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DISCRETE GROUP ACTIONS AND GENERALIZED REAL BOTT MANIFOLDS

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ABSTRACT. We study a class of discrete group actions on the Euclidean space. In particular, we will investigate when the orbit spaces of such group actions are closed manifolds. The answer turns out to be a class of real toric manifolds called generalized real Bott manifolds which are the total spaces of some kind of iterated real projective space bundles. This relation provides a new viewpoint on generalized real Bott manifolds which might be useful for the future study.

1. Introduction

A binary matrix is a matrix with all its entries in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. For any binary matrix A, let $A^i_j \in \mathbb{Z}_2$ denote the (i, j) entry of A and let A^i and A_j be the ith row and j-column vector of A. Let $\mathcal{A}(n)$ be the set of all $n \times n$ binary matrices whose diagonal entries are all zero. For any $A \in \mathcal{A}(n)$, we can define a set of Euclidean motions $s^A_1, \dots, s^A_n$ on the n-dimensional Euclidean space $\mathbb{R}^n$ by:

$$s^A_i(x_1, \dots, x_n) := ((-1)^{A^i_1} x_1, \dots, (-1)^{A^i_{i-1}} x_{i-1}, x_i + \frac{1}{2}, (-1)^{A^i_{i+1}} x_{i+1}, \dots, (-1)^{A^i_n} x_n)$$

So s^A_i is the composition of the reflections about some coordinate hyperplanes in $\mathbb{R}^n$ and a translation in the x_i-direction. Let $\Gamma(A)$ be the discrete subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ generated by $s^A_1, \dots, s^A_n$ and let $M(A) := \mathbb{R}^n/\Gamma(A)$ be the quotient space of the $\Gamma(A)$ action on $\mathbb{R}^n$.

For example, when $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the space $M(A)$ is homeomorphic to the two-dimensional torus, Klein bottle and real projective plane, respectively. They are the most common examples used to demonstrate discrete group actions on Euclidean spaces in the textbooks. In this paper, we will study such $M(A)$'s in all dimensions and answer the following questions.

Question 1.1. For an arbitrary $A \in \mathcal{A}(n)$, when $M(A)$ is homeomorphic to a closed manifold?

Question 1.2. If $M(A)$ is a closed manifold, can we identify it with any known examples of manifolds studied by other people before?

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To answer these two questions, let us first define some auxiliary notations. For any $A \in \mathcal{A}(n)$,

$$\text{let } \tilde{A} := A + I_n, \text{ where } I_n \text{ is the identity matrix.} \tag{1.1}$$

For any $1 \leq j_1 < \cdots < j_s \leq n$, let $\tilde{A}^{j_1 \cdots j_s}$ be the $s \times n$ submatrix of $\tilde{A}$ formed by the j_1th, $\ldots, j_s$th row vectors of $\tilde{A}$, and we define an $s \times s$ submatrix of $\tilde{A}$ by:

$$\tilde{A}^{j_1 \cdots j_s}_{j_1 \cdots j_s} := \begin{pmatrix} \tilde{A}^{j_1}_{j_1} & \cdots & \tilde{A}^{j_1}_{j_s} \\ \cdots & \cdots & \cdots \\ \tilde{A}^{j_s}_{j_1} & \cdots & \tilde{A}^{j_s}_{j_s} \end{pmatrix}. \tag{1.2}$$

The $\tilde{A}^{j_1 \cdots j_s}_{j_1 \cdots j_s}$ is called a *principal minor matrix* of $\tilde{A}$ and its determinant $\det(\tilde{A}^{j_1 \cdots j_s}_{j_1 \cdots j_s})$ is called a *principal minor* of $\tilde{A}$. Note that $\tilde{A}^{j_1 \cdots j_s}_{j_1 \cdots j_s}$ is a submatrix of $\tilde{A}^{j_1 \cdots j_s}$, so $\mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1 \cdots j_s}_{j_1 \cdots j_s}) \leq \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1 \cdots j_s})$.

Then we can answer Question-1 and Question-2 by the following two theorems.

Theorem 1.1. *For any $A \in \mathcal{A}(n)$, the space $M(A)$ is a closed manifold if and only if the matrix $\tilde{A} = A + I_n$ satisfies the following two conditions.*

(a) *for $1 \leq \forall s \leq n$ and $1 \leq j_1 < \cdots < j_s \leq n$, $\mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1 \cdots j_s}_{j_1 \cdots j_s}) = \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1 \cdots j_s})$.*

(b) *any set of distinct row vectors of $\tilde{A}$ are linearly independent over $\mathbb{Z}_2$.*

Theorem 1.2. *For any $A \in \mathcal{A}(n)$, if the space $M(A)$ is a closed manifold, then $M(A)$ must be homeomorphic to a generalized real Bott manifold. Conversely, any n-dimensional generalized real Bott manifold is homeomorphic to $M(A)$ for some $A \in \mathcal{A}(n)$.*

Generalized real Bott manifolds are introduced by Choi-Masuda-Suh in [1] as a special class of examples of *real toric manifolds* (i.e. the set of real points of toric manifolds). An n-dimensional closed smooth manifold M^n is called a *generalized real Bott manifold* if there is a finite sequence of fiber bundles

$$M^n = B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{1.3}$$

where each B_i $(1 \leq i \leq m)$ is the projectivization of the Whitney sum of a finite collection (at least two) of real line bundles over B_{i-1}. The smooth structure of M^n is determined by the bundle structures of $\pi_i : B_i \to B_{i-1}$, $i = 1, \ldots, m$. Suppose the fiber of the bundle $\pi_i : B_i \to B_{i-1}$ in (1.3) is homeomorphic to $\mathbb{R}P^{n_i}$ $(n_i \geq 1)$. Then it is easy to see that M^n is a *small cover* (see [2] for definition) over $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ where Δ^{n_i} is the standard n_i-dimensional simplex and $n_1 + \cdots + n_m = n$. In particular, when $n_1 = \cdots = n_m = 1$, M^n is called a *real Bott manifold*. In addition, it was shown in [1] that any small cover over a product of simplices is homeomorphic to a generalized real Bott manifold (Remark 6.5 in [1]).

Real Bott manifolds have been systematically studied in [3–5]. It was proved in [3] that any real Bott manifold admits a flat Riemannian metric and two real Bott manifolds are homeomorphic or diffeomorphic if and only if their cohomology rings are isomorphic. This is called *cohomological rigidity* of real Bott manifolds (see [6]).

In addition, real Bott manifolds are intimately related to the so-called Bott matrices. A binary square matrix A is called a *Bott matrix* if it is conjugate to a strictly upper triangular binary matrix via a permutation matrix. We use $\mathcal{B}(n)$ to denote the set of all $n \times n$ Bott matrices. Obviously, A is a Bott matrix will imply that all the diagonal entries of A are zero. So $\mathcal{B}(n) \subset \mathcal{A}(n)$. It turns out that when A is a Bott matrix, the action of $\Gamma(A)$ on $\mathbb{R}^n$ is free and $M(A)$ is a real Bott manifold. Conversely, any real Bott manifold can be obtained in this way. This is the viewpoint adopted by [3–5] in their study of real Bott manifolds.

But for an arbitrary $A \in \mathcal{A}(n)$, the action of $\Gamma(A)$ on $\mathbb{R}^n$ is not necessarily free. So it is not so easy to tell which closed manifolds $M(A)$ could represent in general. This is the significance of Theorem 1.2 above. By Theorem 1.2, the set of closed topological manifolds that can be realized by $M(A)$ are exactly all the generalized real Bott manifolds. This gives us another reason why generalized real Bott manifolds are naturally the "extension" of real Bott manifolds.

Remark 1.1. Any generalized real Bott manifold has a regular covering space of the form $S^{n_1} \times \cdots \times S^{n_m}$ (a product of spheres). So a generalized real Bott manifold M^n admits a flat Riemannian metric if and only if M^n is a real Bott manifold.

