = (PQ)_{\text{top}}$, if Mahler's hypothesis holds for $P_1, \ldots, P_n$ (with the same r), then it also holds for the product $P_1 \cdots P_n$.

Under his hypothesis, Mahler [Ma, Satz II] showed that

$$\zeta(s; P) := \sum_{m \in \mathbb{N}_0^r} P(m)^{-s} \qquad (\mathbb{N}_0 = \{0, 1, 2, \ldots\}) \tag{1.2}$$

converges for $\mathrm{Re}(s) > r/\deg P$ and extends to an entire meromorphic function of s, analytic at $s = 0$.[1] Here $\deg P$ is the total degree of P in its r variables, and the complex power in (1.2) uses any continuous branch of $\log P(x)$ for $x \in \mathbb{R}^r_{\geq 0}$. Thus we may define the zeta-regularized product

$$\widehat{\prod}_{m \in \mathbb{N}_0^r} P(m) := \exp(-\zeta'(0; P)),$$

where $\zeta(s; P)$ denotes the meromorphic continuation in s of the right-hand side of (1.2), and the derivative is taken with respect to s.

Given n polynomials $P_1, \ldots, P_n$ satisfying Mahler's hypothesis and having the same number r of variables, we can define their discrepancy

$$F_n = F_n(P_1, \ldots, P_n) := -\zeta'(0; P_1 \cdot P_2 \cdots P_n) + \sum_{j=1}^n \zeta'(0; P_j), \tag{1.3}$$

where the complex powers are taken so that

$$(P_1(m) \cdot P_2(m) \cdots P_n(m))^{-s} = (P_1(m))^{-s} \cdot (P_2(m))^{-s} \cdots (P_n(m))^{-s}. \tag{1.4}$$

[1]Mahler actually took the sum in (1.2) only over $m \in \mathbb{N}^r$. However, if we set k coordinates of $x = (x_1, \ldots, x_r)$ equal to 0 ($k < r$), and consider $P = P(x)$ as a function of the remaining $r - k$ coordinates, P still satisfies Mahler's hypothesis (in $r - k$ variables). Hence Mahler's meromorphic continuation with a sum over $\mathbb{N}^r$ implies the same with a sum over $\mathbb{N}_0^r$. Choosing sums over $\mathbb{N}_0^r$ rather than $\mathbb{N}^r$ is better suited to the integral formula (2.10) below.

In the language of regularized products, (1.3) can be re-written as

$$\exp(F_n) := \frac{\widehat{\prod}_{m\in\mathbb{N}_0^r}\prod_{j=1}^n P_j(m)}{\prod_{j=1}^n \widehat{\prod}_{m\in\mathbb{N}_0^r} P_j(m)}.$$

We will show that the discrepancy $F_n(P_1, \ldots, P_n)$ is a weighted sum of pairwise discrepancies $F_2(P_i, P_j)$. Thus, there is no contribution to the discrepancy from interactions of more than two polynomials.

Theorem 1.1. *Let $P_1, \ldots, P_n \in \mathbb{C}[x]$ be n polynomials in r variables, all satisfying Mahler's hypothesis above. Then*

$$\left(\sum_{j=1}^n \deg P_j\right) F_n(P_1, \ldots, P_n) = \sum_{1\le i<j\le n} (\deg P_i + \deg P_j) F_2(P_i, P_j), \tag{1.5}$$

where $F_n(P_1, P_2, \ldots, P_n)$ is defined by (1.3).

Of course, the (tacit) branch of $\log P_j$ used in either side of (1.5) must be the same.

In Section 2.1, we reduce the theorem to an analogous one for the zeta-integral discrepancy, namely,

$$\left(\sum_{j=1}^n \deg P_j\right) I_n(P_1, \ldots, P_n) = \sum_{1\le i<j\le n} (\deg P_i + \deg P_j) I_2(P_i, P_j), \tag{1.6}$$

where

$$I_n(P_1, \ldots, P_n) := -Z'(0; P_1 \cdot P_2 \cdots P_n) + \sum_{j=1}^n Z'(0; P_j),$$

$$Z(s; P) := \int_{x\in\mathbb{R}_+^n} P(x)^{-s}\, dx \quad (\mathrm{Re}(s) > r/\deg P)\,.$$

This reduction uses an integral formula that relates $\zeta(s; P^{[a]})$ to $Z(s; P^{[a]})$, where $P^{[a]}(x) := P(a + x)$. In Section 2.2, we prove (1.6) for n monic polynomials $P_j(\rho) = \rho + a_j$ of degree 1 in one variable ρ. In Section 2.3, we apply an identity for the value $Z(0; P_1 \cdots P_n)$ and formal properties of the discrepancy to deduce, from the degree 1 case proved in Section 2.2, identity (1.6) for all polynomials in one variable. In Section 2.4, we conclude the proof, using spherical coordinates to deduce (1.6) from the one-variable case proved in Section 2.3.

