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PERIODIC SOLUTIONS OF ABREU'S EQUATION

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Abstract. We solve Abreu's equation with periodic right hand side, in any dimension. This can be interpreted as prescribing the scalar curvature of a torus invariant metric on an Abelian variety.

1. Introduction

Given a smooth periodic function $A : \mathbb{R}^n \to \mathbb{R}$, we would like to find a convex function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = A, \tag{1.1}$$

where u^{ij} is the inverse of the Hessian of u. By an affine transformation we can assume that the fundamental domain for the periodicity is $\Omega = [0,1]^n$. Our main result is the following.

Theorem 1.1. *For any smooth periodic A with mean 0, we can find a smooth periodic function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that*

$$u(x) = \tfrac{1}{2}|x|^2 + \phi(x) \tag{1.2}$$

is a convex solution of equation (1.1). Moreover ϕ is unique up to adding a constant.

The proof shows that the result is true with $\frac{1}{2}|x|^2$ replaced with any strictly convex smooth function f for which f_{ij} is periodic. That A must have mean 0 can be seen by a simple integration by parts.

Equation (1.1) was introduced by Abreu [A] in the study of the scalar curvature of toric varieties. In that case the domain of u is a convex polytope in $\mathbb{R}^n$, and u is required to have prescribed boundary behaviour near the boundary of the polytope, obtained by Guillemin [G]. This equation has been studied extensively by Donaldson [D1, D2, D3] finally solving it in the two-dimensional case, when A is a constant in [D4]. The main difficulty is dealing with the boundary of the polytope, and in our case this does not arise. As a result our problem is significantly simpler and the ideas in [D2] and also Trudinger–Wang [TW2] can be used to solve the equation in any dimension. Other recent works on Abreu's equation include the works of Chen *et al.* [CLS1, CLS2] and Zhou [Z].

We solve the equation using the continuity method, which relies on finding a priori estimates for the solution u. We first set up the continuity method, and prove the uniqueness of the solution in Section 2. Then we need to obtain estimates for the determinant of the Hessian $\det(u_{ij})$ from above and below, which can be done by a

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maximum principle argument adapted from [TW2]. We will show this in Section 3. Then in Section 4, following [TW2] we use the theorem of Caffarelli–Gutierrez [CG] to obtain Hölder estimates for $\det(u_{ij})$, from which higher order estimates follow from Caffarelli [C] and the Schauder estimates. This will complete the proof of Theorem 1.1. An important step in this method is to obtain the strict convexity of the solution u in the sense that we need to control at least one section of the convex function u. For our problem this is quite easy to show in any dimension even independently of the equation, but in [D2, TW2] this is the step that restricts the results to $n = 2$.

We mentioned that Abreu's equation on a convex polytope with suitable boundary data is related to prescribing the scalar curvature on toric varieties. Similarly periodic solutions are related to prescribing the scalar curvature on Abelian varieties. We will explain this in Section 5.

2. The continuity method and uniqueness

Given any smooth periodic A with average 0, we use the continuity method to solve the family of equations

$$(u^{ij})_{ij} = tA, \tag{2.1}$$

where $u(x) = \frac{1}{2}|x|^2 + \phi(x)$ for some periodic function ϕ and we are summing over the repeated indices.

Let us write $S \subset [0, 1]$ for the set of parameters t for which we can solve equation (2.1). Clearly $0 \in S$, since then $\phi = 0$ is a solution, thus S is nonempty. To show S is open, we use the implicit function theorem.

Lemma 2.1. *The set S is open.*

Proof. Since we are only interested in periodic ϕ it is natural to work on the torus T^n. Let us write $C_0^{k,\alpha}(T^n)$ for the space of $C^{k,\alpha}$ functions on T^n with average 0. Define the map

$$\begin{aligned} \mathcal{F} : C_0^{4,\alpha}(T^n) &\to C_0^{0,\alpha}(T^n), \\ \phi &\mapsto (u^{ij})_{ij}, \end{aligned}$$

where $u(x) = \frac{1}{2}|x|^2 + \phi(x)$ as before. Note that if ϕ is periodic then so is $(u^{ij})_{ij}$ and also integration by parts shows that $(u^{ij})_{ij}$ has average 0.

