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MULTILINEAR EMBEDDING ESTIMATES FOR THE FRACTIONAL LAPLACIAN

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Abstract. Three novel multilinear embedding estimates for the fractional Laplacian are obtained in terms of trace integrals restricted to the diagonal. The resulting sharp inequalities may be viewed as extensions of the Hardy–Littlewood–Sobolev inequality, the Gagliardo–Nirenberg inequality and Pitt's inequality.

Sobolev embedding estimates are a central tool for analysis on geometric manifolds. Natural questions arise with the study of multilinear operators and product manifolds that incorporate intrinsic geometric symmetry. New realizations for the fractional Laplacian have emerged as critical elements for resolving challenging issues in nonlinear analysis and conformal geometry. Development of a rigorous framework for central problems in mathematical physics, including the structure and stability of matter and the dynamics of many-body interaction, has suggested new applications for estimates that measure fractional smoothness. Direct methods of approach have proved highly successful, but determining intrinsic connections with the overall framework of Sobolev embedding and fractional integrals is important and useful to gain new insight and increased understanding for the analytical structure. The effort to calculate optimal constants for embedding estimates and convolution integrals underlines not only intrinsic features for exact model problems and encoded geometric information, but lays the groundwork for calculating precise lower-order effects (see [9, 19]). Motivated by current interest to model the many-body dynamics of a Bose gas using the Gross–Pitaevskii hierarchy of density matrices, three new results are given for multilinear embeddings for the fractional Laplacian on $\mathbb{R}^n$ that can be viewed as separate extensions of the Hardy–Littlewood–Sobolev inequality, the Gagliardo–Nirenberg inequality and Pitt's inequality for the Fourier transform with weights. The simplicity of the argument underscores both the naturalness and the novelty of the result. The context for these inequalities has two explicit themes: (1) the development of the Gross–Pitaevskii hierarchy to describe multi-particle dynamics requires control of multilinear smoothing estimates where the evident natural question is to determine the analytic control that the smoothing estimates give in terms of physical measures such as trace integrals (see the operators S_j and R_j and the iterated multiplier argument in [22]); and (2) to determine the behavior of convolution integrals and Young's inequality for multilinear product decomposition of the manifold in the context of Riesz potentials and Stein–Weiss fractional integrals.

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Consider the convolution integral with multi-component decomposition

$$F(w) = \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} G(w-y)\, H(y)\, dy \ , \qquad w \in \mathbb{R}^{mn},$$

(1) "diagonal trace restriction":

$$F(w) \rightsquigarrow F(\underbrace{x, \ldots, x}_{m\,\text{slots}}) \equiv F(x) \ , \qquad x \in \mathbb{R}^n,$$

(2) "multilinear products":

$$F(x) = \int_{\mathbb{R}^{mn}} \prod g_k(x-y_k)\, H(y_1, \ldots, y_m)\, dy,$$

the objective is to determine how the components of G and the nature of H control the size of F. An important point to emphasize initially is that only for special cases will the analysis reduce to an iterative or product function characterization. Here the g_k's will be taken as inputs, including Riesz potentials, so the multilinear map is given by

$$H \in L^p(\mathbb{R}^{mn}) \rightsquigarrow F \in L^q(\mathbb{R}^n).$$

A relatively simple lemma that characterizes this framework can be easily obtained from the classical Young's inequality.

Lemma 1. *For $1 \le p, q < \infty$ with $1 < s_k < p'$ and $m/p' + 1/q = \sum 1/s_k$ (primes denote dual exponents, $1/p + 1/p' = 1$)*

$$\|F\|_{L^q(\mathbb{R}^n)} \le C \prod \|g_k\|_{L^{s_k}(\mathbb{R}^n)} \|H\|_{L^p(\mathbb{R}^{mn})}.$$

Proof. Consider the dual problem

$$\left[\int \Big| \int \prod_k g_k(x-y_k)\, h(x)\, dx \Big|^{p'} dy \right]^{1/p'} \le C \prod \|g_k\|_{L^{s_k}(\mathbb{R}^n)} \|h\|_{L^{q'}(\mathbb{R}^n)}.$$

Choose the sequence $\beta_k = q/s_k - q/p'$ with $\sum \beta_k = 1$ and apply Hölder's inequality on the left-hand side. ☐

Remark 1. (1) Lorentz-space extensions follow using interpolation and Hardy–Littlewood–Sobolev arguments. (2) In the special case where H is radial decreasing and so bounded above by a multiple of $|y|^{-mn/p}$ which allows a splitting of H by non-uniform inverse powers of $|y_k|$, Kenig and Stein [20] show that the allowed range of Lebesgue exponents is full for the indices s_k and that the index q can go below one. This latter result is not possible for the general case. (See their Lemma 7 on page 7 of [20].) (3) The classical notion of trace is expanded here to include any integral which is calculated over the diagonal restriction for variables. (4) In the special case where $p = q = 2$, the optimal value for the constant C is determined by

$$\int_{\mathbb{R}^n} \prod (g_k * g_k)(x)\, dx.$$

(5) More general versions for multilinear fractional integral kernels are treated in Christ [16], and in the conformally invariant setting by Beckner [4] (see Theorem 6 on pages 48–49). (6) A formative treatment for rigorously describing dynamical processes with many-body interaction in macroscopic systems appears in Spohn [25]. (7) Trace

integrals have an intrinsic analytic character which allows facility in making exact calculations. Physical motivation for analytic adaptation of the trace integral is described in [1, 25]. But the results described here are natural since fractional smoothness determines that restriction to a linear sub-variety is well defined (see Chapter 6 in [27], Theorem 11 in [12], the main theorem in [26] and discussion on page 32 in [24]). Implicit recognition of this structure underlies one argument in [22].

