

**FROM QUANTUM SCHUBERT POLYNOMIALS TO k -SCHUR
FUNCTIONS VIA THE TODA LATTICE**

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ABSTRACT. We show that Lapointe–Lascoux–Morse k -Schur functions (at $t = 1$) and Fomin–Gelfand–Postnikov quantum Schubert polynomials can be obtained from each other by a rational substitution. This is based on Kostant’s solution of the Toda lattice and Peterson’s work on quantum Schubert calculus.

1. The theorem

1.1. Quantum Schubert polynomials. Fomin, Gelfand and Postnikov’s quantum Schubert polynomials \mathfrak{S}_w^q are a family of polynomials in the variables x_1, x_2, \dots, x_n and the quantum parameters q_1, q_2, \dots, q_{n-1} indexed by permutations $w \in S_n$. They show [1] that quantum Schubert polynomials represent quantum Schubert classes in the Givental–Kim presentation [3] of the quantum cohomology ring $QH^*(\text{Fl}_n)$ of the flag manifold.

Example 1.1. Let $n = 3$. Then

$$\begin{aligned} \mathfrak{S}_1^q &= 1, & \mathfrak{S}_{s_1}^q &= x_1, & \mathfrak{S}_{s_2}^q &= x_1 + x_2, \\ \mathfrak{S}_{s_1 s_2}^q &= x_1 x_2 + q, & \mathfrak{S}_{s_2 s_1}^q &= x_1^2 - q, & \mathfrak{S}_{s_1 s_2 s_1}^q &= x_1^2 x_2 + q_1 x_1. \end{aligned}$$

1.2. k -Schur functions. Let $k = n - 1$. Lapointe, Lascoux and Morse’s k -Schur functions [9] $s_\lambda^{(k)}$ are a basis of the Hopf subalgebra $\Lambda_{(n)} = \mathbb{C}[h_1, h_2, \dots, h_{n-1}] \subset \Lambda$ of symmetric functions generated by the first $n - 1$ homogeneous symmetric functions. They are indexed by k -bounded partitions, that is, partitions λ with $\lambda_1 \leq k$. Lam [6] showed that k -Schur functions represent affine Schubert classes under the realization of the homology ring $H_*(\text{Gr}) \cong \Lambda_{(n)}$ of the affine Grassmannian $\text{Gr} = \text{Gr}_{SL(n)}$ inside symmetric functions.

Example 1.2. Let $n = 3 = k + 1$. Then

$$\begin{aligned} s_{2^a 1^{2b}}^{(2)} &= h_2^a e_2^b, \\ s_{2^a 1^{2b+1}}^{(2)} &= h_2^a e_2^b h_1, \end{aligned}$$

where $e_2 = h_1^2 - h_2$ is the second elementary symmetric function.

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1.3. The substitution. For $i \in [0, n]$, let R_i denote the rectangular partition i^{n-i} having i columns and $n - i$ rows. Let R'_i denote the partition obtained from R_i by removing the outer corner of the Young diagram of R_i . In the following, let $s_{R_0} = s_{R_n} = 1$ and $s_{R'_0} = s_{R'_n} = 0$.

Define a map $\Phi : \mathbb{C}[x_1, x_2, \dots, x_n, q_1, q_2, \dots, q_{n-1}] \rightarrow \Lambda_{(n)}[s_{R_1}^{-1}, \dots, s_{R_{n-1}}^{-1}]$ by

$$\begin{aligned} x_1 + x_2 + \dots + x_i &\longmapsto \frac{s_{R'_i}}{s_{R_i}} \\ q_i &\longmapsto \frac{s_{R_{i-1}} s_{R_{i+1}}}{s_{R_i}^2}. \end{aligned}$$

Theorem 1.1. *Let $w \in S_n$. Then*

$$\Phi(\mathfrak{S}_w^q) = \frac{s_{\lambda(w)}^{(k)}}{\prod_{i \in \text{Des}(w)} s_{R_i}},$$

where $\lambda(w)$ is a k -bounded partition explicitly described in Section 6 and $\text{Des}(w) = \{i \mid ws_i < w\}$ denotes the descent set of w , where s_i is a simple reflection and $<$ is the Bruhat order.

Because of a factorization result (Theorem 3.3) of Lapointe and Morse, the above theorem determines $s_{\lambda}^{(k)}$ for all λ .

Example 1.3. Take $w = s_1 s_2 s_1$. Then

$$\begin{aligned} \Phi(\mathfrak{S}_{s_1 s_2 s_1}^q) &= \Phi(x_1^2 x_2 + q_1 x_1) \\ &= \left(\frac{h_1}{e_2}\right)^2 \left(\frac{h_1}{h_2} - \frac{h_1}{e_2}\right) + \frac{h_2}{e_2^2} \frac{h_1}{e_2} \\ &= \frac{h_1}{e_2 h_2}. \end{aligned}$$

Since $\text{Des}(w) = \{1, 2\}$ and $\lambda(w) = (1)$, this agrees with $s_1^{(2)} = h_1$.

This theorem allows for the direct comparison of two very different kinds of formulae. On the one hand, the Gromov–Witten invariants of the flag manifold can be studied using the quantum Bruhat graph (related to the strong Bruhat order) and the combinatorics of quantum Schubert polynomials. On the other hand, the structure constants of the Pontryagin product of Schubert classes in the homology of the affine Grassmannian, involve the weak order on the affine symmetric group and the combinatorics of k -Schur functions. Using Theorem 1.1 facts known on one side can be transferred to the other, leading to new insights in both contexts.

