Math. Res. Lett. **19** (2012), *no.* 01, 149–164

ON THE BEILINSON–HODGE CONJECTURE FOR H^2 AND RATIONAL VARIETIES

ANDRE CHATZISTAMATIOU

ABSTRACT. The Beilinson–Hodge conjecture asserts the surjectivity of the cycle map

$$H^n_M(X, \mathbb{Q}(n)) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(-n), H^n(X, \mathbb{Q})),$$

for all integers $n \geq 1$ and every smooth complex algebraic variety X. For $n = 2$, we prove the conjecture if X is rational.

CONTENTS

1. Introduction

The Beilinson–Hodge conjecture (BH(X, n)) asserts the surjectivity of the cycle map

$$\bar{c}_{n,n} : H^n_M(X, \mathbb{Q}(n)) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(-n), H^n(X, \mathbb{Q})),$$

for all integers $n \geq 1$ and every smooth complex algebraic variety X, where the left-hand side is motivic cohomology. Informally speaking, the conjecture means that every holomorphic n-form with logarithmic poles at infinity and rational periods comes from a meromorphic form of the shape

$$\frac{1}{(2\pi i)^n} \sum_j m_j \cdot \frac{df_{j1}}{f_{j1}} \wedge \cdots \wedge \frac{df_{jn}}{f_{jn}},$$

with $f_{jk} \in \mathbb{C}(X)^*$ and $m_j \in \mathbb{Q}$.

Received by the editors May 12, 2011.

It is well known that the conjecture holds for $n = 1$ (see [9, Proposition 2.12]). For $n \geq 2$, Asakura and Saito provided evidence for the conjecture by studying the Noether–Lefschetz locus of Beilinson–Hodge cycles (see [2, 3, 4]). By work of Arapura and Kumar, $\mathrm{BH}(X, n)$ is known to hold for every n provided that X is a semi-abelian variety or a product of curves [1].

In our paper, we consider only the case $n = 2$, and make two observations. First, for a smooth and projective variety X the Beilinson–Hodge conjecture $\mathrm{BH}(\eta, 2)$ for the generic point η of X is equivalent to the injectivity of the cycle map

$$\frac{H^3_M(X, \mathbb{Q}(2))}{H^1_M(X, \mathbb{Q}(1)) \cdot H^2_M(X, \mathbb{Q}(1))} \to \frac{H^3_{\mathcal{H}}(X, \mathbb{Q}(2))}{H^1_{\mathcal{H}}(X, \mathbb{Q}(1)) \cdot H^2_{\mathcal{H}}(X, \mathbb{Q}(1))}$$

to absolute Hodge cohomology (Proposition 3.5). The left-hand side is called the group of *indecomposable* cycles and has been studied by Müller–Stach [16]. In general, indecomposable cycles exist, but the image via the cycle class map is a countable group. By $\mathrm{BH}(\eta, 2)$ we mean the surjectivity of the cycle class map

$$H^2_M(\mathbb{C}(X), \mathbb{Q}(2)) \to \varinjlim_{U \subset X} \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(-2), H^2(U, \mathbb{Q})),$$

where U runs over all open subsets of X, and $\mathbb{C}(X)$ denotes the function field.

The second observation is that if X satisfies $H^1(X, \mathbb{C}) = 0$ then $\mathrm{BH}(U, 2)$ for all open sets $U \subset X$ is equivalent to $\mathrm{BH}(\eta, 2)$ (Proposition 3.6). The statement makes perfect sense when $H^1(X, \mathbb{C}) \neq 0$, but we can prove it only in the case $H^1(X, \mathbb{C}) = 0$.

Combining the two observations we obtain the main theorem of the paper.

Theorem (cf. Theorem 3.1)**.** *Let X be smooth and connected. Let $\bar{X}$ be a smooth compactification of X. We denote by $\mathrm{CH}_0(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ the Chow group of zero cycles on $\bar{X}$. If $\deg : \mathrm{CH}_0(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is an isomorphism then $BH(X, 2)$ holds.*

For the proof we use a theorem of Bloch and Srinivas [7] which states that the indecomposable cycles vanish whenever the assumptions of our theorem are satisfied. The theorem can be applied in the case where X is a rational variety, because $\mathrm{CH}_0(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a birational invariant of $\bar{X}$, and therefore $\mathrm{CH}_0(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{CH}_0(\mathbb{P}^{\dim X}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

2. Cycle class to absolute Hodge cohomology

2.1. Higher Chow groups and absolute Hodge cohomology. Let X be a smooth algebraic variety over the complex numbers.

Absolute Hodge cohomology was introduced by Beilinson ([5], cf. [12, Section 2]). Beilinson constructs for every complex algebraic variety X an object $\underline{R\Gamma}(X, \mathbb{Q})$ in the derived category of mixed Hodge structures $D^b(\mathrm{MHS})$ such that

$$H^i(\underline{R\Gamma}(X, \mathbb{Q})) = H^i(X, \mathbb{Q}), \quad \text{for all } i.$$

Absolute Hodge cohomology $H^{\bullet}_{\mathcal{H}}(X, \mathbb{Q}(\bullet))$ is defined as follows:

$$H^q_{\mathcal{H}}(X, \mathbb{Q}(p)) = \mathrm{Hom}_{D^b(\mathrm{MHS})}(\mathbb{Q}, \underline{R\Gamma}(X, \mathbb{Q})(p)[q]),$$

for all p, q.

The natural spectral sequence

$$E_2^{ij} = \mathrm{Ext}^i_{\mathrm{MHS}}(\mathbb{Q}, H^j(X,\mathbb{Q})(p)) \Rightarrow H^{i+j}_{\mathcal{H}}(X,\mathbb{Q}(p)),$$

and vanishing of Ext^i for $i > 1$, induces short exact sequences

$$0 \to \mathrm{Ext}^1(\mathbb{Q}, H^{q-1}(X,\mathbb{Q})(p)) \to H^q_{\mathcal{H}}(X,\mathbb{Q}(p)) \to \mathrm{Hom}(\mathbb{Q}, H^q(X,\mathbb{Q})(p)) \to 0. \tag{2.1}$$

Note that Homs and Exts are taken in the category of mixed Hodge structures.

If X is smooth and proper then we have a comparison isomorphism with Deligne cohomology

$$H^q_{\mathcal{H}}(X,\mathbb{Q}(p)) \cong H^q_{\mathcal{D}}(X,\mathbb{Q}(p)), \tag{2.2}$$

provided that $q \leq 2p$ [12, Section 2.7].

