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THE HILBERT TRANSFORM DOES NOT MAP $L^1(Mw)$ TO $L^{1,\infty}(w)$

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Abstract. We disprove the following a priori estimate for the Hilbert transform H and the Hardy–Littlewood maximal operator M:

$$\sup_{t>0} tw\{x \in \mathbb{R} : |Hf(x)| > t\} \le C \int |f(x)|Mw(x)\,dx.$$

This is a sequel to paper [1] by the first author, which shows the existence of a weight w and a Haar multiplier operator for which the inequality fails.

1. Introduction and statement of main result

In [2], Fefferman and Stein observed the following a priori estimate for the Hardy–Littlewood maximal operator M:

$$\sup_{t>0} t\, w\{x \in \mathbb{R} : |Mf(x)| > t\} \le C \int |f(x)|Mw(x)\,dx.$$

Here the weight w is a non-negative, locally integrable function, and $w(E)$ denotes the integral of the weight over the set E. A natural question is whether such an inequality holds when the Hardy–Littlewood maximal operator on the left-hand side is replaced by the Hilbert transform. In [3], this question for general Calderón–Zygmund operators is attributed to Muckenhoupt and Wheeden.

Conjecture 1.1 (Muckenhoupt–Wheeden)**.** *Let w be a weight and let M denote the Hardy–Littlewood maximal operator. Let T be a Calderón–Zygmund operator. Then there is a constant C depending only on T such that*

$$\sup_{t>0} tw(\{x \in \mathbb{R} \mid |Tf(x)| > t\}) \le C \int_{\mathbb{R}} |f|Mw(x)dx. \tag{1.1}$$

We refer to [4] for the standard definitions concerning Calderón–Zygmund operators. Early work on this circle of ideas is a paper by Chanillo and Wheeden [5], which proves the inequality (1.1) when T is a square function. This may have been viewed as encouragement towards the Muckenhoupt–Wheeden conjecture at the time. Pérez in [6] proves an analogue of the inequality (1.1) when the Hardy–Littlewood maximal operator is replaced by a larger maximal operator, namely

$$\sup_{t>0} tw(\{x \in \mathbb{R} \mid |Tf(x)| > t\}) \le C \int_{\mathbb{R}} |f|M_{L(\log L)^\epsilon}w(x)dx.$$

He, however, expresses a negative bias towards validity of the conjecture by Muckenhoupt and Wheeden. We also mention the earlier work of Buckley [7], who proves the

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inequality (1.1) for the specific weights $w_\delta(x) = |x|^{-n(1-\delta)}$ for $0 < \delta < 1$. Replacing the Calderón–Zygmund operator by a fractional integral, Carro *et al.* in [8] prove counterexamples to the corresponding analogue of (1.1). In a recent paper [1] by the first author, a dyadic version of the Muckenhoupt–Wheeden conjecture is proved to be false. Building up on this work we disprove the Muckenhoupt–Wheeden conjecture. Our main theorem is:

Theorem 1.1. *For each constant $C > 0$ there is a weight function w on the real line and an integrable compactly supported function f and a $t > 0$ such that*

$$t\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)| Mw(x)\, dx.$$

Following the authors in [9] and [1], we prove Theorem 1.1 as a consequence of Proposition 1.1 below, which in turn is a consequence of the dual Proposition 1.2 further below. For convenience of the reader we include a proof of the reductions to these propositions at the end of this paper.

Proposition 1.1. *For each constant $C > 0$ there is an everywhere positive weight function w on the real line and an integrable compactly supported function f and a $t > 0$ such that*

$$t^2\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)|^2 \left(\frac{Mw(x)}{w(x)}\right)^2 w(x)\, dx. \tag{1.2}$$

Proposition 1.2. *For each constant C there is a non-trivial weight w on the real line such that*

$$\|H(w 1_{[0,1)})\|_{L^2(w/(Mw)^2)} \geq C \|1_{[0,1)}\|_{L^2(w)}.$$

Our construction of the weight w is a somewhat simpler variant of the construction in [1]. Shortly after the completion of a first draft of the present paper, Nazarov *et al.* posted a preprint on the internet [10], proving a stronger variant of Theorem 1.1 by disproving the A_1 conjecture. In particular, they prove that there is no constant C such that the following a priori inequality holds:

$$t\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \leq C \|w\|_{A_1} \int |f(x)| w(x)\, dx. \tag{1.3}$$

Here the A_1 constant is defined as $\|w\|_{A_1} := \|Mw/w\|_\infty$. Their proof involves the Bellmann function technique and is more involved than our argument. We do not see how to easily modify our argument to address inequality (1.3). We close this discussion by mentioning that Lerner *et al.* [3] proves a version of (1.3) with an additional logarithmic factor in the A_1 constant of the weight.