The square-integrable paradigm that represents the motivating step for the arguments developed here is the following representation for the Hardy–Littlewood–Sobolev inequality:

Lemma 2. *For $f \in \mathcal{S}(\mathbb{R}^n)$, $1 < p < 2$ and $\alpha = n(1/p - 1/2)$*

$$\int_{\mathbb{R}^n} |f|^2\,dx \le C_p \Big[\int_{\mathbb{R}^n} |(-\Delta/4\pi^2)^{\alpha/2} f|^p\,dx \Big]^{2/p},$$

$$C_p = \pi^{n/p - n/2} \Big[\Gamma(n/p')/\Gamma(n/p) \Big] \Big[\Gamma(n)/\Gamma(n/2) \Big]^{2/p-1}.$$

This lemma is an equivalent formulation using the fractional Laplacian for the sharp Hardy–Littlewood–Sobolev inequality calculated by Lieb [23].

Consider m copies of $\mathbb{R}^n$ and let f be in the Schwartz class $\mathcal{S}(\mathbb{R}^{mn})$. Define

$$(\mathcal{F}f)(x) = \widehat{f}\,(x) = \int e^{2\pi i x y} f(y)\,dy.$$

Observe that on $\mathbb{R}^n$ with $0 < \lambda < n$

$$\mathcal{F}\big[|x|^{-\lambda}\big] = \pi^{-n/2+\lambda} \left[\frac{\Gamma\big(\frac{n-\lambda}{2}\big)}{\Gamma\big(\frac{\lambda}{2}\big)} \right] |x|^{-(n-\lambda)}.$$

For $f \in \mathcal{S}(\mathbb{R}^{mn})$, Δ_k = standard Laplacian on $\mathbb{R}^n$ in the variable x_k, $0 < \alpha_k < n$, $\alpha = \sum \alpha_k$ for $k = 1$ to m and $(m-1)n < \alpha < mn$, define

$$\Lambda(f;\alpha_1,\dots,\alpha_m) = \int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n} \Big| \prod_{k=1}^m (-\Delta_k/4\pi^2)^{\alpha_k/4} f \Big|^2 dx_1\dots dx_m$$

$$= \int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n} \prod_{k=1}^m |\xi_k|^{\alpha_k} |\widehat{f}\,|^2\,d\xi_1\dots d\xi_m.$$

Theorem 1 (Pitt's inequality)**.** *For $f \in \mathcal{S}(\mathbb{R}^{mn})$ and $n - \beta = mn - \alpha$*

$$\int_{\mathbb{R}^n} |x|^{-\beta} |f(\underbrace{x,\dots,x}_{m\text{ slots}})|^2\,dx \le C_\beta\ \Lambda(f;\alpha_1,\dots,\alpha_m), \tag{1}$$

$$C_\beta = \pi^{-(m-1)n/2+\alpha} \prod_{k=1}^m \left[\frac{\Gamma\big(\frac{n-\alpha_k}{2}\big)}{\Gamma\big(\frac{\alpha_k}{2}\big)} \right] \left[\frac{\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{n-\beta}{2}\big)} \right] \left[\frac{\Gamma\big(\frac{n-\beta}{4}\big)}{\Gamma\big(\frac{n+\beta}{4}\big)} \right]^2.$$

The constant C_β is sharp and no extremals exist for this inequality.

Theorem 2 (Hardy–Littlewood–Sobolev inequality)**.** *For $f \in \mathcal{S}(\mathbb{R}^{mn})$ and $mn - \alpha = 2n/q$*

$$\left[\int_{\mathbb{R}^n} |f(\underbrace{x,\dots,x}_{m\ slots})|^q\,dx\right]^{2/q} \le F_\alpha\ \Lambda(f;\alpha_1,\dots,\alpha_m), \tag{2}$$

$$F_\alpha = \pi^{\alpha/2}\prod_{k=1}^m\left[\frac{\Gamma\big(\frac{n-\alpha_k}{2}\big)}{\Gamma\big(\frac{\alpha_k}{2}\big)}\right]\left[\frac{\Gamma\big(\frac{\alpha-(m-1)n}{2}\big)}{\Gamma\big(n-\frac{mn-\alpha}{2}\big)}\right]\left[\frac{\Gamma(n)}{\Gamma\big(\frac{n}{2}\big)}\right]^{\frac{\alpha-(m-1)n}{n}}.$$

The constant F_α is sharp and extremals are given by

$$f(x_1,\dots,x_m) = \int_{\mathbb{R}^n}\prod_{k=1}^m |x_k - w|^{-(n-\alpha_k/2)}|1+w^2|^{-n/q}\,dw$$

up to conformal automorphism of the factor $|1+w^2|^{-n/q}$.

Remark 2. If $m = 1$, the sharp forms of the classical Pitt's inequality and the Hardy–Littlewood–Sobolev inequality are recovered. By choosing f to be a product function, special cases of the general Pitt's inequality and the Stein–Weiss theorem can be obtained without sharp constants (see Appendix in [8]). For both inequalities, the term on the left-hand side can be viewed as "restriction to the diagonal". Simple iteration allows extension of these estimates to diagonal restriction traces on subblocks of the manifold $\mathbb{R}^{mn}$.