Let us give an outline of the proof. One first obtains an abstract isomorphism between localizations of the quantum cohomology $QH^*(\text{Fl}_n)$ of the flag variety and the homology $H_*(\text{Gr})$ of the affine Grassmannian by the composition of three theorems: (1) the theorem of Kim and of Givental and Kim [3, 4], which identifies $QH^*(\text{Fl}_n)$ with the coordinate ring of the nilpotent Toda lattice; (2) a theorem of Ginzburg [2] and of Peterson which identifies $H_*(\text{Gr})$ with the coordinate ring of the centralizer of a principal nilpotent element in $PGL(n)$; and (3) a theorem of Kostant [5], which solves the nilpotent Toda lattice. The substitution Φ arises in this way.

The quantum and affine Schubert classes are compared using a result of Peterson [13] (also proved in [8, 11]). Using this we obtain the description of Theorem 1.1. The results of Fomin, Gelfand and Postnikov [1] and of Lam [6] allow us to formulate the result explicitly in terms of polynomials.

Remark 1.1. The form of the denominator in Theorem 1.1 probably follows from the affine Grassmannian homology Schubert class factorization result of Magyar [12, Thm. A].

Remark 1.2. Let f^\perp denote the linear operator adjoint to multiplication by a symmetric function f under the Hall inner product. One can get k -Schur functions straight from ordinary Schubert polynomials by directly substituting the ratio $(h_i^\perp \cdot S_{R_m})/S_{R_m}$ of two Schur functions for each elementary symmetric polynomial $e_i(m) = e_i(x_1, x_2, \dots, x_m)$, after the Schubert polynomial is written in terms of products of $e_i(m)$ (see Section 2.3): one may compute the image of \mathfrak{S}_w^q in $\Lambda_{(n)}[s_{R_1}^{-1}, \dots, s_{R_{n-1}}^{-1}]$ by replacing each $E_i^q(m)$ by $(h_i^\perp \cdot S_{R_m})/S_{R_m}$ (see Proposition 5.2). However this is the same as taking the expansion of \mathfrak{S}_w in the $e_{i_1}(1)e_{i_2}(2) \cdots e_{i_{n-1}}(n-1)$ basis and making the substitution $e_i(m) \mapsto (h_i^\perp \cdot S_{R_m})/S_{R_m}$.

1.4. Further directions. In future work, we plan to investigate the generalizations to equivariant (quantum) (co)homology and extensions to other Dynkin types.

A tantalizing open problem is to give a conceptual answer to the question: *does the Toda lattice know about Schubert calculus?* More precisely, do the quantum (=affine) Schubert classes, considered as functions on the nilpotent Toda lattice, have a direct interpretation in terms of the dynamics of the integrable system?

2. Toda lattice and quantum cohomology of flag manifolds

2.1. Toda lattice. The Toda lattice is the Hamiltonian system consisting of n particles with positions \tilde{q}_i and momenta \tilde{p}_i and Hamiltonian

$$H(\tilde{p}, \tilde{q}) = \sum_{i=1}^n \tilde{p}_i^2/2 + \sum_{i=1}^{n-1} e^{\tilde{q}_i - \tilde{q}_{i+1}}.$$

The Toda lattice can be reformulated as the system of differential equations $dL/dt = [L, L_-]$ in terms of the Lax pair

$$L = L(x, q) = \begin{pmatrix} x_1 & -1 & & & \\ q_1 & x_2 & -1 & & \\ & q_2 & x_3 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & q_{n-1} & x_n \end{pmatrix} \quad L_- = \begin{pmatrix} 0 & 0 & & & \\ q_1 & 0 & 0 & & \\ & q_2 & 0 & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & q_{n-1} & 0 \end{pmatrix}$$

where the variables x_i are multiples of \tilde{p}_i , and q_i are multiples of $e^{\tilde{q}_i - \tilde{q}_{i+1}}$. It follows from general theory that $H_k := \text{tr}(L^{k+1}/(k+1))$ gives a complete set of commuting Hamiltonians. Thus, the Toda lattice is a completely integrable system.

2.2. Givental and Kim's description of $QH^*(\mathbf{Fl}_n)$. Let

$$Y_0 = \{L(x, q) \mid L \text{ is nilpotent}\}$$

be the nilpotent Toda leaf. This is the part of phase space, where all Hamiltonians vanish. Let

$$Y_0^\circ = \{L \in Y_0 \mid q_i \neq 0\}$$

be the part of the nilpotent Toda lattice where the quantum parameters are non-zero.

Theorem 2.1. [3, Thm. 1] [4, Thm. I, Section 5]

$$QH^*(\mathbf{Fl}_n) \simeq \mathbb{C}[Y_0] = \mathbb{C}[x_1, \dots, x_n, q_1, \dots, q_{n-1}] / \langle H_k \rangle.$$

2.3. Explicit Schubert representatives. Let $w \in S_n$ be a permutation, and let $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced decomposition of $w^{-1} w_0$, where $w_0 \in S_n$ is the longest permutation. The *Schubert polynomial* $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \dots, x_{n-1}]$ is defined as

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell} (x_1^{n-1} x_2^{n-2} \cdots x_{n-2}),$$

where ∂_i denotes the *divided difference operator*

$$(\partial_i f)(x_1, x_2, \dots, x_{n-1}) = \frac{f(x_1, \dots, x_{n-1}) - f(x_1, \dots, x_{i+1}, x_i, \dots, x_{n-1})}{x_i - x_{i+1}}.$$

Let $e_i(m) = e_i(x_1, x_2, \dots, x_m)$ denote the elementary symmetric functions in m -variables. Let $E_i^q(m)$ be the quantum analogue of the i th elementary symmetric polynomial in variables x_1, x_2, \dots, x_m . It is defined by

$$(2.1) \quad E_i^q(m) = E_i^q(m-1) + x_m E_{i-1}^q(m-1) + q_{m-1} E_{i-2}^q(m-2),$$

where $E_i^q(m) = 0$ if $i < 0$ or $i > m$ and $E_0^q(m) = 1$ for $m \geq 0$. Fomin, Gelfand, and Postnikov [1] define the *quantum Schubert polynomial* \mathfrak{S}_w^q by expanding the ordinary Schubert polynomial \mathfrak{S}_w into the basis $e_{i_1}(1) e_{i_2}(2) \cdots e_{i_{n-1}}(n-1)$ where (i_1, \dots, i_{n-1}) is a tuple of integers such that $0 \leq i_r \leq r-1$ for $1 \leq r \leq n-1$ and then substituting the quantum elementary symmetric polynomial $E_i^q(m)$ for each $e_i(m)$.