Unlike real Bott manifolds, the classification of generalized real Bott manifolds up to homeomorphism is by far less understood, primarily because the cohomological rigidity does not hold for generalized real Bott manifolds. In fact, it was shown in [7] that there exist two generalized real Bott manifolds with the same $\mathbb{Z}_2$-cohomology rings and homotopy groups, but they are not homeomorphic. So we need some new topological invariants to distinguish the homeomorphism types of generalized real Bott manifolds. Since we can now represent any generalized real Bott manifold by a binary matrix $A \in \mathcal{A}(n)$, it is interesting to know if we can classify generalized real Bott manifolds up to homeomorphism in the same way as [5] did for real Bott manifolds.

The paper is organized as following. In Section 2, we will construct a canonical $(\mathbb{Z}_2)^n$-action on $M(A)$ for any $A \in \mathcal{A}(n)$. So $M(A)$ can be constructed from an n-dimensional cube and a $(\mathbb{Z}_2)^n$-valued function on the facets of the cube, called *glue-back construction*. In Section 3, we will study the singularities that might occur in glue-back constructions, which will help us to determine when $M(A)$ is a closed manifold directly from the matrix A. In Section 4, we will see how to realize any generalized real Bott manifold by $M(A)$ and prove Theorems 1.1 and 1.2.

2. Glue-back construction

A manifold with corners W is called *nice* if any codimension-l face of W meets exactly l different *facets* (i.e. codimension-one faces) of W. Let $\mathcal{F}(W)$ denote the set of all facets of W. The reader is referred to [8] or [9] for a detailed introduction to manifolds with corners and related concepts.

Suppose W^n is an n-dimensional nice manifold with corners. Let μ be a $(\mathbb{Z}_2)^m$-valued function on all the facets of W^n i.e., $\mu : \mathcal{F}(W^n) \to (\mathbb{Z}_2)^m$ (m may be different from n). We call μ a $(\mathbb{Z}_2)^m$-*coloring* on W^n. For any proper face f of W^n, let G_f be the subgroup of $(\mathbb{Z}_2)^m$ generated by the following set:

$$\{\mu(F)\,;\, F \text{ is any facet of } W^n \text{ with } F \supseteq f\}.$$

For any $p \in W^n$, let $f(p)$ be the unique face of W^n that contains p in its relative interior. Then we can glue 2^m copies of W^n according to the μ by:

$$M(W^n, \mu) := W^n \times (\mathbb{Z}_2)^m / \sim, \tag{2.1}$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g - g' \in G_{f(p)}$ (see [2, 12]). We call $M(W^n, \mu)$ the *glue-back construction* from (W^n, μ). Moreover, there is a natural action of $(\mathbb{Z}_2)^m$ on $M(W^n, \mu)$ defined by:

$$g \cdot [(p, g_0)] = [(p, g_0 + g)], \ \forall\, p \in W^n, \ \forall\, g, g_0 \in (\mathbb{Z}_2)^m. \tag{2.2}$$

In this paper, we will always assume that $M(W^n, \mu)$ is equipped with the $(\mathbb{Z}_2)^m$-action defined by (2.2). The reader is referred to [12] for a more general form of glue-back construction. In addition, the function μ is called *non-degenerate at a face f* if $\mu(F_{i_1}), \dots, \mu(F_{i_k})$ are linearly independent over $\mathbb{Z}_2$ where $F_{i_1}, \dots, F_{i_k}$ are all the facets of W^n containing f. Moreover, μ is called *non-degenerate on W^n* if μ is non-degenerate at all faces of W^n. Otherwise μ is called *degenerate*.

The glue-back construction was introduced in [2] with the name *small cover* where W^n is a simple polytope P^n and μ is a non-degenerate $(\mathbb{Z}_2)^n$-coloring on P^n (called *characteristic function*). In this case, the natural $(\mathbb{Z}_2)^n$-action on $M(P^n, \mu)$ is *locally standard*, meaning that locally the $(\mathbb{Z}_2)^n$-action is equivariantly homeomorphic to a faithful linear representation of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$.

For a $(\mathbb{Z}_2)^m$-coloring μ on a simple polytope P^n, the non-degeneracy of μ on P^n is equivalent to the non-degeneracy of μ at all vertices of P^n. When μ is non-degenerate on P^n, the space $M(P^n, \mu)$ is always a closed manifold. But if μ is degenerate on P^n, the space $M(P^n, \mu)$ may or may not be a closed manifold (see Examples 3.1 and 3.2).

Next, let us study $M(A)$ from the viewpoint of glue-back construction. First of all, for any $A \in \mathcal{A}(n)$ it is easy to see that the following cube centered at the origin is a fundamental domain of the action of $\Gamma(A)$ on $\mathbb{R}^n$.

$$\mathcal{C}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid -\frac{1}{4} \le x_i \le \frac{1}{4}, \, i = 1, \dots, n\}.$$

For any $1 \le i \le n$, let $\mathbf{F}(i)$ and $\mathbf{F}(-i)$ be the facets of $\mathcal{C}^n$ which lie in the hyperplane $\{x_i = \frac{1}{4}\}$ and $\{x_i = -\frac{1}{4}\}$, respectively. Then $M(A) = \mathbb{R}^n/\Gamma(A)$ is obtained by gluing each $\mathbf{F}(i)$ to $\mathbf{F}(-i)$ by a map $\tau_i^A : \mathbf{F}(i) \to \mathbf{F}(-i)$ where

$$\begin{aligned} \tau_i^A(x_1, \dots, x_n) &= ((-1)^{A^i_1} x_1, \dots, (-1)^{A^i_{i-1}} x_{i-1}, -x_i, (-1)^{A^i_{i+1}} x_{i+1}, \dots, (-1)^{A^i_n} x_n) \\ &= ((-1)^{\tilde{A}^i_1} x_1, \dots, (-1)^{\tilde{A}^i_n} x_n), \quad \text{for } \forall\, (x_1, \dots, x_n) \in \mathbf{F}(i). \end{aligned} \tag{2.3}$$

So we can write $M(A) = \mathcal{C}^n / \langle x \sim \tau_i^A(x), \, \forall\, x \in \mathbf{F}(i), 1 \le i \le n \rangle$.

For any $1 \le i \le n$, let $h_i : \mathcal{C}^n \to \mathcal{C}^n$ be a homeomorphism defined by

$$h_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

Let $H = \langle h_1, \ldots, h_n \rangle \cong (\mathbb{Z}_2)^n$. Then H is a subgroup of the symmetry group of $\mathcal{C}^n$. If we consider H acting on $\mathcal{C}^n$, it is easy to see that the quotient space $\mathcal{C}^n/H$ can be identified with a smaller n-dimensional cube $\mathcal{C}_0^n \subset \mathcal{C}^n$ where

$$\mathcal{C}_0^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, 0 \le x_i \le \frac{1}{4},\ 1 \le \forall\, i \le n\}. \tag{2.4}$$

For each $1 \le j \le n$, let $\bar{F}_j$ be the facet of $\mathcal{C}_0^n$ which lies in the hyperplane $\{x_j = 0\}$. And let $\bar{F}_j^*$ be the opposite facet of $\bar{F}_j$ in $\mathcal{C}_0^n$. In addition, let $\{e_1, \ldots, e_n\}$ be a linear basis of $(\mathbb{Z}_2)^n$ and we can define a $(\mathbb{Z}_2)^n$-coloring λ_A on $\mathcal{C}_0^n$ by:

$$\lambda_A(\bar{F}_j) = e_j,\ 1 \le \forall\, j \le n, \tag{2.5}$$

$$\lambda_A(\bar{F}_j^*) = \sum_{j=1}^{n} \tilde{A}_k^j \cdot e_k,\ 1 \le \forall\, j \le n. \tag{2.6}$$

It is clear that each $\lambda_A(\bar{F}_j^*)$ is non-zero because all the diagonal elements of $\tilde{A}$ are 1. So the value of λ_A at any facet of $\mathcal{C}_0^n$ is non-zero. Note that $\lambda_A(\bar{F}_j^*)$ can be identified with the jth row vector of $\tilde{A}$. So for an arbitrary $A \in \mathcal{A}(n)$, λ_A may not be non-degenerate on $\mathcal{C}_0^n$. In addition, we observe that the action of each h_j on $\mathcal{C}^n$ commutes with any τ_i^A $(1 \le i \le n)$, so we get a well-defined action of H on $M(A)$.