Proof of Theorem 1. Inequality (1) is equivalent to the multilinear fractional integral inequality:

$$\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^{mn}}\prod_{k=1}^m |x-y_k|^{-(n-\alpha_k/2)}f(y_1,\dots,y_m)\,dy\Big|^2\ |x|^{-\beta}\,dx$$

$$\le D_\beta\int_{\mathbb{R}^{mn}}|f(x_1,\dots,x_m)|^2\,dx,$$

$$C_\beta = \pi^{-mn+\alpha}\prod_{k=1}^m\left[\frac{\Gamma\big(\frac{2n-\alpha_k}{4}\big)}{\Gamma\big(\frac{\alpha_k}{4}\big)}\right]^2\ D_\beta.$$

By L^2 duality this is equivalent to

$$\int_{\mathbb{R}^{mn}}\Big|\int_{\mathbb{R}^n}\prod_{k=1}^m |y_k - x|^{-(n-\alpha_k/2)}|x|^{-\beta/2}g(x)\,dx\Big|^2\,dy \le D_\beta\int_{\mathbb{R}^n}|g(x)|^2\,dx.$$

Using rearrangement arguments this inequality is reduced to non-negative radial decreasing functions $g(x)$. The left-hand side becomes

$$\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{mn}} g(x)|x|^{-\beta/2}\prod_{k=1}^m |y_k - x|^{-(n-\alpha_k/2)|}$$

$$\times\prod_{k=1}^m |y_k - w|^{-(n-\alpha_k/2)}|w|^{-\beta/2}g(w)dx\,dw\,dy.$$

Integrating out the y_k variables

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} g(x)|x|^{-\beta/2}|x-w|^{-mn+\alpha}|w|^{-\beta/2}g(w)\,dx\,dw \le E_\beta \int_{\mathbb{R}^n} |g(x)|^2\,dx,$$

$$C_\beta = \pi^{-mn/2+\alpha}\prod_{k=1}^m \Gamma\Big(\frac{n-\alpha_k}{2}\Big)\Big/\Gamma\Big(\frac{\alpha_k}{2}\Big)\ E_\beta.$$

Since $mn-\alpha = n-\beta$, this becomes the classical Stein–Weiss fractional integral:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} g(x)|x|^{-\beta/2}|x-w|^{-(n-\beta)}|w|^{-\beta/2}g(w)\,dx\,dw \le E_\beta \int_{\mathbb{R}^n} |g(x)|^2\,dx$$

with

$$E_\beta = \pi^{n/2}\left[\frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{n-\beta}{2})}\right]\left[\frac{\Gamma(\frac{n-\beta}{4})}{\Gamma(\frac{n+\beta}{4})}\right]^2.$$

See Theorem 3 in [7] and also [5]. Then

$$C_\beta = \pi^{-(m-1)n+\alpha}\prod_{k=1}^m \frac{\Gamma(\frac{n-\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})}\left[\frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{n-\beta}{2})}\right]\left[\frac{\Gamma(\frac{n-\beta}{4})}{\Gamma(\frac{n+\beta}{4})}\right]^2$$

with $mn-\alpha = n-\beta$.

Remark 3. For notation, the Lebesgue measure dx incorporates the dimension of the underlying domain. Observe that as $\beta \to 0$ the constant C_β is unbounded so that the requirement $\beta > 0$ is strict. This reflects that the multilinear estimate is fully at the L^2 spectral level where one would not expect homogeneous Sobolev embedding without weights. The constant C_β is sharp and no extremals exist which follows from reduction to the one variable case in $\mathbb{R}^n$. Note that if f is a product function, for example $f(x)=\prod u(x_k)$, then the inequality reduces to the case of fractional Sobolev embedding on $\mathbb{R}^n$ where the index is an even integer and one can set $\beta = 0$. But the calculation here provides no information on the constant in that case. For large m, some α_k must approach n and so again the constant will be unbounded. Iterative methods are not effective which indicates that the results are clearly multidimensional. The appearance of the factors $\Gamma(\alpha_k/2)$ in the denominator of the constant C_β raises the question of how this constant will behave as one of the α_k's goes to zero. Observe that since $\alpha_k = \sum'(n-\alpha_\ell)+\beta$ is represented as a sum of positive terms with the sum taken for $\ell \ne k$, each term must also approach zero and in such case $C_\beta \to \infty$.

Proof of Theorem 2. Inequality (2) is equivalent to the multilinear fractional integral inequality:

$$\left[\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^{mn}}\prod_{k=1}^m |x-y_k|^{-(n-\alpha_k/2)} f(y_1,\dots,y_m)\,dy\Big|^q dx\right]^{2/q}$$

$$\le G_\alpha \int_{\mathbb{R}^{mn}} |f(x_1,\dots,x_m)|^2\,dx,$$

$$F_\alpha = \pi^{-mn+\alpha}\prod_{k=1}^m\left[\frac{\Gamma(\frac{2n-\alpha_k}{4})}{\Gamma(\frac{\alpha_k}{4})}\right]^2 G_\alpha.$$

By duality this is equivalent to

$$\int_{\mathbb{R}^{mn}} \Big| \int_{\mathbb{R}^n} \prod_{k=1}^m |y_k - x|^{-(n-\alpha_k/2)} g(x)\, dx \Big|^2 dy \le G_\alpha \Big[\int_{\mathbb{R}^n} |g(x)|^p\, dx \Big]^{2/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < 2$ and $mn - \alpha = 2n/q$. As with the calculation for Theorem 1, the left-hand side becomes

$$\int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{mn}} g(x) \prod_{k=1}^m |y_k - x|^{-(n-\alpha_k/2)} \prod_{k=1}^m |y_k - w|^{-(n-\alpha_k/2)} g(w)\, dx\, dw\, dy.$$