The quantum cohomology ring $QH^*(\mathbf{Fl}_n)$ has a $\mathbb{C}[q_1, \dots, q_{n-1}]$ -basis of quantum Schubert classes $\{\sigma^w \mid w \in S_n\}$ labeled by permutations.

Theorem 2.2. [1, Thm. 1.2] Under the isomorphism of Theorem 2.1 we have $\sigma^w \mapsto \mathfrak{S}_w^q \pmod{\langle H_k \rangle}$.

3. Centralizer groups and homology of affine Grassmannian

Let $\text{Gr} = SL_n(\mathbb{C}((t)))/SL_n(\mathbb{C}[[t]])$ denote the affine Grassmannian of $G = SL(n)$. Let $G^\vee = PGL(n)$ denote the Langlands dual of G . Let $e = \sum_{i=1}^{n-1} e_i^\vee$ denote the principal nilpotent element

$$e = \begin{pmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & 0 & \ddots & \\ & & & \ddots & -1 \\ & & & & 0 \end{pmatrix}$$

in the Lie algebra \mathfrak{g}^\vee . Write

$$X = G_e^\vee = \left\{ \begin{pmatrix} 1 & h_1 & h_2 & h_3 & \cdots & h_{n-1} \\ 0 & 1 & h_1 & h_2 & \cdots & h_{n-2} \\ 0 & 0 & 1 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & h_1 & h_2 \\ 0 & 0 & 0 & 0 & 1 & h_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} \subset G^\vee$$

for the centralizer subgroup of e in G^\vee .

The following result is due to Ginzburg [2] and Peterson [13].

Theorem 3.1. [2, Cor. 5.3.2] [13] *There are Hopf isomorphisms*

$$H_*(\text{Gr}) \simeq \mathbb{C}[X] \simeq \mathbb{Z}[h_1, h_2, \dots, h_{n-1}] = \Lambda_{(n)}.$$

3.1. Explicit Schubert representatives. The homology $H_*(\text{Gr}, \mathbb{C})$ has a \mathbb{C} -basis of affine Schubert classes $\{\xi_x \mid x \in \tilde{S}_n/S_n\}$ indexed by cosets of the symmetric group in the affine symmetric group \tilde{S}_n . Recall that the k -Schur functions $s_\lambda^{(k)}$ [9, 10] are labeled by k -bounded partitions: partitions λ satisfying $\lambda_1 \leq k$. The following theorem of the first author was conjectured by the second author.

Theorem 3.2. [6, Thm. 7.1] *Under the isomorphism of Theorem 3.1 we have $\xi_x \mapsto s_{b(x)}^{(k)}$, where the bijection $x \leftrightarrow b(x)$ between \tilde{S}_n/S_n and $(n-1)$ -bounded partitions is described in [7, Prop. 8.15].*

For two partitions λ, μ , we let $\lambda \cup \mu$ be the partition obtained by taking the union of parts of λ and μ . The k -rectangles R_i play a special role in the theory of k -Schur functions because of the following factorization result of Lapointe and Morse:

Theorem 3.3. [10, Thm. 40]

$$s_{\lambda \cup R_i}^{(k)} = s_\lambda^{(k)} s_{R_i}.$$

4. Kostant's solution to the Toda lattice

Since $\mathbb{C}[X] \simeq \mathbb{C}[h_1, h_2, \dots, h_{n-1}]$, via the Jacobi–Trudi formula $s_\lambda = \det(h_{\lambda_i - i + j})$ the Schur functions s_{R_i} can be considered as polynomial functions on X . We define the Zariski-open set $X^\circ \subset X$ by

$$X^\circ = \{g \in X \mid s_{R_i}(g) \neq 0 \text{ for } i \in [1, n-1]\}.$$

Kostant [5] solves the nilpotent Toda lattice by

Theorem 4.1. *There is an isomorphism $\Psi : X^\circ \rightarrow Y_0^\circ$ such that the induced map $\Psi^* : \mathbb{C}[Y_0^\circ] \rightarrow \mathbb{C}[X^\circ]$ is given by $\Psi^* = \Phi$.*

Thus, coordinates on X can be considered as angle coordinates for the nilpotent Toda leaf. The map Ψ is constructed as follows: for $g \in X^\circ$, find a lower unitriangular matrix $n_-(g)$ so that $gn_-(g)$ has the form

$$gn_-(g) = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & * & * & * \\ * & * & * & * \end{pmatrix}.$$

Then $\Psi(g) = n_-^{-1}(g)en_-(g)$. The formula we give in Theorem 1.1 for Φ is a symmetric function translation of Kostant's description of $\Psi(g)$.

Remark 4.1. In fact, $n_-(g)$ has (i, j) -th entry $(-1)^{i-j}(e_{i-j}^\perp \cdot s_{R_j})/s_{R_j}$ for $i > j$, and $n_-(g)^{-1}$ is lower unitriangular with (i, j) entry $(h_{i-j}^\perp \cdot s_{R_{i-1}})/s_{R_{i-1}}$ for $i > j$.

