

CONTACT MONOIDS AND STEIN COBORDISMS

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ABSTRACT. Suppose S is a compact surface with boundary, and let ϕ be a diffeomorphism of S which fixes the boundary pointwise. We denote by $(M_{S,\phi}, \xi_{S,\phi})$ the contact three-manifold compatible with the open book (S, ϕ) . In this paper, we construct a Stein cobordism from the contact connected sum $(M_{S,h}, \xi_{S,h}) \# (M_{S,g}, \xi_{S,g})$ to $(M_{S,hg}, \xi_{S,hg})$. This cobordism accounts for the comultiplication map on Heegaard Floer homology discovered in [3], and illuminates several geometrically interesting monoids in the mapping class group $\text{Mod}^+(S, \partial S)$. We derive some consequences for the fillability of contact manifolds obtained as cyclic branched covers of transverse knots.

1. Introduction

Let M be a closed, oriented three-manifold. In [14], Giroux proves that there is a one-to-one correspondence between contact structures on M up to contactomorphism and abstract open book decompositions of M up to an equivalence called positive stabilization. Giroux’s work allows us to translate questions about tightness and fillability of contact structures into questions about diffeomorphisms of compact surfaces with boundary. In particular, one is tempted to ask whether certain operations which are natural in the latter context, like composition of diffeomorphisms, have natural contact-geometric counterparts. Our paper sets out to answer this question.

Suppose S is a compact, orientable surface with boundary, and let $\text{Mod}^+(S, \partial S)$ denote the set of isotopy classes of orientation-preserving diffeomorphisms of S which restrict to the identity on ∂S . Furthermore, let $(M_{S,\phi}, \xi_{S,\phi})$ denote the contact manifold supported by the open book (S, ϕ) . In this paper, we point out three geometrically significant “contact monoids” in $\text{Mod}^+(S, \partial S)$. Most of our results stem from the following theorem.

Theorem 1.1. $(M_{S,hg}, \xi_{S,hg})$ is the result of contact (-1) -surgery on a Legendrian link \mathbb{L} in the contact connected sum $(M_{S,h}, \xi_{S,h}) \# (M_{S,g}, \xi_{S,g})$.

We may alternately view this theorem as the statement that there exists a Stein two-handle cobordism from the contact connected sum $(M_{S,h}, \xi_{S,h}) \# (M_{S,g}, \xi_{S,g})$ to $(M_{S,hg}, \xi_{S,hg})$. The theorem below is an immediate consequence of Theorem 1.1.

Theorem 1.2. Suppose \mathbf{H} is a property of contact manifolds which is preserved under contact connected sum and contact (-1) -surgery. Then the set of $\phi \in \text{Mod}^+(S, \partial S)$ for which $(M_{S,\phi}, \xi_{S,\phi})$ satisfies the property \mathbf{H} is closed under composition.¹

Examples of such \mathbf{H} are Stein fillability, as well as strong and weak symplectic fillability [8, 11, 23]. In particular, let $\text{Stein}(S, \partial S)$, $\text{Strong}(S, \partial S)$ and $\text{Weak}(S, \partial S)$

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¹If this set also contains the isotopy class of the identity, then it is a monoid.

denote the subsets of $\text{Mod}^+(S, \partial S)$ whose elements give rise to open books supporting Stein fillable, strongly symplectically fillable and weakly symplectically fillable contact manifolds, respectively. Then Theorem 1.2 implies the following.

Theorem 1.3. *Stein($S, \partial S$), Strong($S, \partial S$) and Weak($S, \partial S$) are monoids.*

The contact invariant in Heegaard Floer homology (HF) is well behaved with respect to the maps induced by Stein cobordisms [20]. Specifically, if (M', ξ') is obtained from (M, ξ) by performing contact (-1) -surgery on a Legendrian knot, and W is the corresponding two-handle cobordism from M to M' , then the map

$$F_{-W} : \widehat{\text{HF}}(-M') \rightarrow \widehat{\text{HF}}(-M)$$

sends $c(\xi')$ to $c(\xi)$. In addition, for two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) , the contact invariant $c(\xi_1 \# \xi_2)$ is identified with $c(\xi_1) \otimes c(\xi_2)$ via the isomorphism

$$\widehat{\text{HF}}(-(M_1 \# M_2)) \cong \widehat{\text{HF}}(-M_1) \otimes_{\mathbb{Z}_2} \widehat{\text{HF}}(-M_2).$$

Coupled with these facts, Theorem 1.1 immediately reproduces the following result from [3].