It is worth remarking that throughout the proof we use the existence of the meromorphic continuation of $\zeta(s; P)$ and $Z(s; P)$ [Ma, Satz I, II], but we never need any explicit formula realizing the continuation. The various properties of the discrepancy that we use are derived directly by analytic continuation from simple properties of $\zeta(s; P)$ and $Z(s; P)$ in the half-plane of convergence.

2. Proof

In this section we prove the theorem given in Section 1. We consider the more general Dirichlet series

$$\zeta(s; P; g) := \sum_{m\in\mathbb{N}_0^r} g(m) P(m)^{-s} \quad (\mathrm{Re}(s) > (r + \deg g)/\deg P)\,,$$

and corresponding zeta integrals

$$Z(s;P;g) := \int_{x\in\mathbb{R}^r_{\geq 0}} g(x)P(x)^{-s}\,dx \quad (\mathrm{Re}(s) > (r+\deg g)/\deg P)\,, \tag{2.1}$$

where $g \in \mathbb{C}[x]$ is an arbitrary polynomial in $r \geq 1$ variables, $P \in \mathbb{C}[x]$ satisfies Mahler's hypothesis in r variables (see Section 1), and dx denotes Lebesgue measure on $\mathbb{R}^r$.

Mahler [Ma, Satz I, II] proved

- $\zeta(s;P;g)$ and $Z(s;P;g)$ converge absolutely in the half-plane $\mathrm{Re}(s) > (r + \deg g)/\deg P$.
- $\zeta(s;P;g)$ and $Z(s;P;g)$ extend meromorphically to $s \in \mathbb{C}$, with at most simple poles at rational numbers of the form $s = (r + \deg g - \ell)/\deg P$, where $\ell = 0, 1, 2, \ldots$. However, $\zeta(s;P;g)$ and $Z(s;P;g)$ are analytic at non-positive integers $s = 0, -1, -2, \ldots$.

Note that the (possible) pole set does not depend on the coefficients of P or g.

It is not hard to show [FP, p. 6] that the set of polynomials (over $\mathbb{C}$) of degree d and satisfying Mahler's hypothesis is open in the space of coefficients of polynomials of degree d in r variables. Moreover, Mahler's proof [Ma, Sections 14–15] [FP, Section 3] shows that, outside the s-pole set just described, $\zeta(s;P;g)$ and $Z(s;P;g)$ are analytic not only in s, but also in the coefficients of P (for P near any fixed P_0), provided one chooses the branch of $\log P(x)$ continuously both in $x \in \mathbb{R}^r_{\geq 0}$ and in the coefficients of P. This will prove important below when we take derivatives and integrals with respect to the coefficients.

If $P_1, \ldots, P_n$ satisfy Mahler's hypothesis (see Section 1) and $g \in \mathbb{C}[x]$, Mahler's analytic continuation to $s = 0$ allows us to define

$$F_n(P_1,\ldots,P_n;g) := -\zeta'(0;P_1\cdot P_2\cdots P_n;g) + \sum_{j=1}^{n}\zeta'(0;P_j;g), \tag{2.2}$$

and its zeta-integral analogue,

$$I_n(P_1,\ldots,P_n;g) := -Z'(0;P_1\cdot P_2\cdots P_n;g) + \sum_{j=1}^{n}Z'(0;P_j;g). \tag{2.3}$$

We stress that branches for the (tacit) logarithms of products are taken to satisfy (1.4). We shall prove in Section 2.4

$$d_{n+1}F_n(P_1,\ldots,P_n;g) = \sum_{1\leq i<j\leq n}(d_i+d_j)F_2(P_i,P_j;g), \tag{2.4}$$

where

$$d_j := \deg P_j \ \ (1 \leq j \leq n), \quad d_{n+1} := \sum_{j=1}^{n} d_j,$$

which reduces to the theorem in Section 1 when $g = 1$.

2.1. Reduction to zeta integrals. We shall show in Proposition 2.1 below that (2.4) follows from its zeta-integral analogue

$$d_{n+1} I_n(P_1, \dots, P_n; g) = \sum_{1 \le i < j \le n} (d_i + d_j) I_2(P_i, P_j; g), \tag{2.5}$$

where I_n was defined in (2.3). The proof of (2.5) will occupy Sections 2.2–2.4.

We list some simple properties of I_n.

$$I_n(P, P, \dots, P; g) = 0, \tag{2.6}$$

$$I_n(P_0 \cdot P_1, P_2, \dots, P_n; g) = I_{n+1}(P_0, P_1, \dots, P_n; g) - I_2(P_0, P_1; g), \tag{2.7}$$

$$nI_k(P_1, \dots, P_k; g) = I_{kn}(\overbrace{P_1, \dots, P_1}^{n \text{ times}}, \dots, \overbrace{P_k, \dots, P_k}^{n \text{ times}}; g), \tag{2.8}$$

$$I_n(P_1, \dots, P_n; g) = I_n(P_{\tau(1)}, \dots, P_{\tau(n)}; g), \tag{2.9}$$

where τ is any permutation of $\{1, \dots, n\}$. The above are easily proved by direct substitution into the defining integral (2.1) for $\mathrm{Re}(s) \gg 0$, extending then to $s = 0$ by analytic continuation. Of course, the same branch of $\log P_j$ is used whenever P_j is repeated in the above equations.