Composing Ψ^* with Theorems 2.1 and 3.1, we obtain an isomorphism $QH^*(\mathbb{F}l_n)[q_i^{-1}] \cong H_*(\text{Gr})[s_{R_i}^{-1}]$. We shall now discuss the behavior of Schubert classes under this isomorphism.

5. Isomorphisms in terms of Schubert classes

5.1. Peterson's isomorphism. We first develop some notation allowing us to label affine Schubert classes with extended affine symmetric group elements.¹ Our notation for affine Weyl groups mostly follows that in [8] (see also Appendix A). For explicit affine symmetric group notation, we refer the reader to [7].

Let Q^\vee and P^\vee denote the coroot lattice and coweight lattice of the root system A_{n-1} . Let \hat{S}_n denote the extended affine symmetric group, so that $\hat{S}_n \cong \mathbb{Z}/n\mathbb{Z} \times \tilde{S}_n \cong W \times P^\vee$. Let \hat{S}_n^0 and \tilde{S}_n^0 denote the minimum length coset representatives in \hat{S}_n/S_n and \tilde{S}_n/S_n respectively. An affine permutation $x \in \hat{S}_n$ can be thought of as a bijection $x : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying the periodicity condition $x(i+n) = x(i) + n$, and is determined by the window $[x(1), x(2), \dots, x(n)]$. The two affine permutations $[x(1), x(2), \dots, x(n)]$ and $[x(1) + n, x(2) + n, \dots, x(n) + n]$ are considered identical. Given $(\lambda_1, \lambda_2, \dots, \lambda_n) \in P^\vee = \mathbb{Z}^n/(1, 1, \dots, 1)$, the translation element t_λ has window notation $t_\lambda = [1 + n\lambda_1, 2 + n\lambda_2, \dots, n + n\lambda_n]$. Some distinguished elements in P^\vee are the fundamental coweights $\omega_i^\vee = e_1 + e_2 + \dots + e_i$, and the simple coroots $\alpha_i^\vee = e_i - e_{i+1}$, where $e_i \in \mathbb{Z}^n/(1, 1, \dots, 1)$ denotes the standard basis vectors.

For $x \in \hat{S}_n$, we write $x = zy$ where $z \in \mathbb{Z}/n\mathbb{Z}$ and $y \in \tilde{S}_n^0$. Then we define $\xi_x = \xi_y$ in $H_*(\text{Gr})$.

For $i \in [1, n-1]$ let $w_0^{\omega_i}$ denote the longest minimal length coset representative of $S_n/(S_i \times S_{n-i})$. In one line notation, $w_0^{\omega_i}$ is given by $[n-i+1, n-i+2, \dots, n, 1, 2, \dots, n-i]$. We then set $q_{\omega_i^\vee} = \sigma^{w_0^{\omega_i}}$. With this definition one obtains q_λ for each $\lambda \in P^\vee$, satisfying $q_{\lambda+\mu} = q_\lambda q_\mu$. See Appendix A for further details.

In his study of geometric models for quantum cohomology of partial flag varieties, Peterson relates the quantum cohomology rings with the homology of affine Grassmannians. Peterson's statement [13] and our proof in [8] labels Schubert classes only by non-extended affine Weyl group elements. In Appendix A, we show that the notation can be included to include extended affine Weyl group elements.

Theorem 5.1. *There is an isomorphism $QH^*(\mathbb{F}l_n)[q_i^{-1}] \rightarrow H_*(\text{Gr})[\xi_{t_\lambda}^{-1} \mid \lambda \in P^\vee]$ given by*

$$\sigma^w q_{\mu-\lambda} \longmapsto \xi_{wt_\mu} \xi_{t_\lambda}^{-1},$$

where $\mu, \lambda \in P^\vee$ are antidominant.

¹We could alternatively work with $H_*(\text{Gr}_{PGL(n)})$, but it is simpler to always use $\text{Gr} = \text{Gr}_{SL(n)}$ throughout.

5.2. k -rectangles. Define $d_i \in \tilde{S}_n^0$ to be the unique affine Grassmannian element in the same left $\mathbb{Z}/n\mathbb{Z}$ coset as $t_{-\omega_i^\vee}$. Equivalently, d_i is obtained by affine Dynkin rotation of $w_0^{\omega_i}$. There is a bijection $w \mapsto b(w)$ [7] between \tilde{S}_n^0 and the set of $(n-1)$ -bounded partitions. Under this bijection we have $d_i \mapsto R_i$.

Example 5.1. Let $n = 5$ and $i = 2$ so that $i^* := n - i = 3$. Then $w_0^{\omega_3} = s_2 s_1 s_3 s_2 s_4 s_3$ which in one-line notation is $[3, 4, 5, 1, 2]$. and $d_2 = s_4 s_3 s_0 s_4 s_1 s_0$. The reduced word for d_i is obtained by reading the entries from the French tableau of shape R_i obtained by placing $c - r \pmod n$ into the cell in the r th row and c th column.

3	4
4	0
0	1

By Theorem 3.2, the isomorphism $H_*(\text{Gr}_{SL_n}) \cong \mathbb{Z}[h_1, \dots, h_{n-1}]$ of Theorem 3.1 sends ξ_w to the k -Schur function $s_{b(w)}^{(n-1)}$. Therefore, $\xi_{d_i} \mapsto s_{R_i}$ under the isomorphism of Theorem 3.1. This isomorphism extends to a localized isomorphism

$$(5.1) \quad H_*(\text{Gr}_{SL_n})[\xi_{t_\lambda}^{-1} \mid \lambda \in \tilde{Q}] \cong \mathbb{Z}[h_1, \dots, h_{n-1}][s_{R_1}^{-1}, \dots, s_{R_{n-1}}^{-1}]$$

under which $\xi_{t_{\omega_i^\vee}} = \xi_{-t_{\omega_i^\vee}}^{-1} \mapsto 1/s_{R_i}$. We denote by Φ' the isomorphism

$$QH^*(\text{Fl}_n)[q_1^{-1}, q_2^{-1}, \dots, q_{n-1}^{-1}] \rightarrow \mathbb{Z}[h_1, \dots, h_{n-1}][s_{R_1}^{-1}, \dots, s_{R_{n-1}}^{-1}]$$

given by composing Theorem 5.1 with (5.1). By the above discussion we have $\Phi'(q_{\omega_i^\vee}) = 1/s_{R_i}$.