2.2. Cycle class map. Let $DM_{gm,\mathbb{Q}}$ be Voevodsky's triangulated category of motivic complexes with rational coefficients over $\mathbb{C}$ [17, 15]. Denoting by $\mathrm{Sm}/\mathbb{C}$ the category of smooth complex algebraic varieties, there is a functor

$$\mathrm{Sm}/\mathbb{C} \to DM_{gm,\mathbb{Q}}, \quad X \mapsto M_{gm}(X).$$

Motivic cohomology is defined by

$$H^q_M(X,\mathbb{Q}(p)) = \mathrm{Hom}_{DM_{gm}}(M_{gm}(X), \mathbb{Q}(p)[q]),$$

for X smooth and $p \geq 0, q \in \mathbb{Z}$. There is a comparison isomorphism [18] with Bloch's higher Chow groups

$$H^q_M(X,\mathbb{Q}(p)) \cong \mathrm{CH}^p(X, 2p-q) \otimes \mathbb{Q}.$$

By Levine [14] and Huber [10, 11], we have realizations

$$r_{\mathcal{H}} : DM_{gm} \to D^b(\mathrm{MHS}) \tag{2.3}$$

at our disposal, such that $\underline{R\Gamma}(X,\mathbb{Q})$ is the dual of $r_{\mathcal{H}}(M_{gm}(X))$ and $r_{\mathcal{H}}(\mathbb{Q}(1)) = \mathbb{Q}(1)$. The realizations are triangulated $\otimes$-functors and therefore induce cycle maps

$$c_{p,2p-q} : H^q_M(X,\mathbb{Q}(p)) \to H^q_{\mathcal{H}}(X,\mathbb{Q}(p)), \tag{2.4}$$

which are compatible with the localization sequence. An explicit construction of a cycle map by using currents is presented in [13]. Composition of $c_{p,2p-q}$ with the projection from (2.1) yields

$$\bar{c}_{p,2p-q} : H^q_M(X,\mathbb{Q}(p)) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^q(X,\mathbb{Q})(p)). \tag{2.5}$$

2.3. Beilinson–Hodge conjecture. Let X be a smooth algebraic variety over $\mathbb{C}$, and $n \geq 0$ an integer.

Conjecture 2.1 (Beilinson–Hodge conjecture). $\mathrm{BH}(X,n)$*: The cycle map* $\bar{c}_{n,n}$ (2.5) *is surjective.*

Remark 2.1. If X is smooth then

$$c_{1,1} : H^1_M(X, \mathbb{Q}(1)) \to H^1_{\mathcal{H}}(X, \mathbb{Q}(1))$$

is an isomorphism [9, Proposition 2.12]. In particular, BH(X,1) holds.

3. Beilinson–Hodge conjectures for the generic point

3.1. Coniveau spectral sequences. The main technical tool of our paper is the coniveau spectral sequence for motivic and absolute Hodge cohomology. The existence and construction of the coniveau spectral is well known and follows from the yoga of exact couples as in [6]. Because we could not provide a reference for the case of absolute Hodge cohomology we will explain the construction in this section.

3.1.1. In the following, ? will stand for M or $\mathcal{H}$. For $p \geq 0$ we denote by $X^{(p)}$ the set of codimension p points of X. For a point $x \in X$ (not necessarily closed) we define

$$H^q_?(x, \mathbb{Q}(p)) := \varinjlim_{x \in U \subset X} H^q_?(U, \mathbb{Q}(p)),$$

where U runs over all open neighborhoods of x. For all $n \geq 0$, the coniveau spectral sequence reads:

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}, p \leq n} H^{q-p}_?(x, \mathbb{Q}(n-p)) \Rightarrow H^{p+q}_?(X, \mathbb{Q}(n)). \tag{3.1}$$

The terms $E_1^{p,q}$ with $p > n$ are zero.

3.1.2. In order to construct the coniveau spectral sequence we use the category of finite correspondences $\mathrm{Cor}_{\mathbb{C}}$ [15, Lecture 1]. There is an obvious functor

$$Sm/\mathbb{C} \to \mathrm{Cor}_{\mathbb{C}}, \quad X \mapsto [X].$$

We denote by $\mathcal{H}^b(\mathrm{Cor}_{\mathbb{C}})$ the homotopy category of bounded complexes [17, Section 2.1]. By construction [17, Definition 2.1.1] there is a triangulated functor

$$\mathcal{H}^b(\mathrm{Cor}_{\mathbb{C}}) \to DM_{gm,\mathbb{Q}}. \tag{3.2}$$

Definition 3.1. Let X be smooth and $Y \subset X$ a closed set. We define $c_Y X$ to be the complex

$$[X \backslash Y] \xrightarrow{\jmath} \underset{\deg=0}{[X]}$$

in $\mathcal{H}^b(\mathrm{Cor}_{\mathbb{C}})$. The map $\jmath$ is the open immersion.

Let $Y \subset X$ be a closed subset. For an open subset U of X and a closed subset Y' of U such that $Y \cap U \subset Y'$ we get a morphism of complexes

$$c_{Y'} U \to c_Y X. \tag{3.3}$$

Lemma 3.1. *Let X be smooth. Let Y_1, Y_2 be closed subsets of X with $Y_2 \subset Y_1$.*

(1) *The morphisms*

$$c_{Y_1 \backslash Y_2}(X \backslash Y_2) \to c_{Y_1} X \to c_{Y_2} X \xrightarrow{+1} c_{Y_1 \backslash Y_2}(X \backslash Y_2)[1]$$

induced by (3.3) *and*

$$c_{Y_2}X \to c_{Y_1\backslash Y_2}(X\backslash Y_2)[1]$$

$$\begin{array}{ccc}
[X] & & \\
\uparrow & & \\
[X\backslash Y_2] & \xrightarrow{-\mathrm{id}} & [X\backslash Y_2] \\
 & & \uparrow{\scriptstyle -\mathrm{incl}} \\
 & & [X\backslash Y_1]
\end{array}$$

form a distinguished triangle in $\mathcal{H}^b(Cor_{\mathbb{C}})$.

(2) *If* $Y_2' \subset Y_1'$ *are closed subsets of* X *such that* $Y_i \subset Y_i'$, *for* $i = 1, 2$, *then the morphisms from* (3.3) *induce a morphism of distinguished triangles*

$$\begin{array}{ccccccc}
c_{Y_1\backslash Y_2}(X\backslash Y_2) & \longrightarrow & c_{Y_1}X & \longrightarrow & c_{Y_2}X & \xrightarrow{+1} & \\
\uparrow & & \uparrow & & \uparrow & & \\
c_{Y_1'\backslash Y_2'}(X\backslash Y_2') & \longrightarrow & c_{Y_1'}X & \longrightarrow & c_{Y_2'}X & \xrightarrow{+1} &
\end{array}$$

Proof. For (1). equivalent to 0. There is an obvious isomorphism in $\mathcal{H}^b(\mathrm{Cor}_{\mathbb{C}})$:

$$\mathrm{cone}(c_{Y_1}X \to c_{Y_2}X)[-1] \to c_{Y_1\backslash Y_2}(X\backslash Y_2),$$

$$\begin{array}{ccc}
[X] & & \\
\uparrow & & \\
[X] \oplus [X\backslash Y_2] & \xrightarrow{-\mathrm{pr}_2} & [X\backslash Y_2] \\
{\scriptstyle (\mathrm{incl}, -\mathrm{incl})}\uparrow & & \uparrow \\
[X\backslash Y_1] & \xrightarrow{=} & [X\backslash Y_1],
\end{array}$$

rendering commutative the diagram

$$\begin{array}{ccccccc}
c_{Y_2}X[-1] & \longrightarrow & \mathrm{cone}(c_{Y_1}X \to c_{Y_2}X)[-1] & \longrightarrow & c_{Y_1}X & \longrightarrow & c_{Y_2}X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
c_{Y_2}X[-1] & \longrightarrow & c_{Y_1\backslash Y_2}(X\backslash Y_2) & \longrightarrow & c_{Y_1}X & \longrightarrow & c_{Y_2}X.
\end{array}$$

For (2). Straight-forward. □

Definition 3.2. Let X be smooth and $Y \subset X$ a closed subset. For all $n \geq 0$ and $q \in \mathbb{Z}$ we define

$$H^q_{Y,M}(X, \mathbb{Q}(n)) := \mathrm{Hom}_{DM_{gm}}(c_Y X, \mathbb{Q}(n)[q]),$$
$$H^q_{Y,\mathcal{H}}(X, \mathbb{Q}(n)) := \mathrm{Hom}_{D^b(\mathrm{MHS})}(r_{\mathcal{H}}(c_Y X), \mathbb{Q}(n)[q]).$$

We implicitly used the functor (3.2).