Corollary 1.1 ([3, Theorem 1.4]). *There exists a “comultiplication” map*

$$\widehat{\text{HF}}(-M_{S,hg}) \rightarrow \widehat{\text{HF}}(-M_{S,h}) \otimes_{\mathbb{Z}_2} \widehat{\text{HF}}(-M_{S,g}),$$

which sends $c(\xi_{S,hg})$ to $c(\xi_{S,h}) \otimes c(\xi_{S,g})$.

In particular, the set of $\phi \in \text{Mod}^+(S, \partial S)$ for which $c(\xi_{S,\phi}) \neq 0$ forms a monoid, which prompts the following question. Below, $\text{Tight}(S, \partial S)$ is the set of $\phi \in \text{Mod}^+(S, \partial S)$ for which $(M_{S,\phi}, \xi_{S,\phi})$ is tight.

Question 1.1. *Is $\text{Tight}(S, \partial S)$ a monoid?*

Corollary 1.1 does not provide an answer to Question 1.1, as there are tight contact structures whose contact invariants vanish [12, 19]. In fact, the question of whether tightness is preserved by contact (-1) -surgery remains open for closed² contact manifolds. Interestingly, Theorem 1.2 implies that this seemingly more basic question is actually equivalent to Question 1.1.

The three monoids in Theorem 1.3 have been discovered independently by Baker *et al.*, who constructed their own Stein cobordism from $(M_{S,h}, \xi_{S,h}) \sqcup (M_{S,g}, \xi_{S,g})$ to $(M_{S,hg}, \xi_{S,hg})$ in [1]. Upon hearing of their result, I realized that the cobordism from $(M_{S,h}, \xi_{S,h}) \# (M_{S,g}, \xi_{S,g})$ to $(M_{S,hg}, \xi_{S,hg})$ defined in the last section of my paper with Plamenevskaya [4] carries a very natural Stein structure (which is the one explained here). The proof of Theorem 1.1 in this paper makes use of standard tools in convex surface theory. In contrast, the approach of Baker *et al.* relies on an understanding of the contact structures associated to various cables of the binding of an open book. It would be interesting to determine whether our different approaches yield what are more or less the same Stein cobordisms in the end.

²There is a tight genus four handlebody which becomes overtwisted after contact (-1) -surgery [18].

The methods used in this paper can be applied in other settings as well. For example, suppose that K is a transverse knot in the standard tight contact manifold (S^3, ξ_{std}) . A well-known result of Bennequin asserts that

$$\text{sl}(K) \leq -\chi(\Sigma),$$

where $\text{sl}(K)$ denotes the self-linking number of K and Σ is any Seifert surface for K [5]. We say that K realizes its Bennequin bound if $\text{sl}(K) = -\chi(\Sigma)$ for some Seifert surface Σ . In [16], Hedden proves that if K is fibered and realizes its Bennequin bound then the open book associated to K supports the contact manifold (S^3, ξ_{std}) (see [9] for a more general result). If (S, ϕ) denotes this open book, then (S, ϕ^n) supports the contact manifold obtained by taking the n -fold cyclic cover of (S^3, ξ_{std}) branched along K . Since (S^3, ξ_{std}) is Stein fillable, Theorem 1.3 implies the following.

Corollary 1.2. *If K is a fibered transverse knot in (S^3, ξ_{std}) which realizes its Bennequin bound, then the n -fold cyclic cover of (S^3, ξ_{std}) branched along K is Stein fillable.*

Remark 1.1. The statement in Corollary 1.2 follows independently from the fact that if K is a fibered transverse knot in (S^3, ξ_{std}) which realizes its Bennequin bound, then K is strongly quasi-positive [15] and therefore bounds a complex curve Σ in $B^4 \subset \mathbb{C}^2$ [21]. Hence, the n -fold cyclic cover of (B^4, i) branched along Σ is a holomorphic filling of the n -fold cyclic cover of (S^3, ξ_{std}) branched along K . Lastly, a result of Bogomolov and de Oliveira tells us that this holomorphic filling may be deformed into the blow-up of a Stein filling [6].

Using a slight variation of the main technique in this paper as suggested by Van Horn-Morris, combined with the ideas in [9, Section 3], we can prove a much stronger result which does not assume that K is fibered.