Recall that Φ is defined before Theorem 1.1.

Proposition 5.1. *We have $\Phi = \Phi'$, where both are considered as algebra maps from $QH^*(\text{Fl}_n)[q_i^{-1}]$ to $\mathbb{Z}[h_1, \dots, h_{n-1}][s_{R_i}^{-1}]$.*

Proof. It suffices to check equality on algebra generators q_i and $x_1 + \dots + x_i$. For q_i , defining $\omega_0^\vee = \omega_n^\vee = 0$ we have $\alpha_i^\vee = -\omega_{i+1}^\vee + 2\omega_i^\vee - \omega_{i-1}^\vee$. Therefore

$$(5.2) \quad \Phi'(q_i) = \Phi'(q_{\alpha_i^\vee}) = \frac{s_{R_{i+1}} s_{R_{i-1}}}{s_{R_i}^2} = \Phi(q_i)$$

for $1 \leq i \leq n-1$. We also have

$$\Phi'(x_1 + \dots + x_i) = \Phi'(E_1^q(i)) = \frac{h_1^+ R_i}{R_i} = \Phi(x_1 + \dots + x_i),$$

where we have used Proposition 5.2 for λ a column of size i . □

5.3. Quantum Schur symmetric polynomials. For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ define the quantum Schur function

$$(5.3) \quad S_\lambda^q(x_1, \dots, x_m) = \det(E_{\lambda'_i - i + j}^q(x_1, \dots, x_m)).$$

Proposition 5.2. *For $\lambda_1 \leq n - m$, we have*

$$\Phi'(S_\lambda^q(x_1, \dots, x_m)) = \frac{s_{\lambda'}^+ s_{R_m}}{s_{R_m}}.$$

Proof. Let $1 \leq m \leq n-1$ and λ a partition such that $\ell(\lambda) \leq m$ and $\lambda_1 \leq n-m$, so that $\lambda \subset R_{n-m}$. Let $w_{\lambda,m}$ be the unique m -Grassmannian permutation in S_n of shape λ : in one line notation, $w_{\lambda,m}$ starts with $1 + \lambda_m, 2 + \lambda_{m-1}, \dots, m + \lambda_1$, and ends with the complementary numbers in increasing order. It is known that $s_\lambda(x_1, \dots, x_m) = \det(e_{\lambda'_i - i + j})$ is the ordinary Schubert class of the m -Grassmannian permutation $w_{\lambda,m}$ in $H^*(\text{Gr}(m, n))$. Quantizing this relation (that is, using [1]) yields $\sigma^{w_{\lambda,m}} = \det(E_{\lambda'_i - i + j}^q) = S_\lambda^q(m)$. We have

$$\Phi'(S_\lambda^q(m)) = \Phi'(\sigma^{w_{\lambda,m}}) = \xi_{w_{\lambda,m} t_{-\omega_m^\vee}} \xi_{t_{-\omega_m^\vee}}^{-1}.$$

By direct computation, $w_{\lambda,m} t_{-\omega_m^\vee} = b^{-1}(\lambda^\vee)$ where λ^\vee is the partition obtained by taking the complement of λ in R_{n-m} and then taking the transpose. Therefore,

$$\Phi'(S_\lambda^q(m)) = \frac{s_{\lambda^\vee}}{s_{R_m}}.$$

□

6. Explicit computation of $\lambda(w)$

We now describe the map $w \mapsto \lambda(w)$ of Theorem 1.1 explicitly. For simplicity of notation, we first assume that $w \in S_n$ satisfies $w(1) = 1$.

Let $c_i = s_{n-i} \cdots s_{n-2} s_{n-1} \in S_n$ for $1 \leq i \leq n-1$. Then there is a unique sequence $(m_1, m_2, \dots, m_{n-1})$ of integers such that $0 \leq m_1 \leq n-2$, $0 \leq m_2 \leq n-3$, \dots , $0 \leq m_{n-2} \leq 1$ such that

$$w = c_{n-2}^{m_1} \cdots c_2^{m_{n-3}} c_1^{m_{n-2}}.$$

Then $\lambda(w)$ is the $(n-1)$ -irreducible bounded partition λ with m_i equal to the number of parts of size i . The following example shows how to obtain m_i from w algorithmically. Another description of w in terms of the inversion set of w is given in the proof of Lemma 6.1.

Example 6.1. Let $n = 6$ and $\lambda = (4, 3, 2, 2, 2, 1, 1)$ so that $m_1 = 2$, $m_2 = 3$, $m_3 = 1$, $m_4 = 1$. We have

$$w = (s_2 s_3 s_4 s_5)^2 (s_3 s_4 s_5)^3 (s_4 s_5)^1 (s_5)^1.$$

In one line notation $w = [1, 4, 3, 6, 5, 2]$. To go from w to $\lambda(w)$, we start with $[1, 2, 3, 4, 5, 6]$ and must obtain 4 in position 2. Therefore we must left circular shift twice in the last five positions, that is, $m_1 = 2$. We obtain $[1, 4, 5, 6, 2, 3]$. Next we must get 3 into position 3, which requires three left circular shifts in the last four positions, that is, $m_2 = 3$. We obtain $[1, 4, 3, 5, 6, 2]$. To get 6 to position 4 we need $m_3 = 1$ and we obtain $[1, 4, 3, 6, 2, 5]$. Finally to get 5 to position 5 we need $m_4 = 1$ and we obtain $[1, 4, 3, 6, 5, 2]$ as required.