From (3.3) we obtain a map

$$H^*_{Y,?}(X, \mathbb{Q}(n)) \to H^*_{Y',?}(U, \mathbb{Q}(n)),$$

if U is an open subset of X and $Y \cap U \subset Y'$.

3.1.3. For $p \geq 0$, we denote by $Z^p = Z^p(X)$ the set of closed subsets of X of codimension $\geq p$, ordered by inclusion. Let Z^p/Z^{p+1} denote the ordered set of pairs $(Z, Z') \in Z^p \times Z^{p+1}$ such that $Z \supset Z'$, with the ordering

$$(Z, Z') \geq (Z_1, Z_1'), \quad \text{if } Z \supset Z_1 \text{ and } Z' \supset Z_1'.$$

We can form for all $n \geq 0$ and $p \in \mathbb{Z}$:

$$H^*_{Z^p,?}(X, \mathbb{Q}(n)) := \begin{cases} \varinjlim_{Z \in Z^p} H^*_{Z,?}(X, \mathbb{Q}(n)), & \text{if } p \geq 0, \\ H^*_?(X, \mathbb{Q}(n)), & \text{if } p \leq 0. \end{cases}$$

$$H^*_{Z^p/Z^{p+1},?}(X, \mathbb{Q}(n)) := \begin{cases} \varinjlim_{(Z,Z') \in Z^p/Z^{p+1}} H^*_{Z \setminus Z',?}(X \setminus Z', \mathbb{Q}(n)), & \text{if } p \geq 0, \\ 0, & \text{if } p < 0. \end{cases}$$

In view of Lemma 3.1, we obtain for every $(Z, Z') \in Z^p/Z^{p+1}$ a long exact sequence

$$H^*_{Z',?}(X, \mathbb{Q}(n)) \to H^*_{Z,?}(X, \mathbb{Q}(n)) \to H^*_{Z \setminus Z',?}(X \setminus Z', \mathbb{Q}(n)) \xrightarrow{+1}, \tag{3.4}$$

and we can take the limit to get a long exact sequence

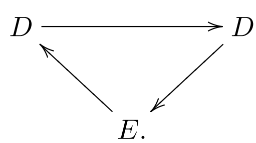
$$H^*_{Z^{p+1},?}(X, \mathbb{Q}(n)) \to H^*_{Z^p,?}(X, \mathbb{Q}(n)) \to H^*_{Z^p/Z^{p+1},?}(X, \mathbb{Q}(n)) \xrightarrow{+1}. \tag{3.5}$$

This also holds for $p < 0$ for trivial reasons. We form an exact couple as follows:

$$D := \bigoplus_{p \in \mathbb{Z}} H^*_{Z^p,?}(X, \mathbb{Q}(n)), \tag{3.6}$$

$$E := \bigoplus_{p \geq 0} H^*_{Z^p/Z^{p+1},?}(X, \mathbb{Q}(n)),$$

and the exact triangle induced by (3.5)



Setting

$$E_1^{p,q} := H^{p+q}_{Z^p/Z^{p+1},?}(X, \mathbb{Q}(n)),$$

the exact couple yields a spectral sequence

$$E_1^{p,q} \Rightarrow H^{p+q}_?(X, \mathbb{Q}(n)), \tag{3.7}$$

for all $n \geq 0$, such that

$$E_\infty^{p,q} = \frac{N^p H^{p+q}_?(X, \mathbb{Q}(n))}{N^{p+1} H^{p+q}_?(X, \mathbb{Q}(n))},$$

with

$$N^p H^i_?(X, \mathbb{Q}(n)) = \text{image}(H^i_{Z^p,?}(X, \mathbb{Q}(n)) \to H^i_?(X, \mathbb{Q}(n))).$$

Lemma 3.2. *Let X be smooth and $n \geq 0$.*

(1) *If $p \leq n$ then*

$$H^{q+p}_{Z^p/Z^{p+1},?}(X, \mathbb{Q}(n)) \cong \bigoplus_{x \in X^{(p)}} H^{q-p}(x, \mathbb{Q}(n-p)), \quad \text{for all } q \in \mathbb{Z}.$$

(2) *If $p > n$ then*

$$H^{q}_{Z^p/Z^{p+1},?}(X, \mathbb{Q}(n)) = 0, \quad \text{for all } q \in \mathbb{Z}.$$

Proof. The set

$$S = \{(Z, Z') \in Z^p/Z^{p+1} \mid Z \backslash Z' \text{ is smooth of pure codimension} = p\}$$

is a cofinal subset of Z^p/Z^{p+1}; thus

$$H^{q+p}_{Z^p/Z^{p+1},?}(X, \mathbb{Q}(n)) = \varinjlim_{(Z,Z') \in S} H^{*}_{Z \backslash Z',?}(X \backslash Z', \mathbb{Q}(n))$$

From the Gysin triangle [17, Proposition 3.5.4] we obtain for all $(Z, Z') \in S$ a natural isomorphism

$$c_{Z \backslash Z'}(X \backslash Z') \cong M_{gm}(Z \backslash Z')(p)[2p],$$

in DM_{gm}.

For (1). After substituting the above equation into the definition and using cancelation, we obtain

$$H^{q+p}_{Z \backslash Z',?}(X \backslash Z', \mathbb{Q}(n)) = H^{q-p}_{?}(Z \backslash Z', \mathbb{Q}(n-p)),$$

for all $(Z, Z') \in S$, and the restriction maps

$$H^{q-p}_{?}(Z \backslash Z', \mathbb{Q}(n-p)) \to \bigoplus_{x \in X^{(p)}} H^{q-p}_{?}(x, \mathbb{Q}(n-p))$$

induce the desired isomorphism.

For (2). Suppose $p > n$. We claim that

$$H^{*}_{Z,?}(X, \mathbb{Q}(n)) = 0, \tag{3.8}$$

for all $Z \in Z^p(X)$. In view of the long exact sequence (3.4) this will prove the claim.

By definition the vanishing of $H^{*}_{Z,?}(X, \mathbb{Q}(n))$ follows if the restriction map

$$H^{*}_{?}(X, \mathbb{Q}(n)) \to H^{*}_{?}(X \backslash Z, \mathbb{Q}(n)) \tag{3.9}$$

is an isomorphism. Set $U := X \backslash Z$. For $? = M$ we can use the comparison isomorphism with higher Chow groups. It is sufficient to prove that the restriction induces an isomorphism of complexes

$$Z^n(X, \bullet) \xrightarrow{\cong} Z^n(U, \bullet), \tag{3.10}$$

where $Z^n(?, \bullet)$ denotes Bloch's cycle complex. Since $X \backslash U$ has codimension $> n$, the map (3.10) is injective. For the surjectivity, let $A \in Z^n(U, m)$ be the class of an irreducible subvariety of $U \times \Delta^m$. By definition A has codimension n and meets all faces $U \times \Delta^i$ properly. Let $\bar{A}$ be the closure of A in $X \times \Delta^m$. Since

$$\bar{A} \cap (X \times \Delta^i) \subset (A \cap (U \times \Delta^i)) \cup ((X \backslash U) \times \Delta^i),$$

and $(X \backslash U) \times \Delta^i$ has codimension $> n$ in $X \times \Delta^i$, we conclude that $\bar{A} \in Z^n(X, m)$.