A nice relation between the Dirichlet series $\zeta(s; P; g)$ and the zeta integral $Z(s; P; g)$ emerges if we insert a translation variable into the polynomial P, so we let

$$P^{[a]}(x) := P(a + x) \quad (a \in \mathbb{C}^r).$$

Note

$$\left(P^{[a]}\right)_{\mathrm{top}} = P_{\mathrm{top}}, \quad (PQ)^{[a]} = P^{[a]} Q^{[a]}.$$

If P satisfies Mahler's hypothesis, then so does $P^{[a]}$ for all a in a sufficiently small ball B around 0 in $\mathbb{C}^r$ [FP, p. 6]. This implies that $P^{[a+t]}$ satisfies Mahler's hypothesis for all $t \in \mathbb{R}^r_{\ge 0}$ and $a \in B$. Moreover, a continuous branch of $\log P^{[a+t]}(x)$ can be defined for $t, x \in \mathbb{R}^r_{\ge 0}$ and $a \in B$.

The relation between zeta-integrals and Dirichlet series we need is [FP, Proposition 4]

$$\int_{t \in [0,1]^r} \zeta(s; P^{[a+t]}; g^{[a+t]})\, dt = Z(s; P^{[a]}; g^{[a]}) \quad (a \in B), \tag{2.10}$$

valid for all s outside the possible pole set given by Mahler.[2] In particular, (2.10) holds for s in a neighbourhood of 0. On taking the derivative with respect to s inside the

[2]Here is a proof of (2.10) for $a = 0$, which suffices (replace P by $P^{[b]}$ and use $(P^{[b]})^{[t]} = P^{[b+t]}$). For $\mathrm{Re}(s) \gg 0$, uniform convergence gives

$$\begin{aligned}
\int_{t \in [0,1]^r} \zeta(s; P^{[t]}; g^{[t]})\, dt &= \int_{t \in [0,1]^r} \sum_{m \in \mathbb{N}_0^r} g(t + m) P(t + m)^{-s}\, dt \\
&= \sum_{m \in \mathbb{N}_0^r} \int_{t \in [0,1]^r} g(t + m) P(t + m)^{-s}\, dt = \sum_{m \in \mathbb{N}_0^r} \int_{t \in m + [0,1]^r} g(t) P(t)^{-s}\, dt \\
&= \int_{\mathbb{R}^r_{\ge 0}} g(t) P(t)^{-s}\, dt = Z(s; P; g),
\end{aligned}$$

proving (2.10) for $\mathrm{Re}(s) \gg 0$. For a general s (outside the possible pole set, which is independent of $t \in [0,1]^r$), (2.10) follows by analytic continuation.

integral in (2.10) (which is certainly permissible by the analyticity in s) and setting $s = 0$, we have

$$\int_{t\in[0,1]^r} F_n(P_1^{[a+t]}, \ldots, P_n^{[a+t]}; g^{[a+t]})\, dt = I_n(P_1^{[a]}, \ldots, P_n^{[a]}; g^{[a]}). \tag{2.11}$$

Fixing $\deg P_j$ and the number r of variables, the map taking the coefficients of the polynomials $P_1, \ldots, P_n$ to the discrepancy $F_n(P_1, \ldots, P_n; g)$ is known to be a polynomial in the non-top coefficients of the P_j (*i. e.* in the coefficients not appearing in any $(P_j)_{\mathrm{top}}$) [DF, p. 37].[3] Since the map $a \to P^{[a]}$ changes only the non-top coefficients of P, we conclude that $F_n(P_1^{[a]}, \ldots, P_n^{[a]}; g^{[a]})$ is a polynomial in a (for $a \in \mathbb{R}^r_{\geq 0}$, or a in some small ball B around 0 in $\mathbb{C}^r$). By (2.11), the same holds for $I_n(P_1^{[a]}, \ldots, P_n^{[a]}; g^{[a]})$.

Proposition 2.1. *Let* $d_j := \deg P_j$, $d_{n+1} := \sum_{j=1}^n d_j$, *and assume*

$$d_{n+1} I_n(P_1, \ldots, P_n; g) = \sum_{1\leq i<j\leq n} (d_i + d_j) I_2(P_i, P_j; g) \tag{2.12}$$

for all polynomials $P_1, \ldots, P_n \in \mathbb{C}[x]$ *in* r *variables satisfying Mahler's hypothesis, and for all* $g \in \mathbb{C}[x]$. *Then for all such polynomials,*

$$d_{n+1} F_n(P_1, \ldots, P_n; g) = \sum_{1\leq i<j\leq n} (d_i + d_j) F_2(P_i, P_j; g).$$

