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MODULAR INVARIANT *D*-MODULES

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Abstract. We study D-modules over the modular curve of level 1 defined as an orbifold, and show that if such D-modules are of rank 1, these monodromy representations map the S-action to the identity. This means that in the orbifold case, there are linear representations of the fundamental groups which do not come from D-modules.

1. Introduction

It is well known that any linear representation of the fundamental group of a complex manifold is obtained as the monodromy of a certain D-module over this manifold. The aim of this paper is to show that the modular curve of level 1 defined as an orbifold does not satisfy this property. Our method is motivated by a result of Nakamura and Schneps (cf. [N, Section 4] and [NS, Section 7]) which concerns the Galois actions on the algebraic fundamental groups of the modular curves X_i of level $i = 1, 2$. We study D-modules over X_1, called *modular invariant D-modules*, using the natural covering map

$$X_2 \cong \mathbb{P}^1 - \{0, 1, \infty\} \to X_1,$$

and show that if a D-module over X_1 has rank 1, then this monodromy representation maps the modular transformation by $S : \tau \leftrightarrow -1/\tau$ to the identity. This implies that there are linear representations of $\pi_1(X_1)$ which are not obtained as the monodromy representations of modular invariant D-modules.

2. *D*-modules over modular curves

2.1. *D*-modules over orbifolds. A complex orbifold has a open covering $\{[U_\lambda/G_\lambda]\}_\lambda$, where G_λ is a finite group acting on a complex manifold U_λ. Denote by $p_\lambda : U_\lambda \to U_\lambda/G_\lambda$ the natural projection to the geometric quotient of U_λ by the action of G_λ. Then M is called a *D-module (of finite rank) over a complex orbifold X* if there exists a open covering $\{[U_\lambda/G_\lambda]\}_\lambda$ of X such that M is a compatible system $(F_\lambda, \nabla_\lambda)$ of vector bundles with meromorphic connection over U_λ/G_λ such that $p_\lambda^*(F_\lambda, \nabla_\lambda)$ is isomorphic to a vector bundle with holomorphic connection over U_λ. For each D-module over X, one can associate naturally its monodromy which is a linear representation of $\pi_1(X)$.

2.2. Modular curves. Let $\zeta_n = \exp(2\pi i/n)$ be an nth root of 1, and for $a, \tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$, put $q^a = \exp(2\pi i a\tau)$. Let $H = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ denote the Poincaré upper-half plane with natural action of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1\}$. The principal congruence subgroup of $SL_2(\mathbb{Z})$ of level 2 is $\Gamma(2) = \mathrm{Ker}(SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/2\mathbb{Z}))$, and $P\Gamma(2)$ is defined as $\Gamma(2)/\{\pm 1\}$. Then $X_1 = H/PSL_2(\mathbb{Z})$ and $X_2 = H/P\Gamma(2)$ are

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called the modular curves of level 1 and 2, respectively. We consider X_1 as a complex orbifold with fundamental group

$$\pi_1(X_1) = \pi_1(X_1; \overrightarrow{01}) \cong PSL_2(\mathbb{Z}),$$

($\overrightarrow{01}$: the tangential base point for the q-coordinate) which has generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with relations $S^2 = (TS)^3 = 1$. Then the generators S, TS of $PSL_2(\mathbb{Z})$ stabilize $\zeta_4 = i, \zeta_6 \in H$ respectively, and X_1 is expressed locally as orbifolds:

$$\begin{cases} [U/\{\pm 1\}], & \text{around the image of } i, \\ [U/\langle \zeta_3 \rangle], & \text{around the image of } \zeta_6, \\ [U] = U, & \text{otherwise,} \end{cases}$$

where $U = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\bar{X}_1 = X_1 \cup \{i\infty\}$ and $\bar{X}_2 = X_2 \cup \{0, 1, i\infty\}$ be the completion of X_1 and X_2 obtained by adding their cusps.