Lemma 2.1. *For any $A \in \mathcal{A}(n)$, $M(A)$ is homeomorphic to $M(\mathcal{C}_0^n, \lambda_A)$ and the action of H on $M(A)$ can be identified with the natural $(\mathbb{Z}_2)^n$-action on $M(\mathcal{C}_0^n, \lambda_A)$.*

Proof. In the definition of $M(\mathcal{C}_0^n, \lambda_A)$, if we only glue the facets $\bar{F}_1, \ldots, \bar{F}_n$ in each $\mathcal{C}_0^n \times \{g\}, g \in (\mathbb{Z}_2)^n$ first according to the rule in (2.1), we will get a big cube which can be identified with the $\mathcal{C}^n$. Then we can think of the boundary of $\mathcal{C}^n$ being tessellated by those facets $\{\bar{F}_i^*\}$ of the 2^n copies of $\mathcal{C}_0^n$ which have not been glued. More specifically, for each $1 \le i \le n$, the facet $\mathbf{F}(i)$ of $\mathcal{C}^n$ is tessellated by the $\bar{F}_i^*$ in all copies of $\mathcal{C}_0^n$ in $\mathcal{C}_0^n \times G_i$, where G_i is the subgroup of $(\mathbb{Z}_2)^n$ generated by $\{e_1, \ldots, \widehat{e_i}, \ldots, e_n\}$, and $\mathbf{F}(-i)$ of $\mathcal{C}^n$ is tessellated by the $\bar{F}_i^*$ in all copies of $\mathcal{C}_0^n$ in $\mathcal{C}_0^n \times (e_i + G_i)$ (see Figure 1 for a two-dimensional example).

To further obtain $M(\mathcal{C}_0^n, \lambda_A)$, we should glue each $\bar{F}_i^* \times \{g\} \subset \mathbf{F}(i)$, $g \in G_i$ to $\bar{F}_i^* \times \{g + \lambda_A(\bar{F}_i^*)\} \subset \mathbf{F}(-i)$ by the map $(x_1, \ldots, x_n) \to ((-1)^{\tilde{A}_1^i} x_1, \ldots, (-1)^{\tilde{A}_n^i} x_n)$

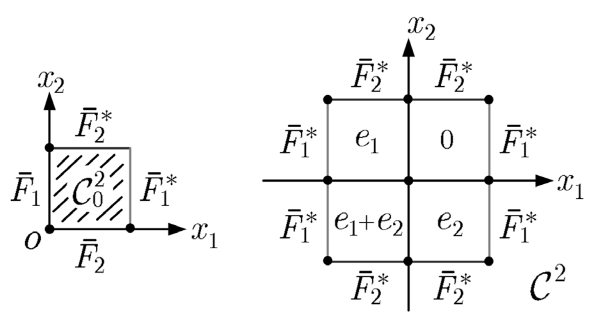

FIGURE 1. Seeing $M(A)$ as a glue-back construction.

which is exactly $\tau_i^A : \mathbf{F}(i) \to \mathbf{F}(-i)$ (see (2.3)). So $M(\mathcal{C}_0^n, \lambda_A)$ and $M(A)$ are the quotient space of $\mathcal{C}^n$ by the same gluing map, hence they are homeomorphic. It is easy to see that the action of H on $M(A)$ can be identified with the natural $(\mathbb{Z}_2)^n$-action on $M(\mathcal{C}_0^n, \lambda_A)$ defined in (2.2). $\square$

When $A \in \mathcal{B}(n)$, we can show that λ_A is a non-degenerate $(\mathbb{Z}_2)^n$-coloring on $\mathcal{C}_0^n$ as following. For each $1 \leq j \leq n$, let u_j be the vertex of $\mathcal{C}_0^n$ on the x_j-axis other than the origin. For any subset $\{j_1, \ldots, j_s\} \subset \{1, \ldots, n\}$, let $u_{j_1\cdots j_s}$ be the vertex of $\mathcal{C}_0^n$ so that all the facets of $\mathcal{C}_0^n$ containing $u_{j_1\cdots j_s}$ are

$$\{\bar{F}_{j_1}^*, \ldots, \bar{F}_{j_s}^*, \bar{F}_{l_1}, \ldots, \bar{F}_{l_{n-s}}\}, \text{ where } \{l_1, \ldots, l_{n-s}\} = \{1, \ldots, n\} \backslash \{j_1, \ldots, j_s\}.$$

By the definition of λ_A, the non-degeneracy of λ_A at a vertex $u_{j_1\cdots j_s}$ corresponds exactly to the non-degeneracy of the matrix $\tilde{A}_{j_1\cdots j_s}^{j_1\cdots j_s}$ (see (1.2)). But since A is a Bott matrix, any principal minor of $\tilde{A}$ is 1 (see [10]), so $\tilde{A}_{j_1\cdots j_s}^{j_1\cdots j_s}$ is non-degenerate.

3. Singularities in glue-back construction

By Lemma 2.1, we can identify the space $M(A)$ with the glue-back construction $M(\mathcal{C}_0^n, \lambda_A)$ for any $A \in \mathcal{A}(n)$. So to judge when $M(A)$ is a closed manifold, we need to understand when singular points might occur in a glue-back construction. Notice that when a $(\mathbb{Z}_2)^n$-coloring λ_A is degenerate on $\mathcal{C}_0^n$, $M(\mathcal{C}_0^n, \lambda_A)$ may not be a manifold. Let us see such an example first.

Example 3.1. Suppose $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathcal{A}(3)$, the $(\mathbb{Z}_2)^3$-coloring λ_A on $\mathcal{C}_0^3$ is:

$$\lambda_A(\bar{F}_1) = e_1,\ \lambda_A(\bar{F}_2) = e_2,\ \lambda_A(\bar{F}_3) = e_3;$$
$$\lambda_A(\bar{F}_1^*) = e_1 + e_3,\ \lambda_A(\bar{F}_2^*) = e_1 + e_2,\ \lambda_A(\bar{F}_3^*) = e_2 + e_3.$$

So the λ_A is degenerate at the vertex u_{123} of $\mathcal{C}_0^3$ (see Figure 2). In this case, $M(\mathcal{C}_0^3, \lambda_A)$ is not a manifold. In fact, the neighborhood of u_{123} in $M(\mathcal{C}_0^3, \lambda_A)$ is homeomorphic to a cone of $\mathbb{R}P^2$. This is because for any triangular section $\bigtriangledown$ of the cube near u_{123}, λ_A induces a $(\mathbb{Z}_2)^3$-coloring $\lambda_A^{\bigtriangledown}$ on the three edges of $\bigtriangledown$ (see the right picture of Figure 2). Obviously, $M(\bigtriangledown, \lambda_A^{\bigtriangledown}) \cong \mathbb{R}P^2$. So $M(\mathcal{C}_0^3, \lambda_A)$ is not a manifold.

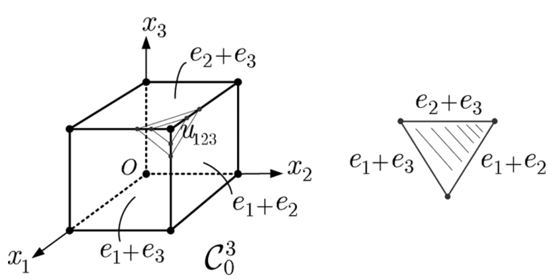

FIGURE 2. A singular point in $M(\mathcal{C}_0^3, \lambda_A)$.