Theorem 1.4. *If K is a transverse knot in a Stein (resp. strongly/weakly symplectically) fillable contact manifold (M, ξ) which realizes its Bennequin bound, then the n -fold cyclic cover of (M, ξ) branched along K is Stein (resp. strongly/weakly symplectically) fillable.*

2. Proof of Theorem 1.1

First, we describe the contact three-manifold, $(M_{S, \phi}, \xi_{S, \phi})$, which is compatible with the open book (S, ϕ) . Let U be the handlebody defined by $U = S \times [-1, 1] / \sim$, where $(x, t) \sim (x, 0)$ for all $x \in \partial S$ (see Figure 1). The oriented curve $\Gamma = \partial S \times \{0\}$ divides $\Sigma = \partial U$ into two pieces, $\Sigma^+ = S \times \{1\}$ and $\Sigma^- = -S \times \{-1\}$. We may therefore view ϕ as a boundary-fixing diffeomorphism of Σ^+ . Note that $\partial\Sigma^+ = \Gamma = -\partial\Sigma^-$, and let $r : \Sigma \rightarrow \Sigma$ be the orientation-reversing involution defined by reflection across Γ .

It is not hard to prove that there exists a unique (up to isotopy) tight contact structure ξ_0 on U for which Σ is convex with dividing set Γ (see [10], for example). Let (U_1, ξ_1) and (U_2, ξ_2) be identical copies of (U, ξ_0) , with $\partial U_1 = \Sigma = \partial U_2$. According to Torisu [22], $(M_{S, \phi}, \xi_{S, \phi})$ is the contact three-manifold obtained by gluing (U_2, ξ_2) to (U_1, ξ_1) via the orientation-reversing diffeomorphism $A_\phi : \partial U_2 \rightarrow \partial U_1$ defined by

$$A_\phi(x) = \begin{cases} r(\phi(x)), & x \in \Sigma^+, \\ r(x), & x \in \Sigma^-. \end{cases}$$

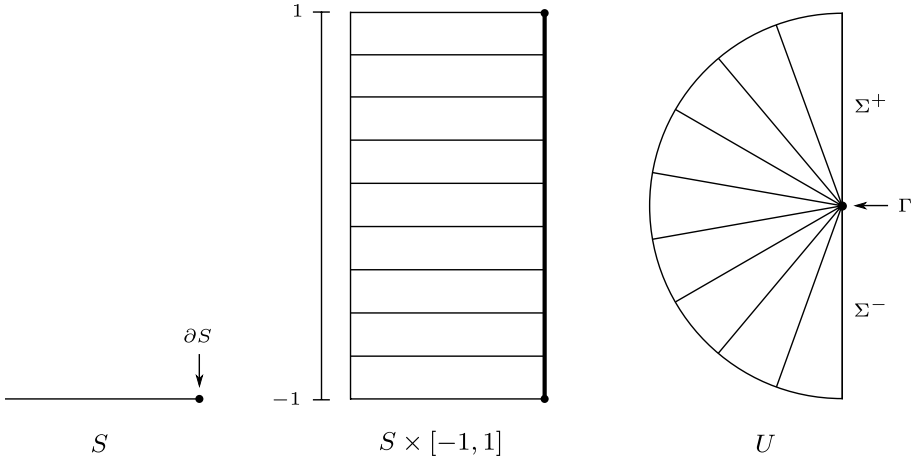


FIGURE 1. The diagram on the left represents the surface S . The diagram in the middle represents $S \times [-1, 1]$; we have drawn some of the $S \times \{t\}$ fibers. The diagram on the right represents the handlebody U obtained from $S \times [-1, 1]$ by collapsing $\partial S \times [-1, 1]$ to $\Gamma = \partial S \times \{0\}$.

(The orientation on $M_{S,\phi}$ is specified by $M_{S,\phi} = U_1 - U_2$.) The fact that A_ϕ sends $\Gamma \subset \partial U_1$ to $\Gamma \subset \partial U_2$ is what makes it possible to glue these two contact structures together, by Giroux's flexibility theorem [13].

Now suppose that ϕ is the composition hg . Let I be the interval $[-\epsilon, \epsilon]$, and let ξ_I be the I -invariant contact structure on $\Sigma \times I$ for which each $\Sigma \times \{s\}$ is convex with dividing set $\Gamma \times \{s\}$.³ Then $(M_{S,hg}, \xi_{S,hg})$ may also be obtained by first gluing (U_2, ξ_2) to $(\Sigma \times I, \xi_I)$ by the diffeomorphism from ∂U_2 to $\Sigma \times \{\epsilon\}$ which sends x to $(A_g(x), \epsilon)$, and then gluing the resulting contact manifold to (U_1, ξ_1) by the diffeomorphism from $\Sigma \times \{-\epsilon\}$ to ∂U_1 which sends $(x, -\epsilon)$ to $A_h(r(x))$. This just amounts to the fact that, in Torisu's description, the convex surface $\Sigma \subset M_{S,hg}$ has an I -invariant neighborhood. See Figure 2 for reference.