Integrating out the y_k variables

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} g(x) |x - w|^{-mn+\alpha} g(w)\, dx\, dw \le H_\alpha \Big[\int_{\mathbb{R}^n} |g(x)|^p\, dx \Big]^{2/p},$$

$$F_\alpha = \pi^{-mn/2+\alpha} \prod_{k=1}^m \Gamma\left(\frac{n-\alpha_k}{2}\right) \Big/ \Gamma\left(\frac{\alpha_k}{2}\right) H_\alpha.$$

Since $mn - \alpha = 2n/q$, this becomes the classical Hardy–Littlewood–Sobolev inequality:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} g(x) |x - w|^{-2n/q} g(w)\, dx\, dw \le H_\alpha \Big[\int_{\mathbb{R}^n} |g(x)|^p\, dx \Big]^{2/p}$$

with

$$H_\alpha = \pi^{n/q} \frac{\Gamma\left(\frac{n}{p} - \frac{n}{2}\right)}{\Gamma\left(\frac{n}{p}\right)} \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{p}\right)} \right]^{2/p-1}.$$

Then

$$F_\alpha = \pi^{\alpha/2} \prod_{k=1}^m \left[\frac{\Gamma\left(\frac{n-\alpha_k}{2}\right)}{\Gamma\left(\frac{\alpha_k}{2}\right)} \right] \left[\frac{\Gamma\left(\frac{\alpha-(m-1)n}{2}\right)}{\Gamma\left(n - \frac{mn-\alpha}{2}\right)} \right] \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \right]^{\frac{\alpha-(m-1)n}{n}}.$$

Extremal functions are determined by the classical inequality. □

The Stein–Weiss lemma (see Appendix in [8]) allows the trace inequality of Theorem 1 to be formulated in more general terms:

Theorem 3 (Stein–Weiss trace). *For $f \in \mathcal{S}(\mathbb{R}^{mn})$ consider*

$$F(x) = |x|^{-\beta/2} \int_{\mathbb{R}^{mn}} \prod_{k=1}^m K_k(x, y_k) f(y_1, \dots, y_m)\, dy_1, \dots, dy_m,$$

where $\{K_k(x,y)\}$ is a family of non-negative kernels defined on $\mathbb{R}^n \times \mathbb{R}^n$, each kernel being continuous on any domain that excludes the diagonal, homogeneous of degree $-\sigma_k$ $(0 < \sigma_k < n)$, $K_k(\delta u, \delta v) = \delta^{-\sigma_k} K_k(u,v)$, and $K_k(Ru, Rv) = K_k(u,v)$ of any $R \in SO(n)$; $0 < \beta < n$ with $2\sigma + \beta - mn = n$ where $\sigma = \sum \sigma_k$, Then

$$\int_{\mathbb{R}^n} |F(x)|^2\, dx \le A_\sigma \int_{\mathbb{R}^{mn}} |f(y_1, \dots, y_m)|^2\, dy, \tag{3}$$

$$A_\sigma = \int_{\mathbb{R}^n} |x|^{-\frac{\beta}{2} - \frac{n}{2}} \prod_k \Big[\int_{\mathbb{R}^n} K_k(x,y) K_k(\xi_1, y)\, dy \Big]\, dx$$

with ξ_1 a unit vector in the first coordinate direction.

Proof. Apply the argument used for the proof of Theorem 1 and observe that the kernel

$$\widehat{K}(x,w) = |x|^{-\beta/2}|w|^{-\beta/2}\prod_k \int_{\mathbb{R}^n} K_k(x,y)K_k(w,y)\,dy$$

satisfies the requirements of the Stein–Weiss lemma. The requirement for this trace estimate to hold is that A_σ is finite. □

In looking to understand how fractional smoothness controls size at the spectral level, and taking into account the dual representation given by the Fourier transform in balancing differentiability versus decay at infinity, the Stein–Weiss integral expresses a realization of the *uncertainty principle*

$$c\int_{\mathbb{R}^n} |f|^2\,dx \le \int_{\mathbb{R}^n} |(-\Delta/4\pi^2)^{\alpha/4}|x|^{\alpha/2}f(x)|^2\,dx. \tag{4}$$

An asymptotic argument gives directly the classical inequality. More broadly, this principle extends to include restriction to a k-dimensional linear sub-variety

$$d\int_{\mathbb{R}^k} |\mathcal{R}f|^2\,dx \le \int_{\mathbb{R}^n} \Big|(-\Delta/4\pi^2)^{\alpha/4}|x|^{\beta/2}f(x)\Big|^2\,dx, \tag{5}$$

where $n-\alpha = k-\beta$, $n \ge k > \beta > 0$ and

$$d = \pi^{-\alpha}\,\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\beta}{2})}\left[\frac{\Gamma(\frac{k+\beta}{4})}{\Gamma(\frac{k-\beta}{4})}\right]^2.$$

Using the principle of the Stein–Weiss trace formulated above, multilinear trace integral embedding estimates described here can be extended to include iterated multiplication of fractional powers with successive alternation between the function side and the Fourier transform side where the optimal constants will be given as closed-form integrals. Although the integral kernel does not have translation invariance on $\mathbb{R}^n$, dilation invariance transforms the problem to repeated convolution integrals on the multiplicative group $\mathbb{R}_+$ (see for example the section on iterated Stein–Weiss integrals in [7]) which facilitates the closed-form computation of sharp constants.