We describe natural models of $\bar{X}_i$ which are examples of the canonical models in Shimura's theory when these are considered as defined over $\mathbb{Q}$. First, the Legendre λ-function gives an isomorphism $\lambda : X_2 \xrightarrow{\sim} \mathbb{P}^1 - \{0, 1, \infty\}$, and this extends to an isomorphism $\bar{\lambda} : \bar{X}_2 \xrightarrow{\sim} \mathbb{P}^1$ mapping the cusps $i\infty, 0, 1$ to $0, 1, \infty$, respectively. Then $\lambda(\tau) = 16q^{1/2} + \cdots$ at $\tau = i\infty$, and one can see that

$$\lambda(T(\tau)) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}, \quad \lambda(S(\tau)) = 1 - \lambda(\tau)$$

by seeing the changes of the $\bar{\lambda}$-values of the cusps under $\tau \mapsto T(\tau), S(\tau)$. Second, the j-function gives a surjective holomorphic map $j : X_1 \to \mathbb{C}$, and this extends to $\bar{j} : \bar{X}_1 \to \mathbb{P}^1$ mapping $\zeta_6, i \in H$ and the cusp $i\infty$ to $0, 1$ and ∞, respectively. Since

$$j = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

$j = 1$ has double roots at

$$\lambda \in R_2 = \{\lambda(i) = 1/2, \lambda(T(i)) = 2, \lambda(S(T(i))) = -1\},$$

and $j = 0$ has triple roots at

$$\lambda \in R_3 = \{\lambda(\zeta_6) = \zeta_6, \lambda(\zeta_3) = \zeta_6^{-1}\}.$$

Let $\pi : X_2 \to X_1$ and $\bar{\pi} : \bar{X}_2 \to \bar{X}_1$ be the natural projections of degree 6. Then $\bar{j} \circ \bar{\pi} \circ (\bar{\lambda})^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ is ramified in $\{0, 1, \infty\} \cup R_2$ with ramification index 2, and is ramified in R_3 with ramification index 3.

2.3. Connection matrices. We recall the definition of connection matrices. Let F be a trivial bundle over $\mathbb{P}^1$ with meromorphic connection of the form $(A_0/z + A_1/(z - 1))\,dz$, where z is the natural coordinate of $\mathbb{P}^1$. Then this connection matrix $\Phi(A_0, A_1)$ is defined as $G_1(z)^{-1} \cdot G_0(z)$, where $G_i(z)$ $(i = 0, 1)$ be the solutions of

$$G'(z) = \left(\frac{A_0}{z} + \frac{A_1}{z - 1} \right) \cdot G(z),$$

such that $\lim_{z\to 0} G_0(z)/z^{A_0} = \lim_{z\to 1} G_1(z)/(1-z)^{A_1} = 1$, where z runs in $(0,1)$ and $\varepsilon^A = \exp(\log(\varepsilon)\cdot A)$ for $\varepsilon > 0$.

Theorem 2.1. *Let M be a D-module over X_1 of finite rank r.*

(1) *Assume that there exist an extension $\bar{M}$ of M to $\bar{X}_1$ as a vector bundle with meromorphic connection having logarithmic pole at $q = 0$, and a trivial bundle F over $\mathbb{P}^1$ with meromorphic connection ∇ such that $\bar{\lambda}^*(F,\nabla) \cong \bar{\pi}^*(\bar{M})$ and that ∇ is holomorphic except $0, 1, \infty$ at which ∇ has logarithmic poles. Denote by*

$$\omega = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right) dz$$

the connection form of ∇. Then the monodromy of M maps S to the connection matrix $\Phi(A_0, A_1)$.