Consider the linearization $\mathcal{L}$ of $\mathcal{F}$ at ϕ which is given by

$$\mathcal{L}(\psi) = (u^{ia}\psi_{ab}u^{bj})_{ij}.$$

Then $\mathcal{L}$ gives a linear elliptic operator

$$\mathcal{L} : C_0^{4,\alpha}(T^n) \to C_0^{0,\alpha}(T^n).$$

The operator is self-adjoint and its kernel is trivial so it is an isomorphism. By the implicit function theorem, this means that if equation (2.1) has a smooth solution for $t = t_0$, then we can solve the equation for all nearby t too. This nearby solution is a priori in $C^{4,\alpha}$, but by elliptic regularity it is actually smooth. Hence S is open. □

The fact that S is closed follows from the a priori estimates for the solution given in Lemma 4.1, which will complete the existence part of Theorem 1.1. For now we will prove the uniqueness of the solution.

Lemma 2.2. *Suppose that ϕ_0 and ϕ_1 are periodic and that $u = \frac{1}{2}|x|^2 + \phi_0$ and $v = \frac{1}{2}|x|^2 + \phi_1$ are convex functions satisfying*

$$(u^{ij})_{ij} = (v^{ij})_{ij} = A.$$

Then $\phi_0 = \phi_1 + c$ for some constant c.

Proof. Consider the functional

$$\mathcal{F}_A(\phi) = \int_\Omega -\log\det(u_{ij}) + A\phi\, d\mu,$$

where Ω is the fundamental domain, $u = \frac{1}{2}|x|^2 + \phi$ as usual and $d\mu$ is the Lebesgue measure. This is analogous to the functional used by Donaldson [D1] which in turn is based on the Mabuchi functional [M].

This functional is convex along the linear path $\phi_t = (1-t)\phi_0 + t\phi_1$, in fact writing $\psi = \phi_1 - \phi_0$ we have

$$\frac{d^2}{dt^2}\mathcal{F}_A(\phi_t) = \int_\Omega (u_t)^{ia}\psi_{ab}(u_t)^{bj}\psi_{ij}\, d\mu \geqslant 0, \tag{2.2}$$

where $u_t = \frac{1}{2}|x|^2 + \phi_t$.

At the same time

$$\frac{d}{dt}\mathcal{F}_A(\phi_t) = \int_\Omega -(u_t)^{ij}\psi_{ij} + A\psi\, d\mu = \int_\Omega \left[A - (u_t^{ij})_{ij}\right]\psi\, d\mu,$$

where we can integrate by parts without a boundary term because of the periodicity. By our assumptions, this derivative vanishes for $t = 0$ and 1, so by the convexity (2.2), the functional $\mathcal{F}_A(\phi_t)$ is constant for $t \in [0,1]$. It then follows from (2.2) that

$$\int_\Omega (u_0)^{ia}\psi_{ab}(u_0)^{bj}\psi_{ij}\, d\mu = 0,$$

and since u_0 is convex and ψ is periodic, this implies that ψ is a constant. □

3. Bounds for the determinant

In this section we will prove the following.

Lemma 3.1. *Suppose that $\phi : \mathbb{R}^n \to \mathbb{R}$ is a periodic smooth function and $u(x) = \frac{1}{2}|x|^2 + \phi(x)$ satisfies Abreu's* equation (1.1). Then

$$c_1 < \det(u_{ij}) < c_2,$$

where $c_1, c_2 > 0$ are constants only depending on A.

First we give a simple C^0 and C^1 bound for ϕ.

Lemma 3.2. *If ϕ is periodic, $\phi(0) = 0$ and $u(x) = \frac{1}{2}|x|^2 + \phi(x)$ is convex, then*

$$|\phi|, |\nabla\phi| < C$$

for some constant C.

Proof. Since ϕ is periodic, it is enough to consider the fundamental domain $\Omega = [0,1]^n$. Since $\frac{1}{2}|x|^2 + \phi$ is convex