To illustrate this framework and outline the strategy needed to treat iterated multilinear embedding forms

$$\int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n} \Big|\prod |x_k|^{\rho_k/2}\prod_j\Big[\prod_k(-\Delta_k/4)^{\alpha_{jk}/4}|x_k|^{\beta_{jk}/2}\Big]f\Big|^2\,dx_1\cdots dx_m,$$

the following theorem includes the critical steps:

Theorem 4 (iterated Stein–Weiss). *For $f \in \mathcal{S}(\mathbb{R}^{mn})$, $0 < \alpha_k < n$, $\alpha = \sum \alpha_k$, $0 < \beta_k < n$, $\beta = \sum \beta_k$, $0 < \rho_k < n$, $\rho = \sum \rho_k$ and $n - \beta - \rho = mn - \alpha$ with*

$0 < \beta + \rho < n$

$$\int_{\mathbb{R}^n} |f(\underbrace{x, \cdots, x}_{m\ slots})|^2\, dx \le C \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \Big| \prod_{k=1}^m |x_k|^{\rho_k/2} (-\Delta_k/4\pi)^{\alpha_k/4}$$
$$\times |x_k|^{\beta_k/2} f \Big|^2 dx_1 \cdots dx_m,$$
$$C = \pi^{-mn+\alpha} \prod_{k=1}^m \left[\frac{\Gamma(\frac{2n-\alpha_k}{4})}{\Gamma(\frac{\alpha_k}{4})} \right]^2 \frac{2^{-mn+\alpha/2}}{\sigma(S^{n-1})} \int_{\mathbb{R}} H(x)\, dx$$

with H defined in the proof below. The constant C is sharp and no extremals exist for this inequality.

Proof. Following the argument given for Theorem 1, and using L^2 duality, the functional to estimate for $h \in L^2(\mathbb{R}^n)$ is

$$\int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{mn}} h(x)|x|^{-\beta/2} \prod_{k=1}^m \left[|y_k|^{-\rho_k} \Big(|y_k - x|\, |y_k - w| \Big)^{-(n-\alpha_k/2)} \right] |w|^{-\beta/2} h(w)\, dx\, dw\, dy.$$

Observe that either by the nature of the Stein–Weiss kernel or by applying the Brascamp–Lieb–Luttinger rearrangement theorem [11], the function h can be taken to be radial. Let $y_k = |y_k|\xi_k$, $x = |x|\eta_1$, $w = |w|\eta_2$. By transferring the analysis first to the multiplicative group $\mathbb{R}_+$ and then to the real line, the functional integral above is equivalent to the form

$$\int_{\mathbb{R} \times \mathbb{R}} g(x) H(x-y) g(y)\, dx\, dy,$$

where $g(x) = h(e^x) e^{nx/2}$ and

$$H(x) = \int_{S^{n-1} \times S^{n-1}} \prod_k B_k(x, n_1, n_2)\, d\eta_1\, d\eta_2$$

with

$$B_k(x, \eta_1, \eta_2)$$
$$= \int_{\mathbb{R} \times S^{n-1}} e^{-(\rho_k - \alpha_k)t} \left[\Big(\cosh\Big(\frac{x}{2} - t\Big) - \xi \cdot \eta_1 \Big) \Big(\cosh\Big(\frac{x}{2} + t\Big) - \xi \cdot \eta_2 \Big) \right]^{-(n/2 - \alpha_k/4)} dt\, d\xi.$$

Differentials on S^{n-1} correspond to standard surface measure. Then by Young's inequality, the constant for bounding the above form in terms in $(\|h\|_2)^2$ is

$$\frac{1}{\sigma(S^{n-1})} \int_{\mathbb{R}} H(x)\, dx$$

which provides the sharp value for C in Theorem 4. □

Bessel potentials provide a framework that extends the result of Theorem 2 in the sense of Gagliardo–Nirenberg estimates. Define

$$\Lambda_*(f;\alpha_1,\dots,\alpha_m) = \int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n} \Big|\prod_{k=1}^m (1-\Delta_k)^{\alpha_k/4} f\Big|^2 dx_1,\dots,dx_m$$

$$= \int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n} \prod_{k=1}^m (1+|\xi_k|^2)^{\alpha_k/2} |\widehat{f}\,|^2\, d\xi_1,\dots,d\xi_m.$$

Bessel potentials are defined by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta}\ e^{-\delta/4\pi}\ \delta^{-(n-\alpha)/2}\ \frac{1}{\delta}\, d\delta$$

with the properties:

$$G_\alpha(x) \ge 0\ ,\quad G_\alpha \in L^1(\mathbb{R}^n)\ ,\quad G_\alpha = \mathcal{F}\left[(1+4\pi^2|\xi|^2)^{-\alpha/2}\right]\ ,\quad \alpha > 0$$

and for $0 < \alpha < n$

$$G_\alpha(x) = \pi^{-n/2}\ 2^{-\alpha}\ \Gamma\Big(\frac{n-\alpha}{2}\Big)\Big/\Gamma\Big(\frac{\alpha}{2}\Big)\ \ |x|^{-n+\alpha} + o(|x|^{-n+\alpha}) \ \text{ as }\ |x| \to 0$$

and

$$G_n(x) \simeq -\left[(4\pi)^{n/2}\Gamma(n/2)\right]^{-1} \ln|x|^2 \ \text{ as }\ |x| \to 0,$$

$$|G_\beta(x)| \le \int_{\mathbb{R}^n} (1+4\pi^2|\xi|^2)^{-\beta/2}\, d\xi \ \text{ for }\ \beta > n$$

and for $\alpha > 0$

$$G_\alpha(x) = O(e^{-\varepsilon|x|}) \ \text{ as }\ |x| \to \infty \ \text{ for some }\ \varepsilon > 0$$

(see Stein [27, p. 132]).