Lemma 6.1. *The element $y = w \prod_{i \in \text{Des}(w)} t_{-\omega_i^\vee}$ lies in \hat{S}_n^0 , and modulo conjugation by an affine Dynkin diagram rotation, is equal to $x \in \tilde{S}_n^0$ where $b(x) = \lambda(w)$.*

Proof. For $1 \leq i \leq n$, let $\text{des}_i(w)$ denote the number of descents of w which lie before i . In one line notation, one has $y(i) = w(i) - \text{des}_i(w)n$. Since $y \in \hat{S}_n^0$ if and only if $y(1) < y(2) < \cdots < y(n)$, the first claim follows.

To obtain $b(x)$, we use [7, Proposition 8.15]. In the current setting, the proposition says that we must show

$$(6.1) \quad \lambda'_r = \#\{j < r \mid y(j) > y(r)\},$$

where we think of y as a bijection from \mathbb{Z} to \mathbb{Z} when evaluating $y(j)$. For $1 \leq i < j \leq n$, define $a(i, j) := \text{des}_j(w) - \text{des}_i(w) - \chi(w(i) > w(j))$. The right-hand side (RHS) of (6.1) is then equal to $\sum_{j>r} a(r, j)$.

It is straightforward to prove by induction the following characterization of the permutations $w = c_{n-2}^{m_1} \cdots c_{n-r}^{m_{r-1}}$: these are exactly the permutations satisfying $w(r) < w(r+1) < \cdots < w(s) > w(s+1) < w(s+2) < \cdots < w(n) < w(r)$. In other words, the word $w(r)w(r+1) \cdots w(s)$ has one cyclic descent. Using this, it is easy to see that successive multiplication by c_{n-r} increases $\sum_{j>r'} a(r', j)$ by 1 for each $r' \in [1, r-1]$. This establishes (6.1). \square

To complete the description of $\lambda(w)$, we remove the condition $w(1) = 1$. For $w \in S_n$ in one line notation $w = [w(1), w(2), \dots, w(n)]$, define $w' = [w(1) - 1 \pmod n, w(2) - 1 \pmod n, \dots, w(n) - 1 \pmod n]$ where the $\pmod n$ function takes values in $\{1, 2, \dots, n\}$. The following result includes a property of quantum Schubert classes, which may be new.

Proposition 6.1. *We have $\lambda(w') = \lambda(w)$ and*

$$q_{\omega_{w^{-1}(1)-1}}^\vee \sigma^{w'} = q_{\omega_{w^{-1}(1)}}^\vee \sigma^w,$$

where $q_{\omega_0^\vee} = q_{\omega_n^\vee} = 1$.

Proof. One has $w' = s_{n-1}s_{n-2} \cdots s_2s_1w$. Now we have $s_{n-1} \cdots s_2s_1 = w_0^{\omega_1} = \tau_1 t_{-\omega_1^\vee}$. We have

$$\Phi(\sigma^{w'}) = \Phi(\sigma^{w_0^{\omega_1^\vee} w}) = \xi_{\omega_0^{\omega_1^\vee} w t_\beta}^\vee \xi_{t_\beta}^{-1}$$

for sufficiently antidominant $\beta \in \tilde{Q}$. We compute

$$w_0^{\omega_1^\vee} w t_\beta = w_0^{\omega_1^\vee} w t_{-w^{-1} \cdot \omega_1^\vee} t_{\beta + w^{-1} \cdot \omega_1^\vee} = w_0^{\omega_1^\vee} t_{-w_1^\vee} w t_{\beta + w^{-1} \cdot \omega_1^\vee} = \tau_1 w t_{\beta + w^{-1} \cdot \omega_1^\vee}.$$

It follows that

$$\Phi(\sigma^{w'}) = \Phi(\sigma^w) \frac{S_{R_{w^{-1}(1)-1}}}{S_{R_{w^{-1}(1)}}}.$$

Now $\text{Des}(w')$ is obtained from $\text{Des}(w)$ by removing $w^{-1}(1) - 1$ and adding $w^{-1}(1)$; hence the result follows. \square

w	$\lambda(w)$
(1, 2)	()

w	$\lambda(w)$
(1, 2, 3)	()
(1, 3, 2)	(1)

w	$\lambda(w)$
(1, 2, 3, 4)	()
(1, 2, 4, 3)	(2)
(1, 3, 2, 4)	(2, 1)
(1, 3, 4, 2)	(1)
(1, 4, 2, 3)	(1, 1)
(1, 4, 3, 2)	(2, 1, 1)

w	$\lambda(w)$
(1, 2, 3, 4, 5)	()
(1, 2, 3, 5, 4)	(3)
(1, 2, 4, 3, 5)	(3, 2)
(1, 2, 4, 5, 3)	(2)
(1, 2, 5, 3, 4)	(2, 2)
(1, 2, 5, 4, 3)	(3, 2, 2)
(1, 3, 2, 4, 5)	(2, 2, 1)
(1, 3, 2, 5, 4)	(3, 2, 2, 1)
(1, 3, 4, 2, 5)	(3, 1)
(1, 3, 4, 5, 2)	(1)
(1, 3, 5, 2, 4)	(2, 1)
(1, 3, 5, 4, 2)	(3, 2, 1)
(1, 4, 2, 3, 5)	(2, 1, 1)
(1, 4, 2, 5, 3)	(3, 2, 1, 1)
(1, 4, 3, 2, 5)	(3, 2, 2, 1, 1)
(1, 4, 3, 5, 2)	(2, 2, 1, 1)
(1, 4, 5, 2, 3)	(1, 1)
(1, 4, 5, 3, 2)	(3, 1, 1)
(1, 5, 2, 3, 4)	(1, 1, 1)
(1, 5, 2, 4, 3)	(3, 1, 1, 1)
(1, 5, 3, 2, 4)	(3, 2, 1, 1, 1)
(1, 5, 3, 4, 2)	(2, 1, 1, 1)
(1, 5, 4, 2, 3)	(2, 2, 1, 1, 1)
(1, 5, 4, 3, 2)	(3, 2, 2, 1, 1, 1)