If S has genus g and r boundary components, then Σ has genus $n = 2g + r - 1$. Let b_1, \dots, b_n be disjoint, properly embedded arcs in S for which $S - \cup_i b_i$ is a disk. For $i = 1, \dots, n$, we define the curve $\beta_i \subset \Sigma$ by

$$\beta_i = b_i \times \{-1\} \cup b_i \times \{1\},$$

where we are again thinking of Σ as $S \times \{1\} \cup -S \times \{-1\}$. (See Figure 3 for an example.) Note that β_i bounds the attaching disk $b_i \times [-1, 1] \subset U$. In particular, U may be recovered from Σ by thickening the surface, attaching two handles to one side along the curves β_i , and then gluing a three-ball to the S^2 boundary component of the resulting manifold.

Let \mathbb{L}_β be the link, contained in the $\Sigma \times I$ portion of $M_{S,hg}$, whose components are the curves $\beta_i \times \{0\} \subset \Sigma \times \{0\}$. The link \mathbb{L}_β is *nonisolating* in the convex surface $\Sigma \times \{0\}$; that is, \mathbb{L}_β is transverse to $\Gamma \times \{0\}$, and the closure of every component of

³The coordinate s on I is not to be confused with the coordinate t on the interval $[-1, 1]$ described previously.

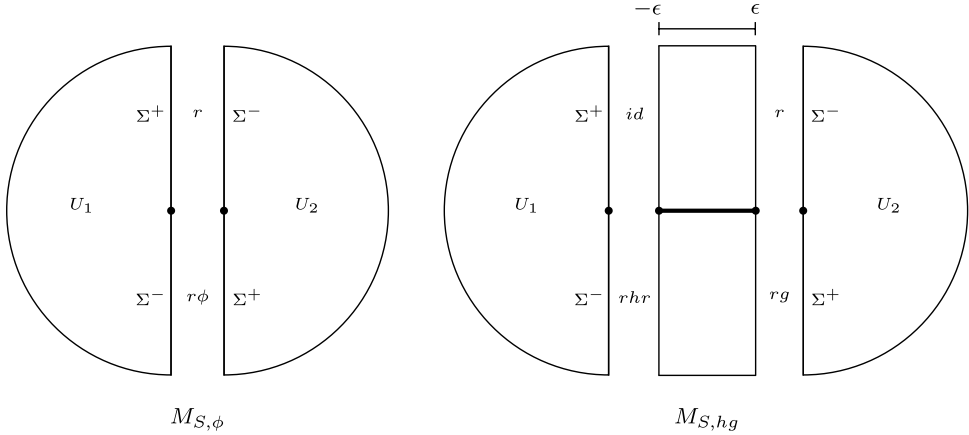


FIGURE 2. The diagram on the left illustrates the process of gluing U_2 to U_1 to form $M_{S,\phi}$. Alternatively, $M_{S,hg}$ can be formed by gluing U_2 to $\Sigma \times I$, and then gluing the result to U_1 , as shown in the diagram on the right.

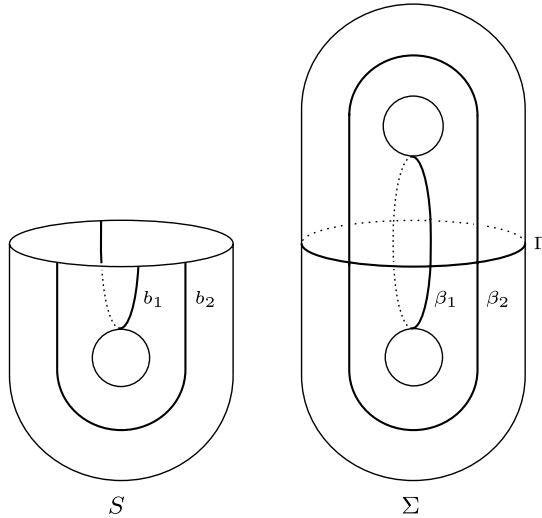


FIGURE 3. In this example, S is a genus one surface with one boundary component. The diagram on the right shows the curves β_1 and β_2 in blue, and the dividing set Γ in red.