Theorem 5 (Gagliardo–Nirenberg inequality)**.** *For $f \in \mathcal{S}(\mathbb{R}^{mn})$, $0 < \alpha_k < n$, $\alpha = \sum \alpha_k$, $k = 1,\dots,m$, $\frac{1}{q} + \frac{1}{p} = 1$ with $2 \le q \le 2n/(mn-\alpha)$ and $mn - \alpha < n$*

$$\left[\int_{\mathbb{R}^n} |f(\underbrace{x,\dots,x}_{m\ slots})|^q\, dx\right]^{2/q} \le C_{\alpha,q}\Lambda_*(f;\alpha_1,\dots,\alpha_m), \tag{6}$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} h(x) \prod_{k=1}^m G_{\alpha_k}(x-w) h(w)\, dx\, dw \le C_{\alpha,q}\Big(\|h\|_{L^p(\mathbb{R}^n)}\Big)^2. \tag{7}$$

Proof. Apply the method used for Theorem 2. For q below the critical index, use Young's inequality to obtain (7). At the critical index $q_* = 2n/(mn-\alpha)$ use the asymptotic behavior of the Bessel potential together with the Hardy–Littlewood–Sobolev inequality. □

Remark 4. This argument gives a sharp estimate for the case $p = q = 2$. Then the sharp constant is given by

$$C_{\alpha,2} = \int_{\mathbb{R}^n} \prod_{k=1}^m G_{\alpha_k}(x)\, dx.$$

For $m = 2$ this constant is $(4\pi)^{-n/2}\,\Gamma((\alpha - n)/2)$. For $m = 1$ this theorem reduces to a Gagliardo–Nirenberg inequality for the fractional Laplacian if $\alpha > 2n/q$. Observe that there exists a constant C so that

$$\prod_{k=1}^{m} (1 + |\xi_k|^2)^{\alpha_k/2} \le C\Big[1 + \prod_{k=1}^{m} |\xi_k|^{\alpha_k}\Big],$$

then

$$\Lambda_*(f;\alpha_1,\ldots,\alpha_m) \le C\Big[\int_{\mathbb{R}^{mn}} |f(x_1,\ldots,x_m)|^2\,dx + \Lambda(f;\alpha_1,\ldots,\alpha_m)\Big].$$

Using a variational argument, this corollary is obtained from Theorem 5.

Corollary 1. *For $f \in \mathcal{S}(\mathbb{R}^{mn})$, $0 < \alpha_k < n$, $\alpha = \sum \alpha_k$, $2 \le q < 2n/(mn - \alpha)$, $mn - \alpha < n$ and $\theta = (mn - \frac{2n}{q})/\alpha$,*

(8)

$$\Big[\int_{\mathbb{R}^n} |f(\underbrace{x,\ldots,x}_{m\,\text{slots}})|^q\,dx\Big]^{2/q} \le D_{\alpha,q}\Big[\int_{\mathbb{R}^{mn}} |f(x_1,\ldots,x_m)|^2\,dx\Big]^{1-\theta}\Big[\Lambda(f;\alpha_1,\ldots,\alpha_m)\Big]^{\theta}.$$

The case $q = 2$ is included here. The parameter θ is restricted: $1 - \frac{1}{n} < \theta < 1$. It is not tractable to calculate sharp values for this constant.

By allowing values of the fractional powers of $(1 - \Delta)$ to increase to the dimension of the space and above, the inequality in Theorem 5 can be extended. Define for two multi-indices of positive numbers

$$\bar\alpha = (\alpha_1,\ldots,\alpha_{m_1}) \;\text{ and }\; \bar\beta = (\beta_{m_1+1},\ldots,\beta_{m_1+m_2}),$$

$$0 < \alpha_k < n,\; n \le \beta_\ell,\; k = 1,\ldots,m_1,\; \ell = (m_1 + 1,\ldots,m_1 + m_2), m = m_1 + m_2,$$

$$\begin{aligned}\Lambda_\#(f;\bar\alpha,\bar\beta) = \int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n} \Big|&\prod_{k=1}^{m_1}(1 - \Delta_k/4\pi^2)^{\alpha_k/4} \\ &\times \prod_{\ell=m_1+1}^{m}(1 - \Delta_\ell/4\pi^2)^{\beta_k/4} f\Big|^2\,dx_1\ldots dx_m \\ = \int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n} &\prod_{k=1}^{m_1}(1 + |\xi_k|^2)^{\alpha_k/2} \prod_{\ell=m_1+1}^{m}(1 + |\xi_\ell|^2)^{\beta_\ell/2}|\hat f|^2\,d\xi_1\ldots d\xi_m.\end{aligned}$$