2. Proof of Proposition 1.2

To prove Proposition 1.2, we construct an appropriate weight w on a "smeared out Cantor set". Our Cantor set is the intersection of sets C_i, called the ith step of the Cantor set construction, and each set C_i is the union of $3^{i(k-1)}$ well separated intervals of length 3^{-ik-1}. Here k is some appropriate large parameter. For each such interval J of C_i we choose a satellite interval $\tilde{J}$, which has length $\epsilon|J|$ for some appropriate small ϵ and is adjacent to J, either to the left or to the right. By "smeared out Cantor set" we then mean the union of all these satellite intervals over all steps of the Cantor set construction.

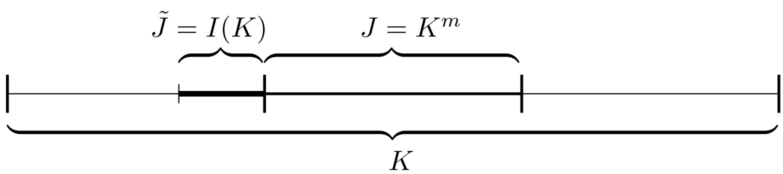
The weight w will be supported on the smeared out Cantor set and constant on each satellite interval. It will be chosen so that it is comparable to Mw on each satellite interval (see (2.3) below). The key estimate then is that on a positive fraction of each satellite interval the function Hw is much larger than w (Lemma 2.1). If we consider for fixed interval J in the Cantor set construction the splitting

$$Hw = H(w1_J) + H(w1_{J^c}),$$

then largeness of Hw on $\tilde{J}$ will be due to the summand $H(w1_J)$. Thanks to the specific construction of the Cantor set, this summand is well approximated by the Hilbert transform of the averaged weight on J, and we have that $H(1_J)$ has size bounded below by $|\log(\epsilon)|$ on $\tilde{J}$. The crux of the matter is to make sure that this large summand is not cancelled by the summand $H(w1_{(J)^c})$. To avoid such cancellation, one uses the liberty to choose $\tilde{J}$ to the left or to the right of J, thereby adjusting the relative sign of the two summands.

We begin with the details of the construction. Recall that a triadic interval I is of the form $[3^j n, 3^j(n+1))$ with integers j, n. Denote by I^m middle third of I, i.e., the triadic interval of one-third the length of I which contains the center of I. Denote the center of I by $c(I)$.

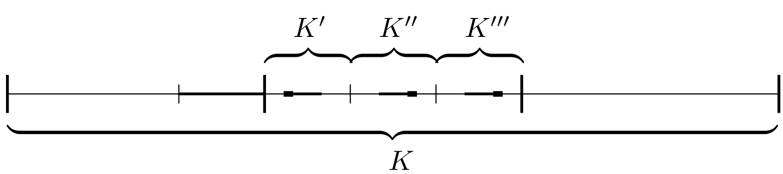
Fix an integer k chosen large enough depending on the constant C in Proposition 1.2 in a manner described further below. The intervals J alluded to in the introductory remarks will be middle thirds of other triadic intervals K, and it is notationally more convenient to work with the parent intervals K. This is illustrated in the first figure which also shows a satellite interval $\tilde{J}$ to the left.



We proceed to define the collections $\mathbf{K}_i$ of intervals of length 3^{-ik}. Define $\mathbf{K}_0$ to be $\{[0,1)\}$ and recursively for $i \geq 1$:

$$\mathbf{K}_i := \left\{ K' : K' \text{ triadic}, \quad |K'| = 3^{-ik}, \quad K' \subset \bigcup_{K \in \mathbf{K}_{i-1}} (K)^m \right\}. \tag{2.1}$$

In words, we pass to the middle third of each interval in $\mathbf{K}_{i-1}$, then we decompose this middle third into 3^{k-1} equally spaced subintervals as indicated in the second figure.



Define $\mathbf{K} := \bigcup_{i\geq 0}\mathbf{K}_i$. Proceeding recursively from the larger to the smaller intervals, we choose for each $K \in \mathbf{K}$ a sign $\epsilon(K) \in \{-1, 1\}$, i.e., $\epsilon(K)$ depends on the values $\epsilon(K')$ with $|K'| > |K|$. The exact choice will be specified later. Then define the interval $I(K)$ to be the triadic interval of length $3^{-k}|K|$ whose right endpoint equals the left endpoint of K^m if $\epsilon(K) = 1$, and whose left endpoint equals the right endpoint of K^m if $\epsilon(K) = -1$. Note that if $K \in \mathbf{K}_i$, then $I(K)$ has the same length as the intervals in $\mathbf{K}_{i+1}$.