Proof. For $a \in \mathbb{R}^r_{\geq 0}$, or a in some small ball B around 0 in $\mathbb{C}^r$, let

$$H(a) := d_{n+1} F_n(P_1^{[a]}, \ldots, P_n^{[a]}; g^{[a]}) - \sum_{1\leq i<j\leq n} (d_i + d_j) F_2(P_i^{[a]}, P_j^{[a]}; g^{[a]}).$$

We have to show that $H(0) = 0$. In fact, we will show that H vanishes identically. By the preceding remarks, H is a polynomial in a. Moreover, from (2.11) we have

$$\begin{aligned} R(H)(a) &:= \int_{t\in[0,1]^r} H(a+t)\, dt \\ &= d_{n+1} I_n\left(P_1^{[a]}, \ldots, P_n^{[a]}; g^{[a]}\right) - \sum_{1\leq i<j\leq n} (d_i + d_j) I_2\left(P_i^{[a]}, P_j^{[a]}; g^{[a]}\right) = 0, \end{aligned}$$

where the last 0 is by our hypothesis (2.12). However, the $\mathbb{C}$-linear map $R : \mathbb{C}[a] \to \mathbb{C}[a]$

$$R(f)(a) := \int_{t\in[0,1]^r} f(a+t)\, dt \quad (f \in \mathbb{C}[a])$$

is injective, since it leaves the top-degree homogeneous part of the polynomial f unchanged.[4] Hence $H(a) = 0$ identically. □

[3]The regularized products in [DF] are actually taken over $m \in \mathbb{N}^r$, rather than over $m \in \mathbb{N}_0^r$. We can pass from the former to the latter as we did in our first footnote. Alternatively, in [DF] one could change the summation set from $\mathbb{N}^r$ to $\mathbb{N}_0^r$ everywhere without affecting any proof.

[4]This implies that R is invertible. In fact, R maps a product of Bernoulli polynomials $B_k(a) := B_{k_1}(a_1) \cdots B_{k_r}(a_r)$ to the monomial $a^k := a_1^{k_1} \cdots a_r^{k_r}$ [FP, Lemma 2.4].

2.2. Monic polynomials of degree 1 in one variable.

Proposition 2.2. *For* $P_j(\rho) = \rho + a_j \in \mathbb{C}[\rho]$ $(1 \le j \le n)$ *satisfying Mahler's hypothesis,* $h \in \mathbb{C}[\rho]$ *and* $n \ge 2$*, we have*

$$nI_n(P_1, \dots, P_n; h) = 2 \sum_{1 \le i < j \le n} I_2(P_i, P_j; h), \tag{2.13}$$

and

$$nZ(0; P_1 \cdots P_n; h) = \sum_{j=1}^{n} Z(0; P_j; h). \tag{2.14}$$

Note that $\deg P_i + \deg P_j = 2$, $n = \sum_{j=1}^n \deg P_j$ in this case, so (2.13) will become our first proved case of (2.5). Formula (2.14) is proved in [FP], but we re-prove it here for completeness. Mahler's hypothesis for P_j (see Section 1) in this case is equivalent to $a_j \notin (-\infty, 0]$, but we shall not require this explicitly.

Proof. We first show that if (2.13) holds for one set of branches P_j^{-s}, then it holds for any other choice $\widetilde{P}_j(\rho)^{-s} = \mathrm{e}^{2\pi i s k_j} P_j(\rho)^{-s}$ for integers k_j. Thus, we need to show

$$\begin{aligned} n\widetilde{I}_n(P_1, \dots, P_n; h) - 2 \sum_{1 \le i < j \le n} \widetilde{I}_2(P_i, P_j; h) \\ = nI_n(P_1, \dots, P_n; h) - 2 \sum_{1 \le i < j \le n} I_2(P_i, P_j; h), \end{aligned} \tag{2.15}$$

where we used a $\widetilde{\ }$ to denote a different choice of branches. Since we can change the branch of one P_ℓ at a time, to prove (2.15) it suffices to do the case where all but one k_ℓ vanish. Say just $k_j \neq 0$. Then,

$$\widetilde{Z}(s; P_j; h) = \mathrm{e}^{2\pi i s k_j} Z(s; P_j; h), \quad \widetilde{Z}(s; P_1 \cdots P_n; h) = \mathrm{e}^{2\pi i s k_j} Z(s; P_1 \cdots P_n; h). \tag{2.16}$$

Differentiating these equations with respect to s and setting $s = 0$, we get

$$I_n - \widetilde{I}_n = 2\pi i k_j \left(Z(0; P_1 \cdots P_n; h) - Z(0; P_j; h) \right).$$

Now (2.15) follows after a short calculation using (2.14) (which we shall prove below without using (2.13)).

Hence, for all j we fix the principal branch of $\log(\rho + a_j)$, i.e., the one for which $\lim_{\rho \to +\infty} \operatorname{Im}(\log P_j(\rho)) = 0$. This makes $I_n(P_1, \dots, P_n; h)$ a symmetric function of the $a_1, \dots, a_n$ (actually, a polynomial, as we shall recall below).