For $? = \mathcal{H}$. In view of (2.1), we need to prove that the restriction induces isomorphisms

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(-n), H^q(X,\mathbb{Q})) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(-n), H^q(U,\mathbb{Q})), \tag{3.11}$$

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-n), H^q(X,\mathbb{Q})) \xrightarrow{\cong} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-n), H^q(U,\mathbb{Q})), \tag{3.12}$$

for all q. In order to prove (3.11) and (3.12) we use the exact sequence

$$\begin{aligned} 0 \to \frac{H^q_{X\backslash U}(X,\mathbb{Q})}{\mathrm{im}(H^{q-1}(U,\mathbb{Q}))} &\to H^q(X,\mathbb{Q}) \to H^q(U,\mathbb{Q}) \\ &\to \ker\left(H^{q+1}_{X\backslash U}(X,\mathbb{Q}) \to H^{q+1}(X,\mathbb{Q})\right) \to 0. \end{aligned} \tag{3.13}$$

Note that $\frac{H^q_{X\backslash U}(X,\mathbb{Q})}{\mathrm{im}(H^{q-1}(U,\mathbb{Q}))}$ and $\ker(H^{q+1}_{X\backslash U}(X,\mathbb{Q}) \to H^{q+1}(X,\mathbb{Q}))$ are Hodge structures of weight $\geq 2p$. Indeed, we claim that $H^*_{X\backslash U}(X,\mathbb{Q}) = H^*_Z(X,\mathbb{Q})$ is a Hodge structure of weight $\geq 2p$, for all $Z \in Z^p(X)$. If Z is smooth and of pure codimension p then

$$H^*_Z(X,\mathbb{Q}) = H^{*-2p}(Z,\mathbb{Q})(-p),$$

and the claim holds. In the general case, we can find an open $V \subset X$ such that $Z \cap V$ is smooth of pure codimension p and $(Z\backslash V) \in Z^{p+1}(X)$. In view of the long exact sequence

$$H^*_{Z\backslash V}(X,\mathbb{Q}) \to H^*_Z(X) \to H^*_{Z\cap V}(V,\mathbb{Q}) \xrightarrow{+1}$$

we can reduce to the smooth case by descending induction on p.

Now, if E is any mixed Hodge structure of weight $\geq 2p$ then

$$\mathrm{Hom}(\mathbb{Q}(-n), E) = 0, \quad \mathrm{Ext}^1(\mathbb{Q}(-n), E) = 0,$$

because $p > n$. Therefore (3.13) implies the statement. □

Proposition 3.1. *Let X be smooth and $? = M$ or $? = \mathcal{H}$. Let $n \geq 0$ be an integer.*

(1) *There is a spectral sequence*

$$E^{p,q}_{1,?} = \bigoplus_{x \in X^{(p)}, p \leq n} H^{q-p}_?(x, \mathbb{Q}(n-p)) \Rightarrow H^{p+q}_?(X, \mathbb{Q}(n))$$

such that

$$E^{p,q}_{\infty,?} = \frac{N^p H^{p+q}_?(X,\mathbb{Q}(n))}{N^{p+1} H^{p+q}_?(X,\mathbb{Q}(n))},$$

with

$$N^p H^*_?(X,\mathbb{Q}(n)) = \bigcup_{\substack{U \subset X \\ \mathrm{cd}(X\backslash U) \geq p}} \ker(H^*_?(X,\mathbb{Q}(n)) \to H^*_?(U,\mathbb{Q}(n))),$$

where U runs over all open subsets with $\mathrm{codim}(X\backslash U) \geq p$.

(2) *The cycle map induces a morphism of spectral sequences*

$$[E^{p,q}_{1,M} \Rightarrow H^{p+q}_M(X,\mathbb{Q}(n))] \to [E^{p,q}_{1,\mathcal{H}} \Rightarrow H^{p+q}_{\mathcal{H}}(X,\mathbb{Q}(n))].$$

Proof. For (1). The statement follows from the spectral sequence (3.7) and Lemma 3.2.

For (2). The realization $r_{\mathcal{H}}$ (2.3) induces a morphism of the exact couples (3.6). □

3.2. E_1 complexes of the coniveau spectral sequence. Let X be smooth and connected, we denote by η the generic point of X. The cycle map induces a morphism between the $E_1^{\bullet,2}$ complexes of the coniveau spectral sequence (Proposition 3.1) for $n = 2$:

$$\begin{array}{ccccc} E_{1,M}^{\bullet,2} : H_M^2(\eta, \mathbb{Q}(2)) & \longrightarrow & \oplus_{x\in X^{(1)}} H_M^1(x, \mathbb{Q}(1)) & \longrightarrow & \oplus_{x\in X^{(2)}} \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow \\ E_{1,\mathcal{H}}^{\bullet,2} : H_{\mathcal{H}}^2(\eta, \mathbb{Q}(2)) & \longrightarrow & \oplus_{x\in X^{(1)}} H_{\mathcal{H}}^1(x, \mathbb{Q}(1)) & \longrightarrow & \oplus_{x\in X^{(2)}} \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(\mathbb{Q}, H^2(\eta, \mathbb{Q})(2)) & \longrightarrow & \oplus_{x\in X^{(1)}} \mathrm{Hom}(\mathbb{Q}, H^1(x, \mathbb{Q})(1)) & \longrightarrow & \oplus_{x\in X^{(2)}} \mathbb{Q}. \end{array} \tag{3.14}$$

We call the complex in the first line $G_M(X, 2)$, the complex in the second line $G_{\mathcal{H}}(X, 2)$, and finally the complex in the third line is called $G_{\mathrm{HS}}(X, 2)$. The complex $G_{\mathrm{HS}}(X, 2)$ is induced by $G_{\mathcal{H}}(X, 2)$ via (2.1).

For $G_M(X, 2)$ the group $H_M^2(\eta, \mathbb{Q}(2))$ is the component in degree $= 0$, and the grading is defined similarly for $G_{\mathcal{H}}(X, 2)$ and $G_{\mathrm{HS}}(X, 2)$. Via Gersten–Quillen resolution we have

$$H^1(G_M(X, 2)) = H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{3.15}$$

where $\mathcal{K}_2$ is Quillen's K-theory Zariski sheaf associated to the presheaf

$$U \mapsto K_2(\mathcal{O}_X(U)).$$

Proposition 3.2. *Let X be smooth and connected.*

(i) *There is a natural isomorphism $H^1(G_M(X, 2)) \xrightarrow{\cong} H_M^3(X, \mathbb{Q}(2))$.*

(ii) *There is a natural injective map $H^1(G_{\mathcal{H}}(X, 2)) \to H_{\mathcal{H}}^3(X, \mathbb{Q}(2))$. We call the image $H_{\mathcal{H},\mathrm{alg}}^3(X, \mathbb{Q}(2))$.*