$\Sigma \times \{0\} - (\Gamma \times \{0\} \cup \mathbb{L}_\beta)$ intersects $\Gamma \times \{0\}$. Therefore, by the Legendrian realization principle, we may assume that \mathbb{L}_β is Legendrian [17]. Moreover, each $\beta_i \times \{0\}$ intersects the dividing set $\Gamma \times \{0\}$ in exactly two places. It follows that $tw(\beta_i \times \{0\}, \Sigma \times \{0\})$, which measures the contact framing of $\beta_i \times \{0\}$ relative to the framing induced by the surface $\Sigma \times \{0\}$, is

$$-\frac{1}{2} \#(\beta_i \times \{0\} \cap \Sigma \times \{0\}) = -1.$$

Therefore, contact (+1)-surgery on \mathbb{L}_β is the same as 0-surgery on \mathbb{L}_β with respect to the framing induced by $\Sigma \times \{0\}$.

For any contact manifold (M, ξ) , and any Legendrian link $\mathbb{L} \subset M$, let us denote by $(M_{\mathbb{L}}, \xi_{\mathbb{L}})$ the contact manifold obtained from M via contact (+1)-surgery on \mathbb{L} .

Proposition 2.1. *The contact manifold $((\Sigma \times I)_{\mathbb{L}_\beta}, (\xi_I)_{\mathbb{L}_\beta})$ is tight.*

Proof. By construction, $((\Sigma \times I)_{\mathbb{L}_\beta}, (\xi_I)_{\mathbb{L}_\beta})$ embeds into $((M_{S,hg})_{\mathbb{L}_\beta}, (\xi_{S,hg})_{\mathbb{L}_\beta})$ for any h and g . So, it is enough to find an h and g for which the latter is tight. Let h and g each be the identity. In this case, $M_{S,hg} = M_{S,\text{id}} \cong \#^n(S^1 \times S^2)$, and $\xi_{S,\text{id}}$ is the unique Stein fillable contact structure on this connected sum. Note that each component $\beta_i \times \{0\}$ of \mathbb{L}_β bounds the disk

$$b_i \times [-1, 1] \cup \beta_i \times [-\epsilon, 0] \subset (U_1 \cong S \times [-1, 1]) \cup (\Sigma \times [-\epsilon, 0]).$$

Since these disks are all disjoint, small neighborhoods of these disks in $U_1 \cup (\Sigma \times I)$ are also disjoint. Moreover, we can assume that these neighborhoods are Darboux balls since $\xi_{S,\text{id}}$ restricts to a tight contact structure on $U_1 \cup (\Sigma \times I)$. Furthermore, the framings induced by these disks agree with the framings induced by $\Sigma \times \{0\}$. Since contact (+1)-surgery on the Legendrian unknot in (S^3, ξ_{std}) with $tb = -1$ is the Stein fillable contact manifold $(S^1 \times S^2, \xi_0)$, $((M_{S,\text{id}})_{\mathbb{L}_\beta}, (\xi_{S,\text{id}})_{\mathbb{L}_\beta})$ is the contact connected sum of $((M_{S,\text{id}}), (\xi_{S,\text{id}}))$ with n copies of $(S^1 \times S^2, \xi_0)$, and is therefore tight. \square

Proposition 2.2. *The contact manifold $((M_{S,hg})_{\mathbb{L}_\beta}, (\xi_{S,hg})_{\mathbb{L}_\beta})$ is the contact connected sum $(M_{S,h}, \xi_{S,h}) \# (M_{S,g}, \xi_{S,g})$.*

Proof. Let $N_i \subset \Sigma \times I$ be a tubular neighborhood of $\beta_i \times \{0\}$ such that ∂N_i is the union of two annuli, $A_i^1 \subset \Sigma \times [-\epsilon, 0]$ and $A_i^2 \subset \Sigma \times [0, \epsilon]$. And let us think of S^1 as the union of two intervals, $S^1 = I_1 \cup I_2$. Topologically, $(\Sigma \times I)_{\mathbb{L}_\beta}$ is obtained from $\Sigma \times I$ by performing 0-surgery on \mathbb{L}_β with respect to the framing induced by $\Sigma \times \{0\}$, as discussed above. $(\Sigma \times I)_{\mathbb{L}_\beta}$ is therefore the result of gluing solid tori $D_i^2 \times S^1$ to $\Sigma \times I - \cup_i \text{int } N_i$ so that $\partial D_i^2 \times I_1$ is glued to A_i^1 , and $\partial D_i^2 \times I_2$ is glued to A_i^2 (see Figure 4). So, $(\Sigma \times I)_{\mathbb{L}_\beta}$ is the union

$$(2.1) \quad (\Sigma \times [-\epsilon, 0] - \cup_i \text{int } N_i) \cup_i (D_i^2 \times I_1) \bigcup (\Sigma \times [0, \epsilon] - \cup_i \text{int } N_i) \cup_i (D_i^2 \times I_2).$$

Each of these two pieces is homeomorphic to the manifold obtained by thickening Σ and attaching two handles to one side of this thickened surface along the curves β_i ; in other words, each piece is the complement of a three-ball in a genus n handlebody, and these pieces are attached along their common S^2 boundary component.