In proving (2.13) we may assume $n \ge 3$, as (2.13) is trivial for $n = 2$. For $\operatorname{Re}(s) \gg 0$, we have for any three *distinct* indices i, j, k,

$$\begin{aligned} &\frac{\partial^3}{\partial a_i \partial a_j \partial a_k} \left(Z(s; P_1 \cdots P_n; h) \right) \\ &\quad = \frac{\partial^3}{\partial a_i \partial a_j \partial a_k} \int_0^\infty h(\rho) \prod_{\ell=1}^{n} (\rho + a_\ell)^{-s} \, d\rho \end{aligned}$$

$$= -s^3 \int_0^\infty h(\rho)\,((\rho+a_i)(\rho+a_j)(\rho+a_k))^{-1} \prod_{\ell=1}^{n} (\rho+a_\ell)^{-s}\, d\rho$$
$$= -s^3 \int_0^\infty \left(h(\rho) \prod_{\substack{1\le \ell\le n\\ \ell\ne i,j,k}} (\rho+a_\ell) \right) \prod_{\ell=1}^{n} (\rho+a_\ell)^{-(s+1)}\, d\rho$$
$$= -s^3 Z(s+1; P_1\cdots P_n; \widetilde{h}), \quad \text{where} \quad \widetilde{h}(\rho) := h(\rho) \prod_{\substack{1\le \ell\le n\\ \ell\ne i,j,k}} (\rho+a_\ell).$$

By analytic continuation,

$$\frac{\partial^3}{\partial a_i \partial a_j \partial a_k} \left(Z(s; P_1\cdots P_n; h) \right) = -s^3 Z(s+1; P_1\cdots P_n; \widetilde{h}). \tag{2.17}$$

for all s outside the pole set. Applying $\frac{\partial}{\partial s}\big|_{s=0}$ to (2.17), and reversing the order of the derivatives (see the third paragraph in Section 2), we find

$$\begin{aligned} &\frac{\partial^3}{\partial a_i \partial a_j \partial a_k} \left(Z'(0; P_1\cdots P_n; h) \right) \\ &\quad = \left(-3s^2 Z(s+1; P_1\cdots P_n; \widetilde{h}) - s^3 Z'(s+1; P_1\cdots P_n; \widetilde{h}) \right) \Big|_{s=0} = 0, \end{aligned} \tag{2.18}$$

because $Z'(s; P_1\cdots P_n; \widetilde{h})$ can have at most a double pole at $s = 1$ (recall that all poles of $Z(s; P_1\cdots P_n; \widetilde{h})$ are simple [Ma, Satz I]).

By [DF, Theorem 4], $F_n(P_1,\ldots,P_n; h)$ is a polynomial in $a_1,\ldots,a_n$. It follows from the integral formula (2.11) that

$$\iota_n(a_1,\ldots,a_n) := I_n(P_1,\ldots,P_n; h).$$

is also a polynomial in $a_1,\ldots,a_n$ (depending on h, but h is fixed in this proof). From (2.18) we see that ι_n is a sum of monomials containing products of at most two distinct variables a_i, a_j. Thus ι_n can be uniquely written as

$$\iota_n(a_1,\ldots,a_n) = C_n + \sum_{i=1}^{n} f_{i,n}(a_i) + \sum_{1\le i<j\le n} p_{i,j,n}(a_i,a_j),$$

where C_n is a constant, $f_{i,n}$ is a polynomial in one variable with no constant term, and $p_{i,j,n}$ is a polynomial in two variables with $p_{i,j,n}(0,a) = 0 = p_{i,j,n}(a,0)$ for all $a \in \mathbb{C}$. But $\iota_n(a_1,\ldots,a_n)$ is symmetric in the a_ℓ, by (2.9). Since the above representation of the polynomial ι_n is unique, we must have

$$\iota_n(a_1,\ldots,a_n) = C_n + \sum_{i=1}^{n} f_n(a_i) + \sum_{1\le i<j\le n} p_n(a_i,a_j),$$

i.e., $f_{i,n} = f_n$ and $p_{i,j,n} = p_n$ independently of i, j. But f_n and p_n have no constant term, and $\iota_n(a_1, a_1, \dots, a_1) = 0$ by (2.6), so $C_n = 0$ and

$$f_n(a) = -\frac{(n-1)p_n(a,a)}{2} \quad (a \in \mathbb{C}) \tag{2.19}$$

$$\iota_n(a_1, \dots, a_n) = \sum_{i=1}^{n} f_n(a_i) + \sum_{1 \le i < j \le n} p_n(a_i, a_j). \tag{2.20}$$