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Appendix A. Extending the Peterson isomorphism

In this appendix, we work in the setting of an arbitrary Weyl group W of a simply-connected algebraic group G . Our notation follows that of [8]. Let $I = \{1, 2, \dots, r\}$ and

$$P^\vee = \bigoplus_{i=1}^r \mathbb{Z}\omega_i^\vee, \quad \tilde{P} = \{\mu \in P^\vee \mid \langle \mu, \alpha_i \rangle \leq 0 \text{ for all } i \in I\},$$

$$Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee, \quad \tilde{Q} = \tilde{P} \cap Q^\vee.$$

Let $\{\sigma^w \in QH^*(G/B) \mid w \in W\}$ denote the Schubert basis of the small quantum cohomology ring of G/B . Let $\{\xi_x \in H_*(\text{Gr}_G) \mid x \in W_{\text{af}}^0\}$ be the Schubert basis of the homology ring of the affine Grassmannian of G .

Lemma A.1. *Let $w \in W$ and $\mu \in Q^\vee$. Then $wt_\mu \in W_{\text{af}}^0$ if and only if $\mu \in \tilde{Q}$ and whenever $wr_i < w$ for some $i \in I$ we have $\langle \mu, \alpha_i \rangle < 0$. In particular, letting $\rho^\vee = \sum_{i \in I} \omega_i^\vee$, for any $w \in W$ we have $wt_\mu \in W_{\text{af}}^0$ for all $\mu \in Q^\vee$ such that $\mu + 2\rho \in \tilde{Q}$.*

Peterson [13] (see also [8]) defined a ring isomorphism

$$(A.1) \quad \begin{aligned} QH^*(G/B)[q_1^{-1}, \dots, q_r^{-1}] &\cong H_*(\text{Gr}_G)[\xi_{t_\mu}^{-1} \mid \mu \in \tilde{Q}] \\ \sigma^w q_{\lambda-\mu} &\mapsto \xi_{wt_\lambda} \xi_{t_\mu}^{-1} \end{aligned}$$

for $w \in W$ and $\lambda, \mu \in \tilde{Q}$ such that $wt_\mu \in W_{\text{af}}^0$. We wish to give a more precise description of the denominators that occur in the RHS of (A.1).

Let $\text{Aut}(I_{\text{af}})$ denote the group of automorphisms of the affine Dynkin diagram. Let $I^s = \text{Aut}(I_{\text{af}}) \cdot \{0\}$ be the set of special nodes where $0 \in I_{\text{af}}$ is the distinguished affine node. There is a bijection $I^s \cong P^\vee/Q^\vee$ such that $i \mapsto -\omega_i^\vee + Q^\vee$ where ω_i^\vee is the fundamental coroot for $i \in I^s \setminus \{0\}$ and $\omega_0^\vee = 0$. For each $i \in I^s$, subtraction by $\omega_i^\vee + Q^\vee$ induces a permutation of I^s denoted τ_i , which extends uniquely to an element $\tau_i \in \text{Aut}(I_{\text{af}})$, which is called the special automorphism associated with $i \in I^s$. It satisfies $\tau_i(i) = 0$. There is a group monomorphism $P^\vee/Q^\vee \rightarrow \text{Aut}(I_{\text{af}})$ such that $-\omega_i^\vee + Q^\vee \mapsto \tau_i$ for $i \in I^s$. We denote the image of this map by $\text{Aut}^s(I_{\text{af}})$, the subgroup of special automorphisms. We write $i \mapsto i^*$ for the element of $\text{Aut}(I_{\text{af}})$ such that $0^* = 0$ and $w_0 r_i w_0 = r_{i^*}$ for $i \in I$. Equivalently, $-w_0 \cdot \alpha_i = \alpha_{i^*}$ or $-w_0 \cdot \omega_i^\vee = \omega_{i^*}^\vee$, or $-\omega_i^\vee + Q^\vee = \omega_{i^*}^\vee + Q^\vee$ for $i \in I^s$.

Let $W_e \cong W \rtimes P^\vee \cong \text{Aut}^s(I_{\text{af}}) \rtimes W_{\text{af}}$ be the extended affine Weyl group. For $w \in W$ and $\mu \in P^\vee$ we have $wt_\mu w^{-1} = t_{w \cdot \mu}$ and for $z \in \text{Aut}^s(I_{\text{af}})$ and $i \in I_{\text{af}}$ we have $zr_i z^{-1} = r_{z(i)}$. We use the level zero action of W_e on P^\vee given by $ut_\lambda \cdot \mu = u \cdot \mu$ for $\lambda, \mu \in P^\vee$ and $u \in W$. Then we have $zt_\mu z^{-1} = t_{z \cdot \mu}$ for all $z \in W_e$ and $\mu \in P^\vee$. With this notation we have

$$(A.2) \quad \tau_i = w_0^{\omega_i} t_{-\omega_i^\vee}, \quad \text{for } i \in I^s,$$

where $w_0^{\omega_i} \in W$ is the shortest element in the coset $w_0 W_{\omega_i}$ where W_{ω_i} is the stabilizer of ω_i and $w_0 \in W$ is the longest element. In particular, if $i = 0$ then $w_0^{\omega_0} = \text{id}$.