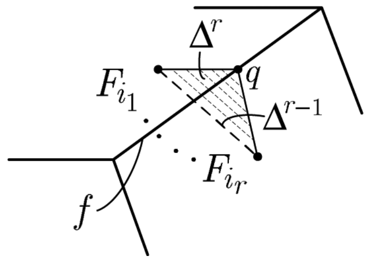

FIGURE 3. Singularities that might occur in $M(P^n, \mu)$.

The cause of the singularity in the above example can be formulated into a general condition on a $(\mathbb{Z}_2)^m$-coloring μ on a simple polytope P^n so that $M(P^n, \mu)$ has some singular points.

Lemma 3.1. *Suppose μ is a $(\mathbb{Z}_2)^m$-coloring on a simple polytope P^n. If there exists a vertex v_0 of P^n and a set of facets $F_{i_1}, \dots, F_{i_r}$ ($3 \le r \le n$) meeting v_0 such that $\mu(F_{i_1}), \dots, \mu(F_{i_{r-1}}) \in (\mathbb{Z}_2)^m$ are linearly independent over $\mathbb{Z}_2$ and $\mu(F_{i_r}) = \mu(F_{i_1}) + \cdots + \mu(F_{i_{r-1}})$, then the space $M(P^n, \mu)$ is not a manifold.*

Proof. In the relative interior of the codimension-r face $f = F_{i_1} \cap \cdots \cap F_{i_r}$, choose a point q and a $(r-1)$-dimensional simplex $\Delta^{r-1} \subset P^n$ near q so that

- Δ^{r-1} intersects ∂P^n transversely;
- all the facets of Δ^{r-1} are $\{F_{i_l} \cap \Delta^{r-1}, 1 \le l \le r\}$;
- all the line segments between q and the points of Δ^{r-1} form an r-dimensional simplex Δ^r with q as a vertex (see Figure 3).

Then we get a natural coloring ν of the facets of Δ^{r-1} induced from μ by

$$\nu(F_{i_l} \cap \Delta^{r-1}) = \mu(F_{i_l}),\ 1 \le l \le r.$$

By our assumption on $\mu(F_{i_1}), \dots, \mu(F_{i_r})$, it is clear that $M(\Delta^{r-1}, \nu)$ is homeomorphic to $\mathbb{R}P^{r-1}$. Similarly, any $(r-1)$-dimensional section of Δ^r that is parallel to Δ^{r-1} gives a $\mathbb{R}P^{r-1}$. So an open neighborhood of q in $M(P^n, \mu)$ is homeomorphic to $(-\varepsilon, \varepsilon)^{n-r} \times \mathrm{Cone}(\mathbb{R}P^{r-1})$. Since $r \ge 3$, the cone on $\mathbb{R}P^{r-1}$ is not homeomorphic to a ball. So the space $M(P^n, \mu)$ is not a manifold at q. $\square$

Notice that if there exists a facet F of P^n with $\mu(F) = 0$, the F in each copy of P^n in $P^n \times (\mathbb{Z}_2)^m$ will not be glued together in $M(P^n, \mu)$ (see (2.1)). Then $M(P^n, \mu)$ will have boundary. So if $M(P^n, \mu)$ is a closed manifold, μ must be non-zero on any facet of P^n. Combining this with Lemma 3.1, we get the following.

Lemma 3.2. *If μ is a $(\mathbb{Z}_2)^m$-coloring on a simple polytope P^n so that the space $M(P^n, \mu)$ is a closed manifold, then μ must satisfy the following conditions.*

(i) *For any facet F of P^n, $\mu(F) \neq 0 \in (\mathbb{Z}_2)^m$.*
(ii) *For any vertex $v = F_1 \cap \cdots \cap F_n$ of P^n, if $\mu(F_{i_1}), \dots, \mu(F_{i_s})$ are maximally linearly independent among $\mu(F_1), \dots, \mu(F_n)$ over $\mathbb{Z}_2$, then any $\mu(F_i)$ ($1 \le i \le n$) must coincide with one of the $\mu(F_{i_1}), \dots, \mu(F_{i_s})$.*

From the above lemma, we can easily derive the following.

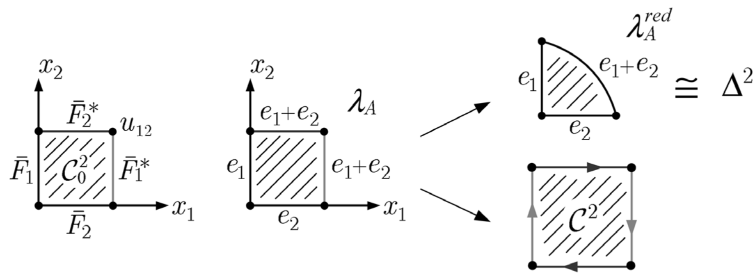

FIGURE 4. Two ways to see $M(\mathcal{C}_0^2, \lambda_A)$.

Corollary 3.1. *Suppose μ is a $(\mathbb{Z}_2)^m$-coloring on a simple polytope P^n so that the space $M(P^n, \mu)$ is a closed manifold. Then at a vertex $v = F_1 \cap \cdots \cap F_n$ of P^n, if $\mu(F_1), \ldots, \mu(F_n) \in (\mathbb{Z}_2)^m$ are all distinct, μ must be non-degenerate at v.*

However, it is possible that a $(\mathbb{Z}_2)^m$-coloring μ on a simple polytope P^n, even degenerate at some vertices, can still make $M(P^n, \mu)$ a closed manifold. Let us see such an example below.

Example 3.2. For the binary matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the $(\mathbb{Z}_2)^2$-coloring λ_A on $\mathcal{C}_0^2$ defined by (2.5) and (2.6) is:

$$\lambda_A(\bar{F}_1) = e_1, \ \lambda_A(\bar{F}_2) = e_2; \ \lambda_A(\bar{F}_1^*) = \lambda_A(\bar{F}_2^*) = e_1 + e_2.$$

So λ_A is degenerate at the vertex $u_{12} = \bar{F}_1^* \cap \bar{F}_2^*$. But it is easy to check that $M(\mathcal{C}_0^2, \lambda_A)$ is homeomorphic to $\mathbb{R}P^2$.

Another way to explain this example is: since $\lambda_A(\bar{F}_1^*) = \lambda_A(\bar{F}_2^*)$, we let the edge $\bar{F}_1^*$ merge with $\bar{F}_2^*$ to form a long edge. And then we get a non-degenerated $(\mathbb{Z}_2)^2$-coloring λ_A^{red} on a two-simple Δ^2 (see the right picture in Figure 4). The corresponding small cover $M(\Delta^2, \lambda_A^{red})$ is homeomorphic to $\mathbb{R}P^2$. Moreover, we have an equivariant homeomorphism from $M(\mathcal{C}_0^2, \lambda_A)$ to $M(\Delta^2, \lambda_A^{\mathrm{red}})$ which is induced by the merging of $\bar{F}_1^*$ with $\bar{F}_2^*$ on $\mathcal{C}_0^2$.

The idea of merging two neighboring edges into one edge in Example 3.2 can be generalized to the following setting.

Definition 3.1 (Smoothing a nice manifold with corners along codimension-two faces)**.** Suppose W^n is a nice manifold with corners and $\mathbf{f} = \{f_1, \ldots, f_k\}$ is a set of codimension-two faces of W^n. When we say *smoothing* W^n *along* $\mathbf{f}$, we mean that we forget $f_1, \ldots, f_k$ as well as all their faces from the manifold with corners structure of W^n. The stratified space we get is denoted by $W^n[\mathbf{f}]$. In other words, we think of $f_1, \ldots, f_k$ as well as all their faces as empty faces in $W^n[\mathbf{f}]$.

Geometrically, we can think of the smoothing of W^n along $\mathbf{f} = \{f_1, \ldots, f_k\}$ as a local deformation of W^n around $f_1, \ldots, f_k$ to make W^n "smooth" at those places, then removing $f_1, \ldots, f_k$ as well as all their faces from the stratification of ∂W^n. This process is similar to the *straightening of angles* introduced in the first chapter of [11].