(iii) *There is a natural isomorphism*

$$H^1(G_{\mathrm{HS}}(X, 2)) \to H_{\mathcal{H},\mathrm{alg}}^3(X, \mathbb{Q}(2)) / \left(H_{\mathcal{H}}^1(\mathbb{C}, \mathbb{Q}(1)) \cdot H_{\mathcal{H}}^2(X, \mathbb{Q}(1))\right).$$

(iv) *The above maps form a commutative diagram*

$$\begin{array}{ccc} H^1(G_M(X, 2)) & \xrightarrow{\cong} & H_M^3(X, \mathbb{Q}(2)) \\ \downarrow & & \downarrow c_{2,1} \\ H^1(G_{\mathcal{H}}(X, 2)) & \xrightarrow{\cong} & H_{\mathcal{H},\mathrm{alg}}^3(X, \mathbb{Q}(2)) \\ \downarrow & & \downarrow \\ H^1(G_{\mathrm{HS}}(X, 2)) & \xrightarrow{\cong} & H_{\mathcal{H},\mathrm{alg}}^3(X, \mathbb{Q}(2)) / H_{\mathcal{H}}^1(\mathbb{C}, \mathbb{Q}(1)) \cdot H_{\mathcal{H}}^2(X, \mathbb{Q}(1)), \end{array}$$

and

$$c_{2,1} : H_M^3(X, \mathbb{Q}(2)) \to H_{\mathcal{H},\mathrm{alg}}^3(X, \mathbb{Q}(2))$$

is surjective.

Proof. Statement (i) is proved in [16].

Proof of (i) and (ii). We use the coniveau spectral sequence (Proposition 3.1)

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} H_?^{q-p}(x,\mathbb{Q}(n-p)) \Rightarrow H_?^{p+q}(X,\mathbb{Q}(n)), \tag{3.16}$$

where ? is M or $\mathcal{H}$, and $0 \le p \le n$. We have

$$G_?(X,2) = E_1^{\bullet,2},$$

for $n=2$. We get $E_2^{1,2} = E_\infty^{1,2}$ and $E_\infty^{2,1} = 0 = E_\infty^{3,0}$ for obvious reasons. Therefore we obtain an exact sequence

$$0 \to E_\infty^{1,2} \to H_?^3(X,\mathbb{Q}(2)) \to E_\infty^{0,3} \to 0, \tag{3.17}$$

with

$$E_\infty^{1,2} = \ker(H_?^3(X,\mathbb{Q}(2)) \to H_?^3(\eta,\mathbb{Q}(2))).$$

For $? = M$ we have $H_M^3(\eta,\mathbb{Q}(2)) = 0$. Indeed, by definition the equality

$$H_M^3(\eta,\mathbb{Q}(2)) = \varinjlim_{\emptyset\neq U\subset X} \mathrm{CH}^2(U,1)$$

holds, and since $\mathrm{CH}^2(U,1)$ is a subquotient of $Z^2(U,1)$ (Bloch's cycle complex), it suffices to show

$$\varinjlim_{\emptyset\neq U\subset X} Z^2(U,1) = 0.$$

Now, $Z^2(U,1)$ is generated by classes $[A]$ of irreducible subvarieties $A \subset U\times\Delta$ such that A has codimension 2 and meets all faces properly. The closure $\overline{\mathrm{pr}_U(A)}$ of the image of A via the projection $\mathrm{pr}_U : U\times\Delta \to U$ has codimension ≥ 1, thus $[A]$ vanishes under the restriction to $U\backslash\overline{\mathrm{pr}_U(A)}$.

For $? = \mathcal{H}$ we define

$$H^3_{\mathcal{H},\mathrm{alg}}(X,\mathbb{Q}(2)) := \ker(H^3_{\mathcal{H}}(X,\mathbb{Q}(2)) \to H^3_{\mathcal{H}}(\eta,\mathbb{Q}(2))) = N^1H^3_{\mathcal{H}}(X,\mathbb{Q}(2)).$$

For (iii). If X is smooth then

$$\mathrm{Pic}(X)\otimes\mathbb{Q} = H_M^2(X,\mathbb{Q}(1)) \cong H_{\mathcal{H}}^2(X,\mathbb{Q}(1)),$$

in view of Lemma 3.3 below. It follows that via the isomorphism $H^1(G_{\mathcal{H}}(X,2)) \cong H^3_{\mathcal{H},\mathrm{alg}}(X,\mathbb{Q}(2))$ the subgroup $H^1_{\mathcal{H}}(\mathbb{C},\mathbb{Q}(1))\cdot H^2_{\mathcal{H}}(X,\mathbb{Q}(1))$ of $H^3_{\mathcal{H},\mathrm{alg}}(X,\mathbb{Q}(2))$ corresponds to the image of $\oplus_{x\in X^{(1)}}\mathbb{C}^*\otimes_{\mathbb{Z}}\mathbb{Q}$ in $H^1(G_{\mathcal{H}}(X,2))$. For every point $x\in X^{(1)}$ we have an exact sequence

$$0 \to \mathbb{C}^*\otimes_{\mathbb{Z}}\mathbb{Q} \to H^1_{\mathcal{H}}(x,\mathbb{Q}(1)) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},H^1(x,\mathbb{Q})(1)) \to 0,$$

and therefore

$$\ker(H^1(G_{\mathcal{H}}(X,2)) \to H^1(G_{\mathrm{HS}}(X,2))) = \mathrm{im}(\oplus_{x\in X^{(1)}}\mathbb{C}^*\otimes_{\mathbb{Z}}\mathbb{Q} \to H^1(G_{\mathcal{H}}(X,2))).$$

This implies the claim.

Statement (iv) is obvious. □

Lemma 3.3. *If X is smooth then*

$$\mathrm{Pic}(X) \otimes \mathbb{Q} = H^2_M(X, \mathbb{Q}(1))$$

and

$$H^2_M(X, \mathbb{Q}(1)) \to H^2_{\mathcal{H}}(X, \mathbb{Q}(1)). \tag{3.18}$$

is an isomorphism.

Proof. For the first equality we note that

$$H^2_M(X, \mathbb{Q}(1)) = \mathrm{CH}^1(X) \otimes \mathbb{Q} = \mathrm{Pic}(X) \otimes \mathbb{Q}.$$

We denote by $A(X)$ the statement: (3.18) is an isomorphism. The first step is to prove $A(X)$ for proper varieties. Indeed, if X is proper then we have

$$H^2_{\mathcal{H}}(X, \mathbb{Q}(1)) \overset{(2.2)}{\cong} H^2_{\mathcal{D}}(X, \mathbb{Q}(1)) \cong H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \otimes \mathbb{Q}.$$

The map (3.18) can be identified with

$$H^1_{\mathrm{Zar}}(X, \mathcal{O}^*_X) \otimes \mathbb{Q} \to H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \otimes \mathbb{Q},$$

which is an isomorphism by GAGA.