For $i \in I$, define $d_i \in W_{\text{af}}$ to be the unique element such that

$$\text{Aut}^s(I_{\text{af}}) t_{-\omega_i^\vee} = \text{Aut}^s(I_{\text{af}}) d_i.$$

Lemma A.2. *$d_i \in W_{\text{af}}^0$. If $i \in I^s$,*

$$(A.3) \quad d_i = \tau_{i^*} w_0^{\omega_{i^*}} \tau_{i^*}^{-1} = \tau_{i^*} t_{-\omega_i^\vee} = w_0^{\omega_{i^*}} t_{-\omega_i^\vee - \omega_{i^*}^\vee}.$$

For general $i \in I$, let $j \in I^s$ be such that $\omega_i \equiv \omega_j \pmod{Q^\vee}$. Then

$$(A.4) \quad d_i = d_j t_{w_0^{\omega_{j^*}} \cdot (-\omega_i^\vee + \omega_j^\vee)}.$$

Proof. Suppose first that $i \in I^s$. Then

$$\begin{aligned} t_{-\omega_i^\vee} &= (w_0^{\omega_i})^{-1} \tau_i \\ &= \tau_i \tau_i^{-1} (w_0^{\omega_i})^{-1} \tau_i \\ &= \tau_i (\tau_{i^*} w_0^{\omega_{i^*}} \tau_{i^*}^{-1}). \end{aligned}$$

By definition d_i satisfies (A.3). We now check that $d_i \in W_{\text{af}}^0$. Using the automorphism τ_{i^*} we see that this holds if and only if $w_0^{\omega_{i^*}}$ is lengthened by right multiplication by

r_k for all $k \neq i^*$. This is true for $k = 0$, since $w_0^{\omega_{i^*}} \in W$ and true for $k \in I \setminus \{i^*\}$ by definition. So $d_i \in W_{\text{af}}^0$.

Let $i \in I$ be general with $j \in I^s$ and $\beta \in Q^\vee$ such that $\omega_i^\vee = \omega_j^\vee + \beta$. We have

$$\begin{aligned} t_{-\omega_i^\vee} &= t_\beta t_{-\omega_j^\vee} \\ &= t_\beta \tau_j d_j \\ &= \tau_j t_{\tau_{j^*} \cdot \beta} d_j \\ &= \tau_j t_{w_0^{\omega_{j^*}} \cdot \beta} d_j. \end{aligned}$$

By definition (A.4) holds. We have $d_i \in W_{\text{af}}^0$ essentially because the factorization $(t_\beta)(t_{-\omega_j^\vee})$ is length additive. \square

Say that $w \in W_{\text{af}}^0$ is i -reducible for $i \in I$, if $\ell(w) = \ell(wd_i) + \ell(d_i)$. Say that $w \in W_{\text{af}}^0$ is irreducible if it is not i -reducible for any $i \in I$.

Proposition A.1. $H_*(\text{Gr}_G)[\xi_{t_\mu}^{-1} \mid \mu \in \tilde{Q}]$ has \mathbb{Z} -basis given by $\xi_w \prod_{i \in I} \xi_{d_i}^{e_i}$ for $w \in W_{\text{af}}^0$ irreducible and $e_i \in \mathbb{Z}$ for $i \in I$.

Lemma A.3. Suppose $w \in W_{\text{af}}^0$ is i -reducible. Let $j \in I^s$ be such that $\omega_i^\vee + Q^\vee = \omega_j^\vee + Q^\vee$. Then $\tau_j w d_i \tau_j^{-1} \in W_{\text{af}}^0$ and

$$\xi_w = \xi_{\tau_j w d_i \tau_j^{-1}} \xi_{d_i}.$$

Magyar has a criterion [12, 4.2] for finding the largest product of elements ξ_{d_i} that factor out of ξ_w for $w \in W_{\text{af}}^0$.

Define W_e^0 be the set of elements of minimum length in their cosets in W_e/W . Then $W_e^0 = \text{Aut}^s(I_{\text{af}}) \times W_{\text{af}}^0$. Note that $W_e^0 \cap P^\vee = \tilde{P}$. For $x \in W_e^0$ let $x = zy$ where $z \in \text{Aut}^s(I_{\text{af}})$ and $y \in W_{\text{af}}^0$. Then define $\xi_x \in H_*(\text{Gr}_G)$ by

$$(A.5) \quad \xi_x = \xi_y.$$

In particular, for all $i \in I^s$,

$$(A.6) \quad \xi_{t_{-\omega_i^\vee}} = \xi_{\tau_i d_i} = \xi_{d_i}.$$

For $\mu \in P^\vee$ let $\lambda \in Q^\vee$ and $i \in I^s$ be the unique elements such that $\mu = \omega_i^\vee + \lambda$. Then define $q_\mu \in QH^*(G/B)$ by

$$(A.7) \quad q_\mu = q_\lambda q_{\omega_i^\vee} = q_\lambda \sigma^{w_0^{\omega_i}}.$$

One may show that $q_\lambda q_\mu = q_{\lambda+\mu}$ for all $\lambda, \mu \in P^\vee$. We may extend the notation of Peterson's isomorphism by writing

$$(A.8) \quad \sigma^w q_{\mu-\lambda} \mapsto \xi_{wt_\mu} \xi_{t_\lambda}^{-1}$$

where $\lambda, \mu \in \tilde{P}$ are such that $wt_\mu \in W_e^0$.

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