Let

$$H_n(a,b) := p_n(a,b) - \frac{p_n(a,a) + p_n(b,b)}{2}, \tag{2.21}$$

and compute

$$\begin{aligned} \sum_{1 \le i < j \le n} H_n(a_i, a_j) &= \sum_{1 \le i < j \le n} p_n(a_i, a_j) - \frac{1}{2} \sum_{1 \le i < j \le n} (p_n(a_i, a_i) + p_n(a_j, a_j)) \\ &= \sum_{1 \le i < j \le n} p_n(a_i, a_j) - \frac{1}{2} \sum_{i=1}^{n} (n-1) p_n(a_i, a_i) \\ &= \sum_{1 \le i < j \le n} p_n(a_i, a_j) + \sum_{i=1}^{n} f_n(a_i) = \iota_n(a_1, \dots, a_n), \end{aligned}$$

where in the last line we used (2.19) and (2.20). Thus,

$$\iota_n(a_1, \dots, a_n) = \sum_{1 \le i < j \le n} H_n(a_i, a_j), \quad H_n(a,a) = 0, \tag{2.22}$$

the second equality following from (2.21). From (2.8) and (2.22) we obtain

$$nI_2(a_1, a_2; h) = \iota_{2n}(\overbrace{a_1, \dots, a_1}^{n \text{ times}}, \overbrace{a_2, \dots, a_2}^{n \text{ times}}) = n^2 H_{2n}(a_1, a_2), \tag{2.23}$$

since $H_{2n}(a_1, a_1) = 0 = H_{2n}(a_2, a_2)$. Using (2.8), (2.22) and (2.23) we get

$$\begin{aligned} \iota_n(a_1, \dots, a_n) &= \frac{1}{2} \iota_{2n}(a_1, a_1, a_2, a_2, \dots, a_n, a_n) \\ &= \frac{1}{2} \sum_{1 \le i < j \le n} 4H_{2n}(a_i, a_j; h) = \frac{2}{n} \sum_{1 \le i < j \le n} I_2(a_i, a_j; h), \end{aligned}$$

proving (2.13).

The proof of (2.14), to which we now turn, is very similar. For distinct i and j we have

$$\frac{\partial^2}{\partial a_i \partial a_j} (Z(s; P_1 \cdots P_n; h)) = s^2 Z(s+1; P_1 \cdots P_n; \widehat{h}),$$

where $\widehat{h}(\rho) := h(\rho)\prod_{\substack{1\le\ell\le n\\ \ell\ne i,j}}(\rho+a_\ell)$. Setting $s=0$ we find

$$\frac{\partial^2}{\partial a_i\partial a_j}\left(Z(0;P_1\cdots P_n;h)\right)=0.$$

Since [DF, Remark 3, pp. 40–41] $\zeta(0;P_1\cdots P_n;h)$ is a polynomial in the a_ℓ, so is $Z(0;P_1\cdots P_n;h)$ (use (2.10) with $s=0$). Hence

$$Z(0;P_1\cdots P_n;h)=c_n+\sum_{i=1}^{n}q_{i,n}(a_i),$$

where c_n is a constant and $q_{i,n}$ is a polynomial in one variable with no constant term. By symmetry in the a_i

$$Z(0;P_1\cdots P_n;h)=\sum_{i=1}^{n}q_n(a_i), \tag{2.24}$$

where $q_n(a) := q_{i,n}(a)+c_n/n$, for any i.[5] Taking $a=a_1=\cdots=a_n$ in (2.24), so $P=P_1=\cdots=P_n$, we have

$$\begin{aligned}Z(0;P;h)&=Z(s;P;h)\big|_{s=0}=Z(ns;P;h)\big|_{s=0}=Z(s;P^n;h)\big|_{s=0}\\&=Z(0;P^n;h)=\sum_{j=1}^{n}q_n(a)=nq_n(a).\end{aligned}$$

In view of (2.24), we are done proving (2.14). □

2.3. Polynomials in one variable. First we assume that the polynomials are monic.

Proposition 2.3. *Let $P_1,\ldots,P_n\in\mathbb{C}[\rho]$ be monic polynomials in one variable satisfying Mahler's hypothesis, and let $h\in\mathbb{C}[\rho]$. Then*

$$d_{n+1}I_n(P_1,\ldots,P_n;h)=\sum_{1\le i<j\le n}(d_i+d_j)I_2(P_i,P_j;h), \tag{2.25}$$

and

$$d_{n+1}Z(0;P_1\cdots P_n;h)=\sum_{j=1}^{n}d_jZ(0;P_j;h), \tag{2.26}$$

where $d_j:=\deg P_j$, $d_{n+1}:=\sum_{j=1}^{n}d_j$.

Proof. Since P_j is a monic polynomial in one variable, it factors as $P_j=\prod_{j=1}^{d_j}L_{j,i}$, where the $L_{j,i}\in\mathbb{C}[\rho]$ are monic of degree 1 and satisfy Mahler's hypothesis. To prove

[5] By (2.16), $Z(0;P;h)$ is independent of the branch of $\log P$ chosen. Thus the symmetry in the a_j of $Z(0;P_1\cdots P_n;h)$ is clear.