Now we define a sequence of absolutely continuous measures on $[0, 1)$, following the steps of the Cantor set construction. We continue to use the same symbol for a measure and its Lebesgue density. Let w_0 be the uniform measure on $[0, 1)^m \cup I([0, 1))$ with total mass 1. Recursively, we define the measure w_i by the following properties: its restriction to the complement of $\bigcup_{K\in\mathbf{K}_i} K$ coincides with the restriction of w_{i-1} to that set. For each $K \in \mathbf{K}_i$ we let $w_i(K) = w_{i-1}(K)$ and we let the restriction of w_i to K be supported on and uniformly distributed on $K^m \cup I(K)$. In other words, we concentrate the measure of each $K \in \mathbf{K}_i$ uniformly onto its middle and satellite intervals.

Let w be the weak limit of the sequence w_i and note that w is supported on $\bigcup_{K\in\mathbf{K}} I(K)$. For $K \in \mathbf{K}_i$, $x \in I(K)$, and any triadic interval I with $|I| \geq |K|$ we have

$$w(x) = \frac{w(I(K))}{|I(K)|} = \frac{w(K)}{|K|} \geq \frac{w(I)}{|I|}. \tag{2.2}$$

We claim that for $K \in \mathbf{K}$ and $x \in (I(K))^m$ we have

$$Mw(x) \leq 6w(x). \tag{2.3}$$

To see this, let I be an interval containing x. If I is contained in $I(K)$, then by the first identity of (2.2) the average of w over I equals $w(x)$. If I is not contained in $I(K)$, then $|I| > |I(K)|/3$ because I contains x which lies in the middle third of $I(K)$. By comparing the average of w over I with the average over two triadic intervals covering I while being no larger than $3|I|$, we obtain from (2.2) that the average of w over I is no more than $6w(x)$. This proves (2.3).

Lemma 2.1. *For $K \in \mathbf{K}_i$, $x \in (I(K))^m$, and $k > 3000$ we have*

$$|Hw(x)| \geq (k/3)w(x).$$

This lemma proves Proposition 1.2, because with (2.3) and since w is constant on every $I(K)$ we have

$$36\|Hw\|^2_{L^2(w/(Mw)^2)} \geq (k^2/9) \sum_{K\in\mathbf{K}} \int_{(I(K))^m} w(y)\, dy = (k^2/27)\|1_{[0,1)}\|^2_{L^2(w)}.$$

Proof of Lemma 2.1. We split the principal value integral for $Hw(x)$ into six summands:

$$p.v. \int_{I(K)} \frac{w(y)}{y-x}\, dy \tag{2.4}$$

$$+ \int_{K^m} \frac{w(y)}{y-x}\, dy \tag{2.5}$$

$$+ \int_{K^c} \left(\frac{w(y)}{y-x} - \frac{w(y)}{y-c(K)} \right) dy \tag{2.6}$$

$$+ \int_{(\bigcup_{\mathbf{K}_i} K')^c} \frac{w(y)}{y-c(K)}\, dy \tag{2.7}$$

$$+ \sum_{K' \in \mathbf{K}_i \setminus \{K\}} \int_{K'} \frac{w(y)}{y-c(K)} - \frac{w(y)}{c(K')-c(K)}\, dy \tag{2.8}$$

$$+ \sum_{K' \in \mathbf{K}_i \setminus \{K\}} \int_{K'} \frac{w(y)}{c(K')-c(K)}\, dy. \tag{2.9}$$

The terms (2.7) and (2.9) remain unchanged if we replace w by w_{i-1} and hence depend only on the choices of $\epsilon(K')$ with $|K'| > |K|$. The integrand of (2.5) is positive or negative depending on $\epsilon(K)$. Specify the choice of $\epsilon(K)$ so that the sign of (2.5) equals the sign of (2.7)+(2.9). If the latter is zero, we may arbitrarily set $\epsilon(K) = 1$. We estimate

$$\begin{aligned} |(2.5)| &\geq \sum_{K' \in \mathbf{K}_{i+1}, K' \subset K^m} \int_{K'} \frac{w(y)}{|y-x|}\, dy \\ &\geq \sum_{K' \in \mathbf{K}_{i+1}, K' \subset K^m} \frac{w(K')}{\sup_{y \in K'} |y-x|} \\ &\geq \sum_{n=1}^{3^k} \frac{1}{n+1} \frac{w(I(K))}{|I(K)|} \geq (k/2) w(x). \end{aligned}$$