Let us denote the left and right pieces in (2.1) by $U_3 - B^3$ and $U_4 - B^3$, respectively, where U_3 and U_4 are genus n handlebodies with $\partial U_3 = -\Sigma \times \{-\epsilon\}$ and $\partial U_4 = \Sigma \times \{\epsilon\}$. According to Giroux [13], their common S^2 boundary component can be made convex in $((\Sigma \times I)_{\mathbb{L}_\beta}, (\xi_I)_{\mathbb{L}_\beta})$ after a small isotopy. By Proposition 2.1, the restriction of $(\xi_I)_{\mathbb{L}_\beta}$ to $U_i - B^3$ is tight, for $i = 3, 4$. Therefore, by Honda's gluing theorem [18, Theorem 2.5], the restriction $(\xi_I)_{\mathbb{L}_\beta}|_{U_i - B^3}$ is isotopic to the contact structure on the complement of a Darboux ball in (U_i, ξ_i) , where ξ_i is the unique tight contact structure on U_i for which ∂U_i is convex with dividing set $\Gamma \times \{-\epsilon\}$ when $i = 3$, and $\Gamma \times \{\epsilon\}$ when $i = 4$. Said differently, $((\Sigma \times I)_{\mathbb{L}_\beta}, (\xi_I)_{\mathbb{L}_\beta})$ is the contact connected sum of identical copies, (U_3, ξ_3) and (U_4, ξ_4) , of the contact handlebody (U, ξ_0) .

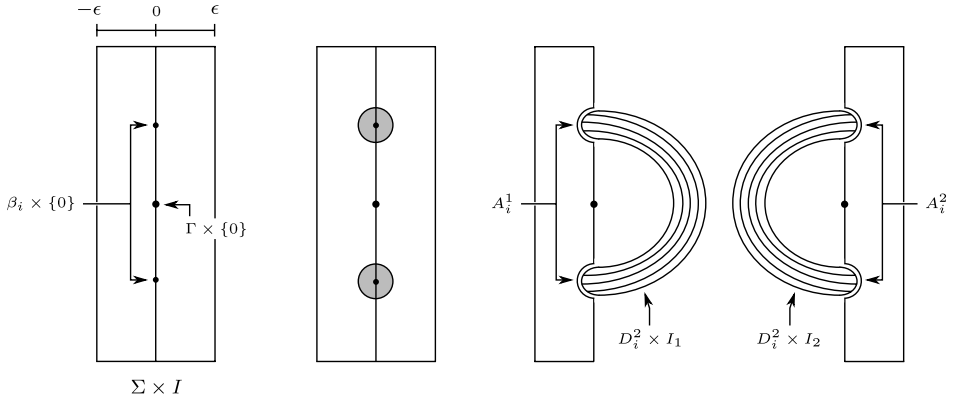


FIGURE 4. The diagram on the left shows the knot $\beta_i \times \{0\} \subset \Sigma \times \{0\}$. The shaded disks in the middle diagram represent the tubular neighborhood N_i . The diagram on the right illustrates the process of performing 0-surgery on $\beta_i \times \{0\}$ by removing N_i and gluing two handles along the annuli A_i^1 and A_i^2 . We have drawn some of the $D^2 \times \{t\}$ fibers in these two handles.

As a result, $((M_{S,hg})_{\mathbb{L}_\beta}, (\xi_{S,hg})_{\mathbb{L}_\beta})$ may be pieced together as follows. First, glue (U_2, ξ_2) to (U_4, ξ_4) by the diffeomorphism from ∂U_2 to $\partial U_4 = \Sigma \times \{\epsilon\}$ which sends x to $(A_g(x), \epsilon)$; this forms $(M_{S,g}, \xi_{S,g})$. Next, glue (U_3, ξ_3) to (U_1, ξ_1) by the diffeomorphism from $-\partial U_3 = \Sigma \times \{-\epsilon\}$ to ∂U_1 which sends $(x, -\epsilon)$ to $A_h(r(x))$; this forms $(M_{S,h}, \xi_{S,h})$. Finally, remove Darboux balls from the U_3 and U_4 portions of $M_{S,h}$ and $M_{S,g}$, and glue the resulting contact manifolds together by a diffeomorphism which identifies the dividing curves on their S^2 boundary components. This process realizes $((M_{S,hg})_{\mathbb{L}_\beta}, (\xi_{S,hg})_{\mathbb{L}_\beta})$ as the contact connected sum of $(M_{S,h}, \xi_{S,h})$ with $(M_{S,g}, \xi_{S,g})$. \square