Theorem 6 (Gagliardo–Nirenberg inequality)**.** *For $f \in \mathcal{S}(\mathbb{R}^{mn})$, $0 < \alpha_k < n$, $\alpha = \sum \alpha_k$, $k = 1,\ldots,m_1$, $\frac{1}{q} + \frac{1}{p} = 1$ with $2 \le q \le 2n/(m_1 n - \alpha)$ and $m_1 n - \alpha < n$*

$$\Big[\int_{\mathbb{R}^n} |f(\underbrace{x,\ldots,x}_{m\;slots})|^q\,dx\Big]^{2/q} \le C_{\bar\alpha,\bar\beta,q}\Lambda_\#(f;\bar\alpha,\bar\beta), \tag{9}$$

(10)

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} h(x)\prod_{k=1}^{m_1} G_{\alpha_k}(x - w)\prod_{\ell=m_1+1}^{m} G_{\beta_\ell}(x - w)h(w)\,dx\,dw \le C_{\bar\alpha,\bar\beta,q}(\|h\|_{L^p(\mathbb{R}^n)})^2.$$

For $p = q = 2$

$$C_{\bar{\alpha},\bar{\beta},2} = \int_{\mathbb{R}^n} \prod_{k=1}^{m_1} G_{\alpha_k}(x) \prod_{\ell=m_1+1}^{m} G_{\beta_\ell}(x)\, dx. \tag{11}$$

Proof. Apply Theorem 5 together with asymptotic estimates for the Bessel potentials for large and small values of $|x|$. If all fractional powers are larger than the dimension n, then the allowed range of p extends to $1 \le p \le 2$ for estimate (10) with $1 < p \le 2$ if some values of the fractional power are equal to n. □

Remark 5. In [15] Chen and Pavlovic "introduce a generalization of Sobolev and Gagliardo–Nirenberg inequalities on the level of marginal density matrices", and their stated result corresponds to the case $q = 2$ with uniform α_k's for inequality (9) above though the methods are entirely different from those used here. Their work is related to obtaining a priori energy bounds for solutions to the Gross–Pitaevskii hierarchy. But there is possible confusion between their notation and that used by Klainerman and Machedon [22] (see also [17, 18, 21]) as they use $\langle \nabla \rangle = \sqrt{1-\Delta}$. To clarify potential issues, it is not possible, even for multivariable functions invariant under the symmetric group, to have global homogeneous Sobolev inequalities of trace type for the index $q = 2$ in contrast to the special case of product functions. Such a result in the general case would force integrability for Riesz potentials and on a conceptual level would "break" the uncertainty principle. This phenomena is similar to the limitation discussed earlier in the context of the Kenig–Stein theorem for fractional integration. The arguments described here clearly illustrate examples where multilinear structure cannot be reduced to the case of product functions or simple iterative processes though the proofs use quadratic functional integration to simplify the calculation where the spirit is similar to application of a Hilbert–Schmidt norm or the Plancherel theorem.

Extension of the Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^n$ to include the multilinear embedding estimates described here suggests that analogous results should hold for the sphere S^n. Following the development outlined in [4] (see p. 62):

Hardy–Littlewood–Sobolev inequality on S^n. *Let F be a smooth function on S^n with corresponding expansion in spherical harmonics, $F = \sum Y_k$; $\alpha = n - 2n/q$ for $q > 2$ and define*

$$B = \left[-\Delta + \left(\frac{n-1}{2}\right)^2\right]^{1/2}, \quad D_\alpha = \frac{\Gamma(B + (1+\alpha)/2)}{\Gamma(B + (1-\alpha)/2)}$$

observe

$$D_\alpha Y_k = \frac{\Gamma(\frac{n}{q'} + k)}{\Gamma(\frac{n}{q} + k)}\, Y_k.$$

Then

$$\left[\|F\|_{L^q(S^n)}\right]^2 \le \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{q})\,\Gamma(\frac{n}{q'} + k)}{\Gamma(\frac{n}{q'})\,\Gamma(\frac{n}{q} + k)} \int_{S^n} |Y_k|^2\, d\xi, \tag{12}$$

$$\left[\|F\|_{L^q(S^n)}\right]^2 \le \frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \int_{S^n} F(D_\alpha F)\, d\xi, \tag{13}$$

where $d\xi$ denotes normalized surface measure on S^n.

This theorem is the sharp Hardy–Littlewood–Sobolev inequality for the n-dimensional sphere as obtained by Lieb [23] in terms of fractional integrals. The representation using spherical harmonics was given by Beckner in [2].

Theorem 7 (Hardy–Littlewood–Sobolev inequality). *For F in the Schwartz class formed over m copies of S^n, and $mn-\alpha = 2n/q$ with $0<\alpha_k<n$, $\alpha=\sum\alpha_k$ and $(m-1)n<\alpha<mn$. Let*

$$\Lambda_S(F,\alpha_1,\dots,\alpha_m) \equiv \prod_{k=1}^m \frac{\Gamma(\frac{n-\alpha_k}{2})}{\Gamma(\frac{n+\alpha_k}{2})} \int_{S^n\times\cdots\times S^n} F\Big(\prod_k D_{\alpha_k} F\Big)\, d\xi_1,\dots,d\xi_m,$$

where D_{α_k} acts on the k^{th} coordinate. Then

$$\Big[\int_{S^n} |F(\underbrace{\xi,\dots,\xi}_{m\ slots})|^q\, d\xi\Big]^{2/q} \le F_{\alpha,S}\Lambda_S(F,\alpha_1,\dots,\alpha_m), \tag{14}$$

$$F_{\alpha,S} = \left[\frac{\Gamma(\frac{n}{q})}{\Gamma(n)}\right]^{m-1} \frac{\Gamma(\frac{n}{2}-\frac{n}{q})}{\Gamma(\frac{n}{p})} \prod_k \frac{\Gamma(\frac{n+\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})}.$$