The remaining terms are small error terms, we estimate with $\delta = |I(K^m)^m|$:

$$\begin{aligned} |(2.4)| &= \left| \int_{I(K) \setminus [x-\delta, x+\delta]} \frac{w(y)}{y-x}\, dy \right| \leq 3w(x), \\ |(2.6)| &\leq 4 \sum_{|K'|=|K|, K' \neq K} \int_{K'} \frac{|x-c(K)|}{|y-c(K)|^2} w(y)\, dy \\ &\leq 8 \sum_{|K'|=|K|, K' \neq K} \frac{|x-c(K)|}{|c(K')-c(K)|^2} w(K') \\ &\leq 16 \sum_{n=1}^{\infty} \frac{1}{(n-3/4)^2} \frac{w(I(K))}{|I(K)|} \leq 200 w(x), \\ |(2.8)| &\leq 4 \sum_{K' \in \mathbf{K}_i} \int_{K'} \frac{|y-c(K')|}{|c(K')-c(K)|^2} w(y)\, dy, \end{aligned}$$

and the last expression is dominated by the same final bound as (2.6). Putting all estimates together, we have

$$\begin{aligned}&|(2.4)+(2.5)+(2.6)+(2.7)+(2.8)+(2.9)|\\&\quad\geq |(2.5)+(2.7)+(2.9)|-|(2.4)|-|(2.6)|-|(2.8)|\\&\quad\geq |(2.5)|-|(2.4)|-|(2.6)|-|(2.8)|\\&\quad\geq (k/2-403)w(x).\end{aligned}$$

This completes the proof of Lemma 2.1 and thus Proposition 1.2. □

3. Remarks

3.1. More general kernels. While our argument seems to rely on the odd symmetry of the Hilbert kernel, the construction can easily be modified to yield the result for singular integrals with even kernels such as for example $\mathrm{Re}(|x|^{-1+\alpha i})$ with $\alpha \neq 0$. For $K \in K_{i-1}$, let $I(K)$ be the interval of length $3^{-k}|K|$ with the same center as K. Depending on the positive or negative bias of $T(w1_{(K)^c})$ choose instead of (2.1) an appropriate collection of 3^{k-1} intervals of length $3^{-k}|K|$ inside K, so that the kernel of the Calderón Zygmund operator for $x \in (I(K))^m$ and $y \in K'$, $K' \in \mathbf{K}_i$, $K' \subset K$ has sufficiently large positive or negative bias.

3.2. Weights in Theorem 1.1. We specify weights satisfying Theorem 1.1. Fix a constant C as in Proposition 1.2 and consider k and the weight w constructed above. We slightly change w to make it positive by adding ce^{-x^2} for sufficiently small c so as to not change the conclusion of Proposition 1.2. We may normalize the measure to be probability measure and call the remaining measure w again. The conclusion of Proposition 1.2 can be written:

$$\left(\int (Hw(x))^2 \frac{w(x)}{(Mw(x))^2}\,dx\right)^{1/2} \geq C. \tag{3.1}$$

Multiplying both sides of (3.1) by the left-hand side of (3.1), setting $f = (Hw)w/(Mw)^2$ and using essential self-duality of H we obtain

$$\left|\int w(x)Hf(x)\,dx\right| \geq C\left(\int f(x)^2\frac{(Mw(x))^2}{(w(x))^2}w(x)\,dx\right)^{1/2}. \tag{3.2}$$

Letting f^* be the non-increasing rearrangement of Hf on $[0,1]$, we may estimate the left hand side of (3.2)

$$\int_0^1 f^*(y)\,dy \leq 2\sup_{y\in[0,1]} y^{1/2}f^*(y) = 2\sup_{t>0} w(\{x : |Hf(x)| \geq t\})^{1/2}t.$$

Hence Proposition 1.1 holds for the constant $C/2$ with the weight w and some existentially chosen t. Now let E be the set on the left hand side of Proposition 1.1 for the given w, f and appropriate t, then we have

$$M(w1_E)(x) = \sup_{x\in I}\frac{\int_I w}{\int_I 1}\frac{\int_I 1_E w}{\int_I w} \leq Mw(x)M_w 1_E(x),$$

where M_w denotes the Hardy–Littlewood maximal function with respect to the weight w. With Hölder's inequality we obtain

$$\int |f(x)|M(w1_E)(x)\,dx \le \left(\int |f(x)|^2 \frac{Mw(x)^2}{w(x)}\,dx\right)^{1/2} \|M_w 1_E\|_{L^2(w)}.$$

With the Hardy–Littlewood maximal theorem with respect to the weight w we can estimate $\|M_w 1_E\|_{L^2(w)}$ by $w(E)^{1/2}$. This shows that Theorem 1.1 holds for the weight $w1_E$.

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