Example 3.3. In Figure 5, we can see the local picture of smoothing a three-dimensional nice manifold with corners along a codimension-two face.

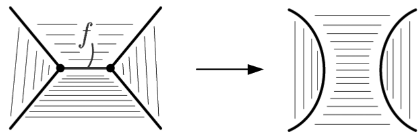

FIGURE 5. Smoothing a three-dimensional nice manifold with corners along a codimension-two face f.

Remark 3.1. Generally speaking, $W^n[\mathbf{f}]$ may not be a nice manifold with corners any more, although W^n is.

Suppose $f_i = F_i^{(1)} \cap F_i^{(2)}$ where $F_i^{(1)}, F_i^{(2)}$ are facets of W^n. Then $F_i^{(1)}$ and $F_i^{(2)}$ will merge into a big facet or part of a big facet in $W^n[\mathbf{f}]$. More generally, two facets F, F' of W^n will become part of a big facet in $W^n[\mathbf{f}]$ if and only if there exists a sequence $F = F_1, F_2, \dots, F_r = F'$ so that for each $1 \leq j \leq r-1$, $F_j \cap F_{j+1} \in \mathbf{f}$. Let $\mathcal{F}(W^n)$ and $\mathcal{F}(W^n[\mathbf{f}])$ denote the set of facets of W^n and $W^n[\mathbf{f}]$, respectively, then we have a natural map

$$\psi_{[\mathbf{f}]} : \mathcal{F}(W^n) \longrightarrow \mathcal{F}(W^n[\mathbf{f}]),$$

where for any facet F of W^n, $\psi_{[\mathbf{f}]}(F)$ is the facet of $W^n[\mathbf{f}]$ which contains F as a set. Obviously, $\psi_{[\mathbf{f}]}$ is surjective.

If μ is a $(\mathbb{Z}_2)^m$-coloring on W^n which satisfies:

$$\mu(F) = \mu(F') \text{ whenever } \psi_{[\mathbf{f}]}(F) = \psi_{[\mathbf{f}]}(F') \text{ for any facets } F, F' \text{ of } W^n,$$

we say that μ is *compatible with* $\psi_{[\mathbf{f}]}$. In this case, μ induces a $(\mathbb{Z}_2)^m$-coloring $\mu[\mathbf{f}]$ on the facets of $W^n[\mathbf{f}]$ by:

$$\mu[\mathbf{f}](\psi_{[\mathbf{f}]}(F)) := \mu(F) \text{ for any facet } F \text{ of } W^n. \tag{3.1}$$

We call $\mu[\mathbf{f}]$ the *induced* $(\mathbb{Z}_2)^m$*-coloring* from μ with respect to the smoothing. If we assume that $W^n[\mathbf{f}]$ is still a nice manifold with corners, then the glue-back construction $M(W^n[\mathbf{f}], \mu[\mathbf{f}])$ can be defined. Notice that the natural $(\mathbb{Z}_2)^m$-action on $M(W^n, \mu)$ and $M(W^n[\mathbf{f}], \mu[\mathbf{f}])$ can be identified through the smoothing of W^n. So we have the following.

Lemma 3.3. *Suppose μ is a $(\mathbb{Z}_2)^m$-coloring on W^n which is compatible with $\psi_{[\mathbf{f}]}$. If $W^n[\mathbf{f}]$ is still a nice manifold with corners, then there is an equivariant homeomorphism from $M(W^n, \mu)$ to $M(W^n[\mathbf{f}], \mu[\mathbf{f}])$.*

Next, let us investigate a special class of smoothings of an n-dimensional cube. Suppose $\{I_1, \dots, I_m\}$ is a *partition* of the set $[n] := \{1, \dots, n\}$, i.e., $I_1, \dots, I_m$ are pairwise disjoint non-empty subsets of $[n]$ with $I_1 \cup \cdots \cup I_m = [n]$. Let $\mathbf{f}_{I_1 \cdots I_m}$ be a set of codimension-two faces of $\mathcal{C}_0^n$ defined by:

$$\mathbf{f}_{I_1 \cdots I_m} := \{\bar{F}_l^* \cap \bar{F}_{l'}^* \,;\, l \text{ and } l' \text{ belong to the same } I_j \text{ for some } 1 \leq j \leq m\} \tag{3.2}$$

Notice that if I_i has only one element, it has no contribution to $\mathbf{f}_{I_1 \cdots I_m}$. Let $\mathcal{C}^n_{I_1 \cdots I_m} := \mathcal{C}_0^n[\mathbf{f}_{I_1 \cdots I_m}]$ be the smoothing of $\mathcal{C}_0^n$ along $\mathbf{f}_{I_1 \cdots I_m}$. So we have a map

$$\psi_{[\mathbf{f}_{I_1 \cdots I_m}]} : \mathcal{F}(\mathcal{C}_0^n) \to \mathcal{F}(\mathcal{C}^n_{I_1 \cdots I_m}).$$

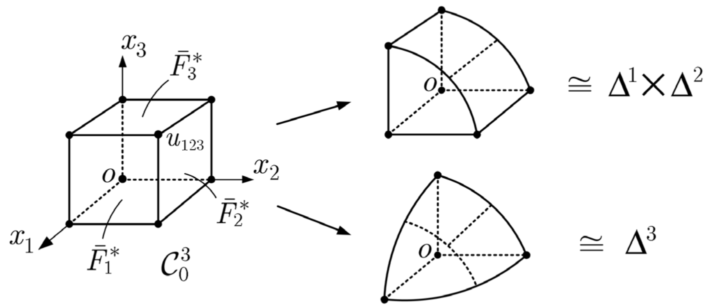

FIGURE 6. Two different smoothings of a cube

It is easy to see that for any $1 \leq j \leq n$, $\bar{F}_j$ does not merge with any other facets in $\mathcal{C}_0^n$, while all the facets in $\{\bar{F}_l^* \,;\, l \in I_i\}$ will merge into one big facet in $\mathcal{C}^n_{I_1 \cdots I_m}$. We denote all the facets of $\mathcal{C}^n_{I_1 \cdots I_m}$ by $\{\tilde{F}_1, \dots, \tilde{F}_n, \tilde{F}^*_{I_1}, \dots, \tilde{F}^*_{I_m}\}$ where:

- $\tilde{F}_j = \psi_{[\mathbf{f}_{I_1 \cdots I_m}]}(\bar{F}_j)$, $1 \leq j \leq n$.
- $\tilde{F}^*_{I_i} = \psi_{[\mathbf{f}_{I_1 \cdots I_m}]}(\bar{F}^*_l)$ for any $l \in I_i$, $1 \leq i \leq m$. In other words, $\tilde{F}^*_{I_i}$ is the merging of all the facets $\{\bar{F}_l^* \,;\, l \in I_i\}$.

It is easy to see that any face of $\mathcal{C}^n_{I_1 \cdots I_m}$ is homeomorphic to a ball.

Example 3.4. In Figure 6, we have two different smoothings of $\mathcal{C}_0^3$. By our notation, the upper one is $\mathcal{C}^3_{\{1\}\{2,3\}} \cong \Delta^1 \times \Delta^2$, and the lower one is $\mathcal{C}^3_{\{1,2,3\}} \cong \Delta^3$ where Δ^i denotes the standard i-dimensional simplex in $\mathbb{R}^i$.