(2.26) note

$$d_{n+1}Z\left(0;\prod_{j=1}^{n}P_j;h\right)=d_{n+1}Z\left(0;\prod_{\substack{1\le j\le n\\1\le i\le d_j}}L_{j,i};h\right)=\sum_{\substack{1\le j\le n\\1\le i\le d_j}}Z(0;L_{j,i};h)$$
$$=\sum_{j=1}^{n}\left(\sum_{i=1}^{d_j}Z(0;L_{j,i};h)\right)=\sum_{j=1}^{n}d_jZ(0;P_j;h),$$

where the second and last equalities used (2.14).

We now turn to (2.25), which we will prove by induction on

$$k:=d_{n+1}-n=\sum_{j=1}^{n}(d_j-1)\ge 0.$$

If $k=0$, then all $d_j=1$ and the proposition reduces to the previous one. Thus we assume (2.25) for all n and all $d_{n+1}-n<k$. For $k\ge 1$, some $d_j>1$. By symmetry, we can suppose $d_1>1$, so we can factor the monic one-variable polynomial $P_1=LQ$, where L and Q are also monic, satisfy Mahler's hypothesis, and $\deg L=1$. We calculate, applying the inductive hypothesis to both $I_{n+1}(L,Q,P_2,\dots,P_n;h)$ and $I_3(L,Q,P_j;h)$ $(2\le j\le n)$, dropping now the fixed polynomial h from the notation,

$$d_{n+1}I_n(P_1,\dots,P_n)-\sum_{1\le i<j\le n}(d_i+d_j)I_2(P_i,P_j)$$
$$=d_{n+1}\left(I_{n+1}(L,Q,P_2,\dots,P_n)-I_2(L,Q)\right)\qquad\text{(use (2.7))}$$
$$-\sum_{2\le i<j\le n}(d_i+d_j)I_2(P_i,P_j)-\sum_{j=2}^{n}(d_1+d_j)I_2(LQ,P_j)$$
$$=d_1I_2(L,Q)+\sum_{j=2}^{n}(1+d_j)I_2(L,P_j)+\sum_{j=2}^{n}(d_1-1+d_j)I_2(Q,P_j)\qquad\text{(induction)}$$
$$-d_{n+1}I_2(L,Q)-\sum_{j=2}^{n}(d_1+d_j)\left(I_3(L,Q,P_j)-I_2(L,Q)\right)$$
$$=\sum_{j=2}^{n}\left(d_1I_2(L,Q)+(1+d_j)I_2(L,P_j)+(d_1-1+d_j)I_2(Q,P_j)\right)$$
$$-\sum_{j=2}^{n}(1+(d_1-1)+d_j)I_3(L,Q,P_j)\qquad\left(\text{recall } d_{n+1}:=\sum_{j=1}^{n}d_j\right)$$
$$=\sum_{j=2}^{n}\Big(d_1I_2(L,Q)+(1+d_j)I_2(L,P_j)+(d_1-1+d_j)I_2(Q,P_j)\qquad\text{(induction)}$$
$$-d_1I_2(L,Q)-(1+d_j)I_2(L,P_j)-(d_1-1+d_j)I_2(Q,P_j)\Big)=0,$$

as claimed. $\square$

Now we drop the assumption that the P_j be monic.

Proposition 2.4. *Let* $P_1, \dots, P_n \in \mathbb{C}[\rho]$ *be polynomials in one variable satisfying Mahler's hypothesis, and let* $h \in \mathbb{C}[\rho]$*. Then*

$$d_{n+1} I_n(P_1, \dots, P_n; h) - \sum_{1 \le i < j \le n} (d_i + d_j) I_2(P_i, P_j; h) = 0, \tag{2.27}$$

and

$$d_{n+1} Z(0; P_1 \cdots P_n; h) = \sum_{j=1}^{n} d_j Z(0; P_j; h), \tag{2.28}$$

where $d_j := \deg P_j, \;\; d_{n+1} := \sum_{j=1}^{n} d_j$.

Proof. If $\tilde{P} = \lambda P$, where $P \in \mathbb{C}[\rho]$ satisfies Mahler's hypothesis and $\lambda \in \mathbb{C} - \{0\}$, then

$$Z(s; \tilde{P}; h) = \lambda^{-s} Z(s; P; h), \tag{2.29}$$

provided the branches are chosen so that $\tilde{P}(x)^{-s} = \lambda^{-s} P(x)^{-s}$ for $x \in \mathbb{R}_{\ge 0}$. Hence $Z(0; \tilde{P}; h) = Z(0; P; h)$, so (2.28) follows from the monic case proved in the previous proposition.