Proof of Theorem 1.1. According to Ding and Geiges [7, Proposition 8], Proposition 2.2 implies that $(M_{S,hg}, \xi_{S,hg})$ is the result of contact (-1) -surgery on a link in the contact connected sum $(M_{S,h}, \xi_{S,h}) \# (M_{S,g}, \xi_{S,g})$. \square

3. Fillability of cyclic branched covers

The essential idea in the proof of Theorem 1.1 is that we can find curves on the convex surface $\Sigma \times \{0\} \subset M_{S,hg}$ which each intersect the dividing set twice and which are attaching curves for the handlebody $S \times [-1, 1]$. These conditions guarantee that contact $(+1)$ -surgery on these curves is the same as 0-surgery with respect to the framing induced by Σ , and, therefore, that such surgery results in the appropriate connected sum. This idea can be applied more generally to prove results like Theorem 1.4, as below.

Proof of Theorem 1.4. Suppose that K is a transverse knot in a tight contact manifold (M, ξ) , and that S is a Seifert surface for K for which $\text{sl}_\xi(K) = -\chi(S)$. Following the discussion in [9, Section 3], we may perturb S so that it is convex with dividing set Γ disjoint from ∂S [9]. This convex S has an I -invariant neighborhood $N = S \times [-1, 1]$

whose convex boundary, after rounding corners, is $\Sigma = DS$, the double of S . In [9, Section 3], the authors show that the dividing set on Σ is given by

$$\mathbf{\Gamma} = \Gamma \cup \bar{\Gamma} \cup C,$$

where Γ is the dividing set of S as it sits on $S \times \{1\}$, $\bar{\Gamma}$ is the dividing set of S as it sits on $S \times \{-1\}$, and $C = \partial S \times \{1/2\}$ is a curve isotopic to K .

As before, let b_1, \dots, b_n be disjoint, properly embedded arcs in S for which $S - \cup_i b_i$ is a disk, and define curves $\beta_i = \partial(b_i \times [-1, 1])$ on Σ . Now, it is not necessarily true that each β_i intersects $\mathbf{\Gamma}$ twice. To remedy this, we let p_1, \dots, p_k denote the points of intersection between the b_i and Γ , and consider instead the complementary handlebody

$$N' = N - \cup_j (\nu(p_j) \times [-1, 1]),$$

where each $\nu(p_j) \times [-1, 1]$ is a standard neighborhood of the Legendrian arc $p_j \times [-1, 1]$ in N . The boundary $\Sigma' = \partial N'$ is obtained from Σ by attaching k tubes from $S \times \{1\}$ to $S \times \{-1\}$ corresponding to the p_j . Removing the $\nu(p_j)$ from S cuts each b_i into properly embedded arcs $b_{i,1}, \dots, b_{i,n_i}$ in $S - \cup_j \nu(p_j)$. Then the $\beta_{i,l} = \partial b_{i,l} \times [-1, 1]$ are attaching curves for the handlebody N' and can each be made to intersect the new dividing curves $\mathbf{\Gamma}' \subset \Sigma'$ twice (see Figure 5).

Note that the contact manifold $(\Sigma^2(M, K), \xi^2(M, K))$ obtained by taking the double cover of (M, ξ) branched along K is formed by gluing together two copies of $M - \text{int}(N)$ along their boundaries (which are copies of Σ) so that the dividing curves match up. Therefore, by gluing together two copies of $M - \text{int}(N')$ along their boundaries (which are copies of Σ'), one obtains a contact connected sum

$$(3.1) \quad (\Sigma^2(M, K), \xi^2(M, K)) \# (\#^k(S^1 \times S^2, \xi')),$$

for some contact structure ξ' on $\#^k(S^1 \times S^2)$. Let $\beta'_{i,l}$ denote the copy of $\beta_{i,l}$ in this glued manifold, and let

$$\mathbb{L} = \cup_{i=1}^n \cup_{l=1}^{n_i} \beta'_{i,l}.$$

Then contact (+1)-surgery on \mathbb{L} produces the contact connected sum $(M, \xi) \# (M, \xi)$, as before. In other words, the manifold in (3.1) is obtained from $(M, \xi) \# (M, \xi)$