Theorem 3.1. *For any partition $\{I_1, \dots, I_m\}$ of the set $[n] := \{1, \dots, n\}$, the $\mathcal{C}^n_{I_1 \cdots I_m}$ is homeomorphic to $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ as a manifold with corners, where $n_i = |I_i|$, $1 \leq i \leq m$ and $n_1 + \cdots + n_m = n$.*

Proof. We will borrow some notations in [1]. Let $\{v_0^i, \dots, v_{n_i}^i\}$ be the set of all vertices of Δ^{n_i}. Then each vertex of $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ can be uniquely written as a product of vertices from Δ^{n_i}'s. Hence all the vertices of $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ are:

$$\{\tilde{v}_{j_1 \dots j_m} = v^1_{j_1} \times \cdots \times v^m_{j_m} \mid 0 \leq j_i \leq n_i,\ i = 1, \dots, m\}.$$

Any facet of $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ is the product of a codimension-one face of some Δ^{n_i} and the remaining simplices. So all the facets of $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ are:

$$\mathcal{F}(\Delta^{n_1} \times \cdots \times \Delta^{n_m}) = \{F^i_{k_i} \mid 0 \leq k_i \leq n_i,\ i = 1, \dots, m\},$$

where $F^i_{k_i} = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f^i_{k_i} \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_m}$ and $f^i_{k_i}$ is the codimension-one face of the simplex Δ^{n_i} which is opposite to the vertex $v^i_{k_i}$. So there are total of $m + n$ facets in $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$. The n facets meeting the vertex $\tilde{v}_{j_1 \dots j_m}$ are:

$$\mathcal{F}(\Delta^{n_1} \times \cdots \times \Delta^{n_m}) - \{F^i_{j_i} \mid i = 1, \dots, m\}.$$

In particular, the n facets meeting the vertex $\tilde{v}_{0 \dots 0}$ are:

$$\mathcal{F}(\Delta^{n_1} \times \cdots \times \Delta^{n_m}) - \{F^i_0 \mid i = 1, \dots, m\} = \{F^1_1, \dots, F^1_{n_1}, \dots, F^m_1, \dots, F^m_{n_m}\}.$$

Next, we define a map Θ from the set of all facets of $\mathcal{C}^n_{I_1\cdots I_m}$ to the set of all facets of $\Delta^{n_1}\times\cdots\times\Delta^{n_m}$. Without loss of generality, we can assume that:

$$I_1=\{1,\dots,n_1\},\ I_2=\{n_1+1,\dots,n_1+n_2\},\dots \\ \dots,I_m=\{n_1+\cdots+n_{m-1}+1,\dots,n\}. \tag{3.3}$$

First, we define Θ maps the facets of $\mathcal{C}^n_{I_1\cdots I_m}$ meeting at the origin to the facets of $\Delta^{n_1}\times\cdots\times\Delta^{n_m}$ meeting at $\tilde{v}_{0\cdots 0}$ by:

$$\begin{gathered}\Theta(\tilde{F}_1)=F^1_1,\dots,\Theta(\tilde{F}_{n_1})=F^1_{n_1},\\ \Theta(\tilde{F}_{n_1+1})=F^2_1,\dots,\Theta(\tilde{F}_{n_1+n_2})=F^2_{n_2},\\ \cdots\qquad\cdots\qquad\cdots\\ \Theta(\tilde{F}_{n_1+\cdots+n_{m-1}+1})=F^m_1,\dots,\ \Theta(\tilde{F}_{n_1+\cdots+n_{m-1}+n_m})=F^m_{n_m},\end{gathered}$$

where $n_1+\cdots+n_{m-1}+n_m=n$. For the remaining facets of $\mathcal{C}^n_{I_1\cdots I_m}$, we define:

$$\Theta(\tilde{F}^*_{I_i})=F^i_0,\ 1\le i\le m.$$

By the definition of $\mathcal{C}^n_{I_1\cdots I_m}$, it is easy to check that Θ induces an isomorphism between the face lattices of $\mathcal{C}^n_{I_1\cdots I_m}$ and $\Delta^{n_1}\times\cdots\times\Delta^{n_m}$. In addition, since any face of $\mathcal{C}^n_{I_1\cdots I_m}$ is homeomorphic to a ball, so $\mathcal{C}^n_{I_1\cdots I_m}$ is homeomorphic to $\Delta^{n_1}\times\cdots\times\Delta^{n_m}$ as a manifold with corners. $\square$

By Lemma 3.3, if a $(\mathbb{Z}_2)^m$-coloring μ on $\mathcal{C}^n_0$ is compatible with $\psi_{[\mathbf{f}_{I_1\cdots I_m}]}$, then $M(\mathcal{C}^n_0,\mu)$ is homeomorphic to $M(\mathcal{C}^n_{I_1\cdots I_m},\mu[\mathbf{f}_{I_1\cdots I_m}])$. This result will be used in the next section.

4. Representing generalized real Bott manifolds by $M(A)$

Suppose M^n is an n-dimensional generalized real Bott manifold. In the rest of this paper, we will ignore the smooth structure on M^n and only treat it as a closed topological manifold. We can think of M^n as a small cover over $\Delta^{n_1}\times\cdots\times\Delta^{n_m}$ where $n_1+\cdots+n_m=n$. Let λ_{M^n} be the $(\mathbb{Z}_2)^n$-coloring on $\Delta^{n_1}\times\cdots\times\Delta^{n_m}$ determined by M^n. By Theorem 3.1, we can identify $\Delta^{n_1}\times\cdots\times\Delta^{n_m}$ with $\mathcal{C}^n_{I_1\cdots I_m}$, where $I_1,\dots,I_m$ are defined by (3.3). Then we think of λ_{M^n} as a $(\mathbb{Z}_2)^n$-coloring on $\mathcal{C}^n_{I_1\cdots I_m}$, and we have a homeomorphism

$$M^n\cong M(\mathcal{C}^n_{I_1\cdots I_m},\lambda_{M^n}).$$

By our discussion in Section 3, the facets of $\mathcal{C}^n_{I_1\cdots I_m}$ are $\{\tilde{F}_1,\dots,\tilde{F}_n,\tilde{F}^*_{I_1},\dots,\tilde{F}^*_{I_m}\}$. Since λ_{M^n} is non-degenerate, we can assume $\lambda_{M^n}(\tilde{F}_j)=e_j$ for each $1\le j\le n$, where $\{e_1,\dots,e_n\}$ is a linear basis of $(\mathbb{Z}_2)^n$. And we assume

$$\lambda_{M^n}(\tilde{F}^*_{I_i})=\mathbf{a}_i\in(\mathbb{Z}_2)^n,\ 1\le i\le m.$$

If we consider each $\mathbf{a}_i$ as a row vector, we have an $m\times n$ binary matrix $\mathbf{\Lambda}$.

$$\mathbf{\Lambda}=\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_m\end{pmatrix},\quad \text{where each } \mathbf{a}_i\in(\mathbb{Z}_2)^n.$$

We write $\mathbf{a}_i = (\mathbf{a}_i^1, \ldots, \mathbf{a}_i^j, \ldots, \mathbf{a}_i^m)$
$= ([a_{i1}^1, \ldots, a_{in_1}^1], \ldots, [a_{i1}^j, \ldots, a_{in_j}^j], \ldots, [a_{i1}^m, \ldots, a_{in_m}^m]),$

where $\mathbf{a}_i^j = [a_{i1}^j, \ldots, a_{in_j}^j] \in (\mathbb{Z}_2)^{n_j}$ for each $j = 1, \ldots, m$. Then we have:

$$\boldsymbol{\Lambda} = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^1 & \cdots & \mathbf{a}_1^m \\ \vdots & \cdots & \vdots \\ \mathbf{a}_m^1 & \cdots & \mathbf{a}_m^m \end{pmatrix} = \begin{pmatrix} a_{11}^1 & \cdots & a_{1n_1}^1 & \cdots & a_{11}^m & \cdots & a_{1n_m}^m \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1}^1 & \cdots & a_{mn_1}^1 & \cdots & a_{m1}^m & \cdots & a_{mn_m}^m \end{pmatrix}. \tag{4.1}$$