Proof. Let $T_{\alpha_k} = [\Gamma(\frac{n-\alpha_k}{2})/\Gamma(\frac{n+\alpha_k}{2})]D_{\alpha_k}$; then T_{α_k} is a positive-definite self-adjoint invertible operator and $T_{\alpha_k}^{-1}$ can be realized as a fractional integral operator on S^n:

$$(T_{\alpha_k}^{-1}G)(\xi) = 2^{n-\alpha_k}\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\,\frac{\Gamma(\frac{n+\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})}\int_{S^n}|\xi-\eta|^{-(n-\alpha_k)}G(\eta)\,d\eta.$$

Inequality (14) is equivalent to the inequality

$$\Big[\int_{S^n}\Big|\Big(\prod_k T_{\alpha_k}^{-1/2}F\Big)(\xi)\Big|^q d\xi\Big]^{2/q} \le F_{\alpha,S}\int_{S^n\times\cdots\times S^n}|F(\xi_1,\dots,\xi_m)|^2\,d\xi_1,\dots,d\xi_m$$

and by duality this is equivalent to

$$\int_{S^n\times\cdots\times S^n}\Big|\Big(\prod_k T_{\alpha_k}^{-1/2}G\Big)(\xi_1,\dots,\xi_m)\Big|^2 d\xi_1,\dots,\xi_m \le F_{\alpha,S}\Big[\int_{S^n}|G(\xi)|^p\,d\xi\Big]^{2/p},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<p<2$ and $mn-\alpha=2n/q$. As with earlier calculations, the left-hand side becomes

$$\int_{S^n\times S^n} G(\eta)\prod_k T_{\alpha_k}^{-1}(\xi,\eta)G(\xi)\,d\xi\,d\eta \le F_{\alpha,S}\Big[\int_{S^n}|G(\xi)|^p\,d\xi\Big]^{2/p},$$

$$\begin{aligned}\prod_k T_{\alpha_k}^{-1}(\xi,\eta) &= \prod_k 2^{n-\alpha_k}\,\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\,\frac{\Gamma(\frac{n+\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})}\,|\xi-\eta|^{-(n-\alpha_k)}\\ &= 2^{2n/q}\left[\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right]^m\prod_k\frac{\Gamma(\frac{n+\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})}\,|\xi-\eta|^{-2n/q}.\end{aligned}$$

Re-writing the previous inequality

$$A_\alpha\int_{S^n\times S^n}G(\eta)|\xi-\eta|^{-2n/q}G(\xi)\,d\xi\,d\eta \le F_{\alpha,S}\Big[\int_{S^n}|G(\xi)|^p\,d\xi\Big]^{2/p}, \tag{15}$$

where

$$A_\alpha = 2^{2n/q} \left[\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right]^m \prod_k \frac{\Gamma(\frac{n+\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})}.$$

Now inequality (15) is the classical Hardy–Littlewood–Sobolev inequality on the sphere, and by comparison with the sharp constant (see [3])

$$F_{\alpha,S} = A_\alpha \; 2^{-2n/q} \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n-\frac{2n}{q}}{2})}{\Gamma(\frac{n}{p})},$$

$$F_{\alpha,S} = \left[\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right]^{m-1} \frac{\Gamma(\frac{n}{2}-\frac{n}{q})}{\Gamma(\frac{n}{p})} \prod_k \frac{\Gamma(\frac{n+\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})}. \tag{16}$$

With the calculation of the sharp constant $F_{\alpha,S}$, the proof of Theorem 7 is complete. □

Remark 6. While the inequalities in Theorems 2 and 7 directly depend on the conformally invariant Hardy–Littlewood–Sobolev inequality, these inequalities are not in themselves conformally invariant. This circumstance may be reflected in the lack of limiting phenomena for the allowed range of Lebesgue exponents.

The arguments given here, and especially for Theorem 7, allow a fairly general formulation of the essential idea embodied in these theorems. Suppose $\{T_k\}$ is a family of positive-definite self-adjoint invertible operators acting on smooth function classes on an n-dimensional manifold equipped with a suitable measure. For simplicity, assume that T_k^{-1} can be realized by action of an integral kernel. Then the preceding theorems may be reformulated in the following form:

Theorem 8 (Multilinear trace)**.** *For $f \in \mathcal{S}(M \times \cdots \times M)$ and some suitable $q \geq 2$ with $\frac{1}{p}+\frac{1}{q}=1$ and T_k acting on the k^{th} variable*

$$\left[\int_M |f(\underbrace{x,\ldots,x}_{m\ slots})|^q \, dx\right]^{2/q} \leq E_T \int_{M\times\cdots\times M} f\Big(\prod_k T_k\, f\Big)\, dx_1,\ldots,dx_m. \tag{17}$$

E_T is the optimal constant for the inequality

$$\int_{M\times M} h(w) \prod_k T_k^{-1}(x,w) h(x)\, dx\, dw \leq E_T \left[\int_M |h|^p \, dx\right]^{2/p}. \tag{18}$$

Proof. Just follow the steps in the previous argument. □

Remark 7. In the context of the conformally invariant structure discussed in the author's papers [4, 6], and more recent treatments of the fractional Laplacian ([13, 14]), it is natural to consider the multilinear trace estimate in the setting where M is a hyperbolic manifold. Broader questions can be addressed in that context, and they will be treated in a forthcoming paper [10].

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