(2) *Assume that $r = 1$. Then the monodromy maps S to 1, and T to a cubic root of 1.*

Proof. First, we prove (1). Put $V = \mathbb{C}^r$, and identify $V \times I$ $(I = [0,1])$ with the pull back by $\bar{\pi}\circ(\bar{\lambda})^{-1}|_I$ of the trivial bundle F. By that $\lambda(-1/\tau) = 1-\lambda(\tau)$ and that the connection form of ∇ has the residues A_p at $p = 0, 1$, the transformation by S along the line $i\mathbb{R} \subset H$ from $i\infty$ to 0 gives an element of $\mathrm{End}_{\mathbb{C}}(V)$ represented as

$$\lim_{\varepsilon\to 0}(\varepsilon^{-A_1} S \varepsilon^{A_0}),$$

where $S : V \cong V \times \{\varepsilon\} \to V \times \{1-\varepsilon\} \cong V$. Therefore, using iterated integrals of ω,

$$\lim_{\varepsilon\to 0}(\varepsilon^{-A_1} S \varepsilon^{A_0}) = \lim_{\varepsilon\to 0}\left\{\varepsilon^{-A_1}\left(\sum_{n=0}^{\infty}\int_{\varepsilon}^{1-\varepsilon}\underbrace{\omega\cdots\omega}_{n}\right)\varepsilon^{A_0}\right\},$$

which is $\Phi(A_0, A_1)$.

Second, we prove (2). By a theorem of Frobenius, there is an extension $\bar{M}$ of M by gluing the trivial line bundle around $q = 0$ with meromorphic connection of the form Adq/q, where $\exp(2\pi i A)$ is the monodromy of M around $q = 0$. Since $(\pi\circ\lambda^{-1})^*(M)$ is isomorphic to a D-module over $\mathbb{P}^1 - \{0,1,\infty\}$, there are a line bundle F' over $\mathbb{P}^1$ and a meromorphic connection ∇' on F' holomorphic except $0, 1, \infty$ at which ∇' has logarithmic poles such that $(\bar{\pi}\circ(\bar{\lambda})^{-1})^*(\bar{M})$ is isomorphic to (F', ∇'). Represent the fiber of F' around ∞ as $W \cong \mathbb{C}$. Then changing the trivialization of F' around ∞ by $a \mapsto a\cdot z^n$ $(a \in W)$ for certain $n \in \mathbb{Z}$, (F', ∇') becomes a trivial line bundle with meromorphic connection (F, ∇) over $\mathbb{P}^1$ satisfying the desired property. Since M is of rank 1, A_0 and A_1 given in (1) are commutative, and hence $G_0(z) = z^{A_0}(1-z)^{A_1} = G_1(z)$ which implies that S is mapped to $\Phi(A_0, A_1) = 1$. Therefore, by the relation $(TS)^3 = 1$, T is mapped to a cubic root of 1. □

Corollary 2.1. *There is a representation $\pi_1(X_1) \to \mathbb{C}^\times$ which is not obtained as the monodromy of any D-module over X_1.*

Proof. Let $\rho : \pi_1(X_1) = \langle T, S\rangle \to \mathbb{C}^\times$ be the representation which maps T, S to $e^{2\pi i/6}, -1$ respectively. Then by Theorem 2.1 (2), ρ cannot be obtained as the monodromy of any D-module over X_1. □

2.4. The cubic root of *j*. By results of Kac and Peterson [KP] and of Tsuchiya *et al.* [TUY], the conformal field theory for the family of elliptic curves gives rise to examples of D-modules over X_1 (satisfying the assumption of Theorem 2.1 (1)) whose sections are described by the characters for affine Lie algebras. For example, the cubic root

$$\begin{aligned} j^{1/3}(\tau) &= q^{-1/3}\left(1+\sum_{n=1}^{\infty}\binom{1/3}{n}(q\cdot j(\tau)-1)^n\right) \\ &= q^{-1/3}(1+248q+4124q^2+\cdots) \end{aligned}$$

of $j(\tau)$ satisfies the differential equation

$$\frac{d}{d\lambda}j^{1/3} = \left(-\frac{2}{3}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)+\sum_{\gamma\in R_3}\frac{1}{\lambda-\gamma}\right)j^{1/3}$$

associated with a D-module over X_1 of rank 1, and satisfies the functional equations

$$j^{1/3}(T(\tau)) = e^{-2\pi i/3}\cdot j^{1/3}(\tau), \quad j^{1/3}(S(\tau)) = j^{1/3}(\tau).$$

By a result of Kac [K], $j^{1/3}$ becomes the character for the affine Lie algebra of type E_8.

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