So the matrix $\boldsymbol{\Lambda}$ can be viewed as an $m \times m$ matrix whose entries in the jth column are vectors in $(\mathbb{Z}_2)^{n_j}$. Such a matrix $\boldsymbol{\Lambda}$ is called a *vector matrix* (see [1]). In addition, for given $1 \leq k_1 \leq n_1, \ldots, 1 \leq k_m \leq n_m$, let $\boldsymbol{\Lambda}_{k_1 \cdots k_m}$ be the $m \times m$ submatrix of $\boldsymbol{\Lambda}$ whose jth column is the k_jth column of the following $m \times n_j$ matrix.

$$\begin{pmatrix} \mathbf{a}_1^j \\ \vdots \\ \mathbf{a}_m^j \end{pmatrix} = \begin{pmatrix} a_{11}^j & \cdots & a_{1k_j}^j & \cdots & a_{1n_j}^j \\ \vdots & & \vdots & & \vdots \\ a_{m1}^j & \cdots & a_{mk_j}^j & \cdots & a_{mn_j}^j \end{pmatrix} \tag{4.2}$$

So we have: $\boldsymbol{\Lambda}_{k_1 \cdots k_m} = \begin{pmatrix} a_{1k_1}^1 & \cdots & a_{1k_m}^m \\ \vdots & & \vdots \\ a_{mk_1}^1 & \cdots & a_{mk_m}^m \end{pmatrix}.$

A *principal minor* of the $m \times n$ matrix $\boldsymbol{\Lambda}$ in (4.1) means a principal minor of an $m \times m$ matrix $\boldsymbol{\Lambda}_{k_1 \cdots k_m}$ for some $1 \leq k_1 \leq n_1, \ldots, 1 \leq k_m \leq n_m$. And the determinant of $\boldsymbol{\Lambda}_{k_1 \cdots k_m}$ itself is also considered as a principal minor of $\boldsymbol{\Lambda}$. This generalizes the usual definition of principal minors of a square matrix.

The lemma 3.2 in [1] says that the $(\mathbb{Z}_2)^n$-coloring λ_{M^n} is non-degenerate at all vertices of $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ is exactly equivalent to all principal minors of $\boldsymbol{\Lambda}$ being 1. This implies:

(c1) $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are distinct.
(c2) in the vector $\mathbf{a}_i = (\mathbf{a}_i^1, \ldots, \mathbf{a}_i^m)$, we must have $\mathbf{a}_i^i = (1, 1, \ldots, 1)$ for any $1 \leq i \leq m$.

Now, from the $\boldsymbol{\Lambda}$ in (4.1), we define an $n \times n$ binary matrix $\tilde{A}$ by: the first row to n_1-th row vectors of $\tilde{A}$ are all $\mathbf{a}_1$, the (n_1+1)th row to (n_1+n_2)th row vectors of $\tilde{A}$ are all $\mathbf{a}_2$, …, the $(n_1 + \cdots + n_{m-1} + 1)$th row to the nth row vectors of $\tilde{A}$ are all $\mathbf{a}_m$. Using the transpose of a matrix, we can write $\tilde{A}$ as:

$$\tilde{A} = (\overbrace{\mathbf{a}_1^t, \ldots, \mathbf{a}_1^t}^{n_1}, \overbrace{\mathbf{a}_2^t, \ldots, \mathbf{a}_2^t}^{n_2}, \ldots, \overbrace{\mathbf{a}_m^t, \ldots, \mathbf{a}_m^t}^{n_m})^t \tag{4.3}$$

Then the condition (c2) above implies that all the diagonal entries of $\tilde{A}$ are 1. For this matrix $\tilde{A}$, define

$$A = \tilde{A} - I_n. \tag{4.4}$$

So A is an $n \times n$ binary matrix with zero diagonal.

Theorem 4.1. *For any generalized real Bott manifold M^n, the matrix A defined by* (4.4) *satisfies that $M(A)$ is homeomorphic to M^n.*

Proof. By the definition of A in (4.4), the $(\mathbb{Z}_2)^n$-coloring λ_A on $\mathcal{C}_0^n$ satisfies:

$$\lambda_A(\bar{F}_l^*) = \mathbf{a}_i = \lambda_{M^n}(\tilde{F}_{I_i}^*),\ \forall\, l \in I_i.$$

So λ_A is compatible with the map $\psi_{[\mathbf{f}_{I_1\cdots I_m}]} : \mathcal{F}(\mathcal{C}_0^n) \to \mathcal{F}(\mathcal{C}_{I_1\cdots I_m}^n)$ where $\mathbf{f}_{I_1\cdots I_m}$ is defined by (3.2). Obviously, the induced $(\mathbb{Z}_2)^n$-coloring $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ on $\mathcal{C}_{I_1\cdots I_m}^n$ from λ_A coincides with λ_{M^n}. So we have

$$M(A) \overset{\text{Lem 2.1}}{\cong} M(\mathcal{C}_0^n, \lambda_A) \overset{\text{Lem3.3}}{\cong} M(\mathcal{C}_{I_1\cdots I_m}^n, \lambda_A[\mathbf{f}_{I_1\cdots I_m}]) = M(\mathcal{C}_{I_1\cdots I_m}^n, \lambda_{M^n}) \cong M^n.$$

Moreover, notice that the above homeomorphisms are all equivariant, so $M(A)$ with the action of H is equivariantly homeomorphic to M^n. □

Proof of Theorem 1.2. Since Theorem 4.1 has shown that any generalized real Bott manifold can be realized as $M(A)$ for some $A \in \mathcal{A}(n)$ up to homeomorphism, it remains to prove that if $M(A)$ is a closed manifold, it must be homeomorphic to a generalized real Bott manifold. By Lemma 2.1, we can identify $M(A)$ with $M(\mathcal{C}_0^n, \lambda_A)$. Now consider the $(\mathbb{Z}_2)^n$-coloring λ_A around the vertex $u_{12\cdots n}$ on $\mathcal{C}_0^n$. Since all the facets of $\mathcal{C}_0^n$ meeting $u_{12\cdots n}$ are $\bar{F}_1^*, \dots, \bar{F}_n^*$, so by re-indexing the coordinates of $\mathbb{R}^n$, we can assume that:

$$\begin{aligned}
\lambda_A(\bar{F}_1^*) = \cdots = \lambda_A(\bar{F}_{n_1}^*) &= \mathbf{a}_1,\\
\lambda_A(\bar{F}_{n_1+1}^*) = \cdots = \lambda_A(\bar{F}_{n_1+n_2}^*) &= \mathbf{a}_2,\\
\cdots \qquad \cdots \qquad \cdots&\\
\lambda_A(\bar{F}_{n_1+\cdots+n_{m-1}}^*) = \cdots = \lambda_A(\bar{F}_{n_1+\cdots+n_{m-1}+n_m}^*) &= \mathbf{a}_m,
\end{aligned}$$

where $n_1+\cdots+n_m = n$ and $\mathbf{a}_1, \dots, \mathbf{a}_m$ are distinct non-zero elements of $(\mathbb{Z}_2)^n$. Then $\tilde{A} = A + I_n$ is in the form (4.3). Now, let $I_1, \dots, I_m$ be the partition of $\{1, \dots, n\}$ defined by (3.3) and $\mathbf{f}_{I_1\cdots I_m}$ be the set of codimension-two faces of $\mathcal{C}^n$ defined by (3.2). Obviously, the $(\mathbb{Z}_2)^n$-coloring λ_A is compatible with the smoothing of $\mathcal{C}_0^n$ along $\mathbf{f}_{I_1\cdots I_m}$. So by Lemma 3.3, $M(\mathcal{C}_0^n, \lambda_A) \cong M(\mathcal{C}_{I_1\cdots I_m}^n, \lambda_A[\mathbf{f}_{I_1\cdots I_m}])$, where $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ is the induced $(\mathbb{Z}_2)^n$-coloring on the facets of $\mathcal{C}_{I_1\cdots I_m}^n$ from λ_A. By definition,

$$\lambda_A[\mathbf{f}_{I_1\cdots I_m}](\tilde{F}_{I_i}^*) = \mathbf{a}_i,\ 1 \le i \le m \text{ (see (3.1))}.$$

$$\text{Let } \mathbf{\Lambda}^A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}, \quad \text{where each } \mathbf{a}_i \in (\mathbb{Z}_2)^n. \tag{4.5}$$