Now, let $Z \subset X$ be a smooth closed subset of X. We claim that

$$A(X) \implies A(X \backslash Z). \tag{3.19}$$

If $\mathrm{codim}(Z) \geq 2$ then the restriction maps are isomorphisms:

$$H^2_M(X, \mathbb{Q}(1)) \xrightarrow{\cong} H^2_M(X \backslash Z, \mathbb{Q}(1)),$$
$$H^2_{\mathcal{H}}(X, \mathbb{Q}(1)) \xrightarrow{\cong} H^2_{\mathcal{H}}(X \backslash Z, \mathbb{Q}(1)),$$

(see (3.9)). Thus we may assume that Z is smooth of pure codimension $= 1$. In this case we have a morphism of exact sequences

$$\begin{array}{ccccccc}
H^0_M(Z, \mathbb{Q}(0)) & \longrightarrow & H^2_M(X, \mathbb{Q}(1)) & \longrightarrow & H^2_M(X \backslash Z, \mathbb{Q}(1)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow & & \downarrow & & \\
H^0_{\mathcal{H}}(Z, \mathbb{Q}(0)) & \longrightarrow & H^2_{\mathcal{H}}(X, \mathbb{Q}(1)) & \longrightarrow & H^2_{\mathcal{H}}(X \backslash Z, \mathbb{Q}(1)) & \longrightarrow & 0,
\end{array}$$

and $A(X)$ implies $A(X \backslash Z)$.

By using the existence of smooth compactifications with strict normal crossing divisor at infinity, the statement (3.19) and $A(X)$ for X proper, imply that $A(X)$ holds for any smooth variety X. □

Remark 3.1. For a smooth projective variety X we know that

$$H^1(G_{\mathrm{HS}}(X, 2)) \xrightarrow{\cong} H^3_{\mathcal{H},\mathrm{alg}}(X, \mathbb{Q}(2)) / \left(H^1_{\mathcal{H}}(\mathbb{C}, \mathbb{Q}(1)) \cdot H^2_{\mathcal{H}}(X, \mathbb{Q}(1)) \right)$$

is a countable group [16]. This is a consequence of the fact that deformations a' of a class $a \in H^3_M(X, \mathbb{Q}(2))$ have the same image via $c_{2,1}$ modulo the group $H^1_{\mathcal{H}}(\mathbb{C}, \mathbb{Q}(1)) \cdot H^2_{\mathcal{H}}(X, \mathbb{Q}(1))$. There exist examples of K3-surfaces X such that $H^1(G_{\mathrm{HS}}(X, 2)) \neq 0$ [16].

3.2.1. In the following we will write $H^1_?(\mathbb{C},\mathbb{Q}(1))$ for $H^1_?(\mathrm{Spec}(\mathbb{C}),\mathbb{Q}(1))$, with $? \in \{M,\mathcal{H}\}$. For a proper smooth connected variety X, we have a commutative diagram

$$\begin{array}{ccccc} H^1_M(X,\mathbb{Q}(1)) & \xrightarrow[\text{[15, Theorem 4.1]}]{\cong} & H^0(X,\mathcal{O}^*_X)\otimes\mathbb{Q} & \xrightarrow{=} & \mathbb{C}^*\otimes\mathbb{Q} \\ \downarrow & & \downarrow\cong & & \downarrow = \\ H^1_{\mathcal{H}}(X,\mathbb{Q}(1)) & \xrightarrow[(2.2)]{\cong} & H^0(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}})\otimes\mathbb{Q} & \xrightarrow{=} & \mathbb{C}^*\otimes\mathbb{Q}. \end{array} \tag{3.20}$$

In particular, $H^1_?(\mathbb{C},\mathbb{Q}(1)) \to H^1_?(X,\mathbb{Q}(1))$ is an isomorphism.

Definition 3.3. Let X be smooth, connected and projective. We denote by

$$\mathrm{image}(H^1_M(\mathbb{C},\mathbb{Q}(1))\cdot H^2_M(X,\mathbb{Q}(1))) =: H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}} \subset H^3_M(X,\mathbb{Q}(2))$$

the subgroup of decomposable cycles. In the same way we define $H^3_{\mathcal{H}}(X,\mathbb{Q}(2))_{\mathrm{dec}}$.

Lemma 3.4. *If X is smooth, projective, and $H^1(X)=0$, then the maps*

$$H^1_M(\mathbb{C},\mathbb{Q}(1))\otimes_{\mathbb{Q}} H^2_M(X,\mathbb{Q}(1))) \to H^3_M(X,\mathbb{Q}(2)),$$
$$H^1_{\mathcal{H}}(\mathbb{C},\mathbb{Q}(1))\otimes_{\mathbb{Q}} H^2_{\mathcal{H}}(X,\mathbb{Q}(1))) \to H^3_{\mathcal{H}}(X,\mathbb{Q}(2))$$

are injective. In particular,

$$H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}} \to H^3_{\mathcal{H}}(X,\mathbb{Q}(2))_{\mathrm{dec}}$$

is an isomorphism.

Proof. By using the cycle map it is sufficient to prove the statement for absolute Hodge cohomology. The assumption $H^1(X)=0$ implies

$$H^2_{\mathcal{H}}(X,\mathbb{Q}(1))) \cong \mathrm{Hom}(\mathbb{Q}(-1),H^2(X,\mathbb{Q})).$$

The pure Hodge structure $H^2(X,\mathbb{Q})$ is polarizable and therefore

$$\mathrm{Hom}(\mathbb{Q}(-1),H^2(X,\mathbb{Q}))\otimes\mathbb{Q}(-1) \subset H^2(X,\mathbb{Q})$$

is a direct summand which we call $\mathrm{Hg}^{1,1}$. We get

$$\begin{aligned}(\mathrm{Hg}^{1,1}\otimes_{\mathbb{Q}}\mathbb{C})/(2\pi i)^2\cdot\mathrm{Hg}^{1,1}_{\mathbb{Q}} &= \mathrm{Ext}^1(\mathbb{Q}(-2),\mathrm{Hg}^{1,1}) \\ &\subset \mathrm{Ext}^1(\mathbb{Q}(-2),H^2(X,\mathbb{Q})) \subset H^3_{\mathcal{H}}(X,\mathbb{Q}(2)),\end{aligned}$$

and clearly $H^1_{\mathcal{H}}(\mathbb{C},\mathbb{Q}(1))\otimes_{\mathbb{Q}} H^2_{\mathcal{H}}(X,\mathbb{Q}(1)))$ is mapping isomorphically onto $(\mathrm{Hg}^{1,1}\otimes_{\mathbb{Q}}\mathbb{C})/(2\pi i)^2\cdot\mathrm{Hg}^{1,1}_{\mathbb{Q}}$. Because of (3.20) and Lemma 3.3 we conclude that

$$H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}} \to H^3_{\mathcal{H}}(X,\mathbb{Q}(2))_{\mathrm{dec}}$$

is an isomorphism. □

Proposition 3.3. *Let X be smooth and connected. Restriction to the generic point yields the following equalities:*

$$H^2_M(X,\mathbb{Q}(2)) \xrightarrow{\cong} H^0(G_M(X,2)), \tag{3.21}$$

$$H^2_{\mathcal{H}}(X,\mathbb{Q}(2)) \xrightarrow{\cong} H^0(G_{\mathcal{H}}(X,2)). \tag{3.22}$$

Proof. We use the coniveau spectral sequence (Proposition 3.1) for $n = 2$. We have

$$E_{2,?}^{0,2} = H^0(G_?(X,2)),$$

for $? = M$ and $? = \mathcal{H}$. Note that $E_{1,?}^{2,q} = 0$ for $q \neq 2$. Thus $E_{2,?}^{0,2} = E_{\infty,?}^{0,2}$ and $E_{\infty,?}^{2,0} = 0$. Moreover, $E_{1,?}^{1,1} = 0$, because $H_?^0(U,\mathbb{Q}(1)) = 0$ for every U. It follows that $E_{\infty,?}^{1,1} = 0$ and:

$$H_?^2(X,\mathbb{Q}(2)) = E_{\infty,?}^{0,2} = E_{2,?}^{0,2}.$$

□

Lemma 3.5. *Let X be smooth and connected. We denote by η the generic point of X. If*

$$H_M^2(\eta,\mathbb{Q}(2)) \to H_{\mathcal{H}}^2(\eta,\mathbb{Q}(2))$$

is surjective then

$$H_M^2(X,\mathbb{Q}(2)) \to H_{\mathcal{H}}^2(X,\mathbb{Q}(2))$$

is surjective.