To prove (2.27), we first show that its left-hand side is unchanged if one of the P_j is replaced by $\tilde{P}_j = \lambda P_j$. Say $j = 1$, to simplify notation. From (2.29) and definition (2.3) of I_n, we find

$$I_n(P_1, P_2, \dots, P_n; h) - I_n(\tilde{P}_1, P_2, \dots, P_n; h) = \big(Z(0; P_1 \cdots P_n; h) - Z(0; P_1; h)\big) \log \lambda.$$

This (also applied to $I_2(\tilde{P}_1, P_j; h)$), (2.28) and some calculation show (dropping h)

$$\begin{aligned} d_{n+1} I_n(P_1, P_2, \dots, P_n) &- 2 \sum_{1 \le i < j \le n} (d_i + d_j) I_2(P_i, P_j) \\ &= d_{n+1} I_n(\tilde{P}_1, P_2, \dots, P_n) - 2 \sum_{2 \le i < j \le n} (d_i + d_j) I_2(P_i, P_j) \\ &\quad - 2 \sum_{j=2}^{n} (d_1 + d_j) I_2(\tilde{P}_1, P_j). \end{aligned}$$

Thus, starting with non-monic P_j's on the left-hand side of (2.27), we can replace them one by one by monic polynomials $\tilde{P}_j$ without changing the value of the left-hand side of (2.27). Once they are all monic we are done by Proposition 2.3. □

2.4. Polynomials in several variables. In this subsection, we pass from one to several variables and thereby conclude the proof. Let $P \in \mathbb{C}[x]$ be a polynomial in r variables satisfying Mahler's hypothesis, and let

$$S_+^{r-1} := \left\{ \sigma \in \mathbb{R}_{\ge 0}^r \,\middle|\, \sum_{j=1}^{r} \sigma_j^2 = 1 \right\}.$$

Spherical coordinates (ρ, σ) on $\mathbb{R}^r_{\geq 0}$ (so $x = \rho\sigma, dx = \rho^{r-1}\,d\rho\,d\sigma$) allow us to write for $\mathrm{Re}(s) \gg 0$

$$\begin{aligned} Z(s; P; g) &:= \int_{x\in\mathbb{R}^r_{\geq 0}} g(x)P(x)^{-s}\,dx \\ &= \int_{\sigma\in S^{r-1}_+} \int_{\rho=0}^{\infty} \rho^{r-1} g(\rho\sigma)P(\rho\sigma)^{-s}\,d\rho\,d\sigma \quad = \int_{\sigma\in S^{r-1}_+} Z(s; P_\sigma; g_{\sigma,r})\,d\sigma, \end{aligned}$$

where the one-variable polynomials P_σ and $g_{\sigma,r}$ are defined by

$$P_\sigma(\rho) := P(\rho\sigma), \quad g_{\sigma,r}(\rho) = \rho^{r-1} g(\rho\sigma). \tag{2.30}$$

For $\rho \neq 0$, $(P_\sigma)_{\mathrm{top}}(\rho) = \rho^{\deg P} P_{\mathrm{top}}(\sigma) \neq 0$ by Mahler's hypothesis, so P_σ satisfies Mahler's hypothesis in one variable. Also, the degree in ρ of P_σ satisfies

$$\deg P_\sigma = \deg P \quad (\sigma \in S^{r-1}_+), \tag{2.31}$$

and $\deg g_{\sigma,r} \leq r - 1 + \deg g$. Thus, the set of poles of $Z(s; P_\sigma; g_{\sigma,r})$ is contained in the possible pole set of $Z(s; P; g)$ given by Mahler (see the second paragraph of Section 2). By analytic continuation we therefore have for all s outside this set of possible poles

$$Z(s; P; g) = \int_{\sigma\in S^{r-1}_+} Z(s; P_\sigma; g_{\sigma,r})\,d\sigma. \tag{2.32}$$

Taking the s-derivative at $s = 0$ inside the integral gives

$$I_n(P_1, \ldots, P_n; g) = \int_{\sigma\in S^{r-1}_+} I_n\left((P_1)_\sigma, \ldots, (P_n)_\sigma; g_{\sigma,r}\right)\,d\sigma. \tag{2.33}$$

Theorem 2.1. *Let $P_1, \ldots, P_n \in \mathbb{C}[x]$ be n polynomials in r variables, all satisfying Mahler's hypothesis (see Section 1), and let $g \in \mathbb{C}[x]$. Then*

$$\left(\sum_{j=1}^{n} \deg P_j\right) F_n(P_1, \ldots, P_n; g) = \sum_{1\leq i<j\leq n} (\deg P_i + \deg P_j) F_2(P_i, P_j; g),$$

where $F_n(P_1, \ldots, P_n; g)$ is defined by (2.2).

Proof. As before, write $d_j := \deg P_j$, $d_{n+1} := \sum_{j=1}^n d_j$. By Proposition 2.1, to prove the theorem it suffices to prove

$$d_{n+1} I_n(P_1, \ldots, P_n; g) = \sum_{1\leq i<j\leq n} (d_i + d_j) I_2(P_i, P_j; g). \tag{2.34}$$

For a fixed $\sigma \in S^{r-1}_+$, let $(P_j)_\sigma$ be as in (2.30). Proposition 2.4 and (2.31) give

$$d_{n+1} I_n\left((P_1)_\sigma, \ldots, (P_n)_\sigma; g_{\sigma,r}\right) = \sum_{1\leq i<j\leq n} (d_i + d_j) I_2\left((P_i)_\sigma, (P_j)_\sigma; g_{\sigma,r}\right).$$

In view of (2.33), integrating over $\sigma \in S^{r-1}_+$ yields (2.34). □

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