By Theorem 3.1, $\mathcal{C}_{I_1\cdots I_m}^n \cong \Delta^{n_1} \times \cdots \times \Delta^{n_m}$. So to prove $M(A)$ is homeomorphic to a generalized real Bott manifolds, it suffices to show that the $(\mathbb{Z}_2)^n$-coloring $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ is non-degenerate at all vertices of $\mathcal{C}_{I_1\cdots I_m}^n$. Recall that for any vertex $u_{j_1\cdots j_s}$ of $\mathcal{C}_0^n$, all the facets of $\mathcal{C}_0^n$ meeting at $u_{j_1\cdots j_s}$ are:

$$\bar{F}_{j_1}^*, \dots, \bar{F}_{j_s}^*, \bar{F}_{l_1}, \dots, \bar{F}_{l_{n-s}},\ \text{where } \{l_1, \dots, l_{n-s}\} = \{1, \dots, n\} \backslash \{j_1, \dots, j_s\}$$

A critical observation here is that: $\lambda_A(\bar{F}_j^*) \neq \lambda_A(\bar{F}_l)$ for $\forall\, j \in \{j_1, \dots, j_s\}$ and $\forall\, l \in \{l_1, \dots, l_{n-s}\}$ (see the definition of λ_A in (2.5) and (2.6)). So when we smooth $\mathcal{C}_0^n$

into $\mathcal{C}^n_{I_1\cdots I_m}$, for any vertex $\tilde{v}$ of $\mathcal{C}^n_{I_1\cdots I_m}$, the value of $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ on all the facets meeting at $\tilde{v}$ are distinct. Then by our assumption that $M(\mathcal{C}^n_{I_1\cdots I_m}, \lambda_A[\mathbf{f}_{I_1\cdots I_m}]) \cong M(\mathcal{C}^n_0, \lambda_A) \cong M(A)$ is a closed manifold, Corollary 3.1 asserts that $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ must be non-degenerate at each $\tilde{v}$. So $M(A)$ is homeomorphic to a generalized real Bott manifold. □

Proof of Theorem 1.1. By the argument in Theorem 1.2, any $A \in \mathcal{A}(n)$ determines a partition $I_1, \dots, I_m$ of $\{1, \dots, n\}$ and a $(\mathbb{Z}_2)^n$-coloring $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ on $\mathcal{C}^n_{I_1\cdots I_m} \cong \Delta^{n_1} \times \cdots \times \Delta^{n_m}$. Moreover at any vertex $\tilde{v}$ of $\mathcal{C}^n_{I_1\cdots I_m}$, the value of $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ on all the facets meeting at $\tilde{v}$ are distinct. So Corollary 3.1 implies that $M(A) \cong M(\mathcal{C}^n_{I_1\cdots I_m}, \lambda_A[\mathbf{f}_{I_1\cdots I_m}])$ is a closed manifold if and only if $\lambda_A[\mathbf{f}_{I_1\cdots I_m}]$ is non-degenerate at all vertices of $\mathcal{C}^n_{I_1\cdots I_m}$, which is also equivalent to saying that all the principal minors of the $m \times n$ matrix $\mathbf{\Lambda}^A$ (see (4.5)) being 1.

Now, let us compare the $n \times n$ matrix $\tilde{A}$ in the form (4.3) and the matrix $\mathbf{\Lambda}^A$. Observe that for any given $1 \le k_1 \le n_1, \dots, 1 \le k_m \le n_m$, the $m \times m$ submatrix $\mathbf{\Lambda}^A_{k_1\cdots k_m}$ of $\mathbf{\Lambda}^A$ (see (4.2)) is exactly the submatrix $\tilde{A}^{j_{k_1}\cdots j_{k_m}}_{j_{k_1}\cdots j_{k_m}}$ of $\tilde{A}$ where

$$j_{k_1} = k_1,\ j_{k_2} = n_1 + k_1, \dots, j_{k_m} = n_1 + \cdots + n_{m-1} + k_m. \tag{4.6}$$

Next, we assume any principal minor of $\mathbf{\Lambda}^A$ is 1 and see whether it implies $\tilde{A}$ should satisfy (a) and (b). Indeed, under this assumption it is clear that $\mathbf{a}_1, \dots, \mathbf{a}_m$ are linearly independent over $\mathbb{Z}_2$, so $\tilde{A}$ must satisfy (b). Moreover, for any $1 \le j_1 < \cdots < j_s \le n$, if the row vectors $\tilde{A}^{j_1}, \dots, \tilde{A}^{j_s}$ are pairwise distinct, then by the above observation, the matrix $\tilde{A}^{j_1\cdots j_s}_{j_1\cdots j_s}$ can be realized as a submatrix of a submatrix $\mathbf{\Lambda}^A_{k_1\cdots k_s}$ of $\mathbf{\Lambda}^A$ for some $1 \le k_1 \le n_1, \dots, 1 \le k_m \le n_m$. Then by our assumption $\det(\tilde{A}^{j_1\cdots j_s}_{j_1\cdots j_s}) = 1$, $\mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1\cdots j_s}) = \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1\cdots j_s}_{j_1\cdots j_s}) = s$. Otherwise, let $\tilde{A}^{j_{i_1}}, \dots, \tilde{A}^{j_{i_r}}$ be all the different vectors among $\tilde{A}^{j_1}, \dots, \tilde{A}^{j_s}$, $1 \le r \le s$. Then we have: $\mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1\cdots j_s}) = \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_{i_1}\cdots j_{i_r}}) = \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_{i_1}\cdots j_{i_r}}_{j_{i_1}\cdots j_{i_r}}) = r$. On the other hand, since $\tilde{A}^{j_{i_1}\cdots j_{i_r}}_{j_{i_1}\cdots j_{i_r}}$ is submatrix of $\tilde{A}^{j_1\cdots j_s}_{j_1\cdots j_s}$ and $\tilde{A}^{j_1\cdots j_s}_{j_1\cdots j_s}$ is submatrix of $\tilde{A}^{j_1\cdots j_s}$, so $\mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_{i_1}\cdots j_{i_r}}_{j_{i_1}\cdots j_{i_r}}) \le \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1\cdots j_s}_{j_1\cdots j_s}) \le \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1\cdots j_s})$. Hence $\mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_1\cdots j_s}_{j_1\cdots j_s}) = r$ too. So $\tilde{A}$ satisfies (a).

Finally, let us assume that $\tilde{A}$ satisfies (a) and (b) and see whether it will force any principal minor of $\mathbf{\Lambda}^A$ to be 1. By the above observation, we can identify any submatrix $\mathbf{\Lambda}^A_{k_1\cdots k_m}$ of $\mathbf{\Lambda}^A$ with the submatrix $\tilde{A}^{j_{k_1}\cdots j_{k_m}}_{j_{k_1}\cdots j_{k_m}}$ of $\tilde{A}$ defined by (4.6). Notice that the row vectors $\tilde{A}^{j_{k_1}}, \dots, \tilde{A}^{j_{k_m}}$ of $\tilde{A}$ in this case are all distinct. So by the property (b), $\tilde{A}^{j_{k_1}}, \dots, \tilde{A}^{j_{k_m}}$ are linearly independent over $\mathbb{Z}_2$. Then property (a) of $\tilde{A}$ implies that $\mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_{k_1}\cdots j_{k_m}}_{j_{k_1}\cdots j_{k_m}}) = \mathrm{rank}_{\mathbb{Z}_2}(\tilde{A}^{j_{k_1}\cdots j_{k_m}}) = m$, so $\det(\tilde{A}^{j_{k_1}\cdots j_{k_m}}_{j_{k_1}\cdots j_{k_m}}) = 1$. Similarly, we can show that any principal minor of $\mathbf{\Lambda}^A_{k_1\cdots k_m}$ must be 1. So we are done. □

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