Proof. We use Proposition 3.3 and need to prove that

$$H^0(G_M(X,2)) \to H^0(G_{\mathcal{H}}(X,2))$$

is surjective. Since $H_M^1(x,\mathbb{Q}(1)) = H_{\mathcal{H}}^1(x,\mathbb{Q}(1))$ for every point $x \in X$ of codimension $= 1$ this follows immediately from diagram (3.14). □

Proposition 3.4. *If X is smooth, connected and projective then*

$$H^0(G_{\mathrm{HS}}(X,2)) = 0.$$

Proof. Let $\eta \in X$ be the generic point. For

$$a \in H^0(G_{\mathrm{HS}}(X,2)) \subset \mathrm{Hom}(\mathbb{Q}(-2), H^2(\eta,\mathbb{Q})),$$

we can find an effective divisor D such that a is induced by a cohomology class $a' \in \mathrm{Hom}(\mathbb{Q}(-2), H^2(X\backslash D,\mathbb{Q}))$. Let S be a closed subset of X of codimension ≥ 2, such that $D\backslash S$ is smooth. Denoting $X' := X\backslash S$ and $D' = D\backslash S$, we claim that $a' \mid_{X'\backslash D'}$ maps to zero in $H^1(D',\mathbb{Q})(-1)$ via the boundary map of the localization sequence for singular cohomology. Indeed, the map

$$\mathrm{Hom}(\mathbb{Q}(-1), H^1(D',\mathbb{Q})) \to \bigoplus_{x\in X^{(1)}} \mathrm{Hom}(\mathbb{Q}(-1), H^1(x,\mathbb{Q}))$$

is injective and therefore the claim follows from $a \in H^0(G_{\mathrm{HS}}(X,2))$.

Now $a' \mid_{X'\backslash D'}$ defines an extension of Hodge structures:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{im}\left(H^0(D',\mathbb{Q})(-1)\right) & \longrightarrow & H^2(X',\mathbb{Q}) & \longrightarrow & H^2(X'\backslash D',\mathbb{Q}) & \longrightarrow & H^1(D',\mathbb{Q})(-1) \\
 & & = \big\uparrow & & \cup \big\uparrow & \square & \big\uparrow a'|_{X'\backslash D'} & & \big\uparrow \\
0 & \longrightarrow & \mathrm{im}\left(H^0(D',\mathbb{Q})(-1)\right) & \longrightarrow & E & \longrightarrow & \mathbb{Q}(-2) & \longrightarrow & 0,
\end{array}$$

where $\mathrm{im}(H^0(D',\mathbb{Q})(-1)) = \mathrm{im}(\mathbb{Q}(-1)^{\pi_0(D')})$ is the image in $H^2(X',\mathbb{Q})$. We note that $H^2(X',\mathbb{Q}) = H^2(X,\mathbb{Q})$ is a pure Hodge structure of weight $= 2$, and therefore the same holds for E. Thus the extension is trivial and $a' \mid_{X'\backslash D}$ lifts to

$$\mathrm{Hom}(\mathbb{Q}(-2), H^2(X',\mathbb{Q})) = 0.$$

This proves that $a' \mid_{X'\backslash D} = 0$ and implies $a = 0$. □

3.3. An exact sequence for projective varieties with vanishing H^1.

Lemma 3.6. *Let X be smooth, projective and connected. Suppose that $H^1(X)=0$. We denote by η the generic point of X. There is an exact sequence*

$$H^2_M(\eta,\mathbb{Q}(2)) \to H^2_{\mathcal{H}}(\eta,\mathbb{Q}(2)) \to H^3_M(X,\mathbb{Q}(2)) \to H^3_{\mathcal{H}}(X,\mathbb{Q}(2)).$$

Proof. Via Proposition 3.2 we identify

$$H^3_M(X,\mathbb{Q}(2)) \cong H^1(G_M(X,2)), \quad H^1(G_{\mathcal{H}}(X,2)) \subset H^3_{\mathcal{H}}(X,\mathbb{Q}(2))$$

and need to show that there is an exact sequence

$$H^2_M(\eta,\mathbb{Q}(2)) \to H^2_{\mathcal{H}}(\eta,\mathbb{Q}(2)) \to H^1(G_M(X,2)) \to H^1(G_{\mathcal{H}}(X,2)).$$

We work with diagram (3.14). The map

$$H^2_{\mathcal{H}}(\eta,\mathbb{Q}(2)) \to H^1(G_M(X,2)) \tag{3.23}$$

is defined by using $E^{1,2}_{1,M} = E^{1,2}_{1,\mathcal{H}}$. The assumptions on X imply that $H^2_{\mathcal{H}}(X,\mathbb{Q}(2))=0$ and therefore $H^0(G_{\mathcal{H}}(X,2))=0$ by Proposition 3.3. Thus the map (3.23) has precisely the image of $H^2_M(\eta,\mathbb{Q}(2))$ as kernel. Since $E^{1,2}_{1,M} = E^{1,2}_{1,\mathcal{H}}$, the map $H^1(G_M(X,2)) \to H^1(G_{\mathcal{H}}(X,2))$ has the image of $H^2_{\mathcal{H}}(\eta,\mathbb{Q}(2))$ as kernel. □

3.4. Main theorem.

Theorem 3.1. *Let X be smooth and connected. Let $\bar{X}$ be a smooth compactification of X. We denote by $\mathrm{CH}_0(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ the Chow group of zero cycles on $\bar{X}$. If $\deg : \mathrm{CH}_0(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is an isomorphism then $BH(X,2)$ holds.*

Proof. In view of Lemma 3.5 it is sufficient to show that $\bar{c}_{2,2} : H^2_M(\eta,\mathbb{Q}(2)) \to H^2_{\mathcal{H}}(\eta,\mathbb{Q}(2))$ is surjective, where η is the generic point on X.

Now, choose a smooth projective model Y of η. Since $\bar{X}$ and Y are birational we conclude that $\mathrm{CH}_0(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. It follows that $H^1(Y)=0$.

We claim that

$$H^3_M(Y,\mathbb{Q}(2))_{\mathrm{dec}} = H^3_M(Y,\mathbb{Q}(2)).$$

This implies the theorem by using Lemma 3.6, because

$$H^3_M(Y,\mathbb{Q}(2))_{\mathrm{dec}} \subset H^3_{\mathcal{H}}(Y,\mathbb{Q}(2))_{\mathrm{dec}}$$

by Lemma 3.4.