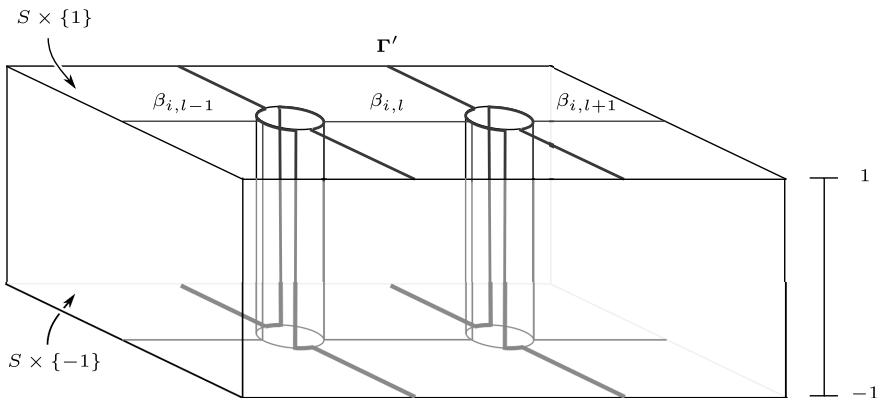


FIGURE 5. A portion of the handlebody N' , obtained by boring tunnels out of N . The new dividing curve $\mathbf{\Gamma}'$ is obtained after rounding corners in the usual way.

via l Stein 2-handle additions. So, if (M, ξ) is at least weakly fillable, then the same is true of the manifold in (3.1); in this case, ξ' must be the unique Stein fillable contact structure on $\#^k(S^1 \times S^2)$. Moreover, $(\Sigma^2(M, K), \xi^2(M, K))$ is obtained from the manifold in (3.1) by k Stein 2-handle additions. So, in the end, $(\Sigma^2(M, K), \xi^2(M, K))$ is obtained from $(M, \xi) \# (M, \xi)$ via $(k + l)$ Stein 2-handle additions. As a result, we find that as long as (M, ξ) is Stein (resp. strongly/weakly symplectically) fillable, then so is $(\Sigma^2(M, K), \xi^2(M, K))$.

Note that if \tilde{K} and \tilde{S} are the lifts of K and S in $\Sigma^2(M, K)$, then

$$\text{sl}_{\xi^2(M, K)}(\tilde{K}) = -\chi(\tilde{S}).$$

We may therefore apply a similar construction to conclude that there is a Stein two-handle cobordism from $(\Sigma^2(M, K), \xi^2(M, K)) \# (M, \xi)$ to $(\Sigma^3(M, K), \xi^3(M, K))$, the three-fold cyclic cover of (M, ξ) branched along K . In general, if (M, ξ) is at least weakly fillable, then there is Stein cobordism from $(\Sigma^n(M, K), \xi^n(M, K)) \# (M, \xi)$ to $(\Sigma^{n+1}(M, K), \xi^{n+1}(M, K))$. This completes the proof of Theorem 1.4. \square

Theorem 1.4 ultimately rests on the fact that K may be “protected” from the dividing curves on S whenever $\text{sl}(K) = -\chi(S)$ [1]. That is, S is isotopic to a convex surface with for which there is a component C of the dividing set such that C and K cobound an annulus with characteristic foliation consisting of arcs from C to K . This is what allows us to conclude that the dividing set on $\Sigma = DS$ is of the form $\Gamma \cup \bar{\Gamma} \cup C$.

It is an interesting problem to find a more general criterion which ensures that a transverse knot $K \subset (S^3, \xi_{\text{std}})$ is the protected boundary of some Seifert surface. Van Horn-Morris and I hope to return to this problem in a future paper. For now, note that the proof of Theorem 1.4 provides an obstruction: if the n -fold cyclic branched cover of K is not Stein-fillable, then K does not have protected boundary. The proposition below is an application of this obstruction.

Proposition 3.1. *Let B be the transverse three-braid in (S^3, ξ_{std}) with braid word given by $(\sigma_1 \sigma_2)^3 \sigma_1 \sigma_2^{-a_1} \cdots \sigma_1 \sigma_2^{-a_m}$, where the $a_i \geq 0$ and some $a_j \neq 0$. Then B is not the protected boundary of any Seifert surface when $4 + m - \sum a_i < 0$.*

Proof. From the proof of Theorem 1.4, it is enough to observe that the branched double cover $(\Sigma^2(S^3, B), \xi^2(S^3, B))$ is not Stein fillable when $4 + m - \sum a_i < 0$. This fact appears in [3]. \square

Remark 3.1. One should contrast Proposition 3.1 with the fact that $(\Sigma^n(S^3, B), \xi^n(S^3, B))$ is tight for all n [2].

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