In view of Proposition 3.2 and (3.15) it is sufficient to prove that the cokernel of

$$\mathbb{C}^* \otimes_{\mathbb{Z}} \mathrm{Pic(X)} \to H^1(X,\mathcal{K}_2)$$

is torsion. This is proved in [7, Theorem 3(i)]. □

3.5. An exact sequence for projective varieties.

Proposition 3.5. *Let X be smooth, projective and connected. We denote by η the generic point of X. There is an exact sequence*

$$\begin{aligned} H^2_M(\eta,\mathbb{Q}(2)) &\to \mathrm{Hom}(\mathbb{Q}(-2), H^2(\eta,\mathbb{Q})) \to H^3_M(X,\mathbb{Q}(2))/H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}} \\ &\to H^3_{\mathcal{H}}(X,\mathbb{Q}(2))/H^3_{\mathcal{H}}(X,\mathbb{Q}(2))_{\mathrm{dec}}. \end{aligned}$$

Proof. Via Proposition 3.2 we identify

$$H^1(G_M(X,2)) \cong H^3_M(X,\mathbb{Q}(2)),$$
$$H^1(G_{\mathrm{HS}}(X,2)) \subset H^3_{\mathcal{H}}(X,\mathbb{Q}(2))/H^3_{\mathcal{H}}(X,\mathbb{Q}(2))_{\mathrm{dec}}.$$

We work with diagram (3.14). For the map

$$\mathrm{Hom}(\mathbb{Q}(-2),H^2(\eta,\mathbb{Q})) \to H^3_M(X,\mathbb{Q}(2))/H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}}$$

we observe that

$$0 \to \bigoplus_{x\in X^{(1)}} \mathbb{C}^*\otimes\mathbb{Q} \to \bigoplus_{x\in X^{(1)}} H^1_M(x,\mathbb{Q}(1)) \to \bigoplus_{x\in X^{(1)}} \mathrm{Hom}(\mathbb{Q},H^1(x,\mathbb{Q})(1)) \to 0$$

is exact and

$$H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}} = \mathrm{im}(\bigoplus_{x\in X^{(1)}} \mathbb{C}^*\otimes\mathbb{Q} \to H^3_M(X,\mathbb{Q}(2))).$$

The assumptions on X imply $H^0(G_{\mathrm{HS}}(X,2))=0$ by Proposition 3.4. The rest of the proof involves only diagram chasing. □

Remark 3.2. Proposition 3.5 has also been proved by de Jeu and Lewis in [8, Corollary 4.14], and more generally with integral coefficients in [8, Corollary 6.5].

Proposition 3.6. *Let X be smooth, projective and connected. We suppose that $H^1(X,\mathbb{Q})$ vanishes. The following statements are equivalent.*

1. *$BH(U,2)$ holds for all open subsets U of X.*
2. *$BH(\eta,2)$ holds for the generic point η of X.*

Proof. Only (2) ⇒ (1) is interesting. Proposition 3.5 implies that

$$H^3_M(X,\mathbb{Q}(2))/H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}} \to H^3_{\mathcal{H}}(X,\mathbb{Q}(2))/H^3_{\mathcal{H}}(X,\mathbb{Q}(2))_{\mathrm{dec}}$$

is injective. From Lemma 3.4, we see that

$$H^3_M(X,\mathbb{Q}(2))_{\mathrm{dec}} \to H^3_{\mathcal{H}}(X,\mathbb{Q}(2))_{\mathrm{dec}}$$

is an isomorphism. Thus

$$H^3_M(X,\mathbb{Q}(2)) \to H^3_{\mathcal{H}}(X,\mathbb{Q}(2))$$

is injective. It follows from Lemma 3.6 that

$$H^2_M(\eta,\mathbb{Q}(2)) \to H^2_{\mathcal{H}}(\eta,\mathbb{Q}(2))$$

is surjective. Lemma 3.5 applied to U implies the claim. □

Question 3.1. *Suppose X is smooth, connected, projective, but $H^1(X)\neq 0$. We denote by η the generic point of X. What is the relation between $BH(U,2)$, for all U open in X, and the surjectivity of*

$$H^2_M(\eta,\mathbb{Q}(2)) \to \mathrm{Hom}(\mathbb{Q}(-2),H^2(\eta,\mathbb{Q}))?$$

Acknowledgments

This work has been supported by the SFB/TR 45 "Periods, moduli spaces and arithmetic of algebraic varieties".

It is a pleasure to thank Hélène Esnault for her strong encouragement. I thank Donu Arapura, Florian Ivorra and Manish Kumar for useful discussions.

References

[1] D. Arapura and M. Kumar, *Beilinson-Hodge cycles on semiabelian varieties*, Math. Res. Lett. **16**(4) (2009), 557–562.

[2] M. Asakura and S. Saito, *Noether–Lefschetz locus for Beilinson–Hodge cycles. I*, Math. Z. **252**(2) (2006), 251–273.

[3] ———, *Beilinson's Hodge conjecture with coefficients for open complete intersections*, in 'Algebraic Cycles and Motives,' **2**, *London Math. Soc. Lecture Note Ser.*, **344**, Cambridge University Press, Cambridge, 2007, 3–37.

[4] ———, *Maximal components of Noether–Lefschetz locus for Beilinson–Hodge cycles*, Math. Ann. **341**(1) (2008), 169–199.

[5] A. A. Beĭlinson, *Notes on absolute Hodge cohomology*, in 'Applications of algebraic K-theory to algebraic geometry and number theory,' Part I, II (Boulder, Colo., 1983), Contemp. Math. **55**, Amer. Math. Soc., Providence, RI, 1986, 35–68.

[6] S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201 (1975).

[7] S. Bloch and V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105**(5) (1983), 1235–1253.

[8] R. de Jeu and J. D. Lewis, *Beilinson's Hodge conjecture for smooth varieties*, `ArXiv:1104.4364v1`.

[9] H. Esnault and E. Viehweg, *Deligne-Beĭlinson cohomology*, in Beĭlinson's conjectures on special values of L-functions, *Perspect. Math.* **4**, Academic Press, Boston, MA, 1988, 43–91.

[10] A. Huber, *Realization of Voevodsky's motives*, J. Algebraic Geom. **9**(4) (2000), 755–799.

[11] ———, *Corrigendum to: "Realization of Voevodsky's motives" [J. Algebraic Geom.* **9** *(2000), no. 4, 755–799; MR1775312]*, J. Algebraic Geom. **13**(1) (2004), 195–207.

[12] U. Jannsen, *Deligne homology, Hodge-D-conjecture, and motives*, in 'Beĭlinson's conjectures on special values of L-functions,' *Perspect. Math.* **4**, Academic Press, Boston, MA, 1988, 305–372.

[13] M. Kerr and J. D. Lewis, *The Abel–Jacobi map for higher Chow groups, II*, Invent. Math. **170**(2) (2007), 355–420.

[14] M. Levine, *Mixed motives*, in 'Handbook of K-theory,' **1**, **2**, Springer, Berlin, 2005, 429–521.

[15] C. Mazza, V. Voevodsky, and C. Weibel, Lecture notes on motivic cohomology, Clay Math. Monogr. 2, American Mathematical Society, Providence, RI, 2006, ISBN 978-0-8218-3847-1; 0-8218-3847-4.

[16] S. J. Müller-Stach, *Constructing indecomposable motivic cohomology classes on algebraic surfaces*, J. Algebraic Geom. **6**(3) (1997), 513–543.

[17] V. Voevodsky, *Triangulated categories of motives over a field*, in 'Cycles, transfers, and motivic homology theories,' Ann. Math. Stud. **143**, Princeton University Press, Princeton, NJ, 2000, 188–238.

[18] ———, *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not. **2002**(7) (2002), 351–355.

Fachbereich Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany
E-mail address: `a.chatzistamatiou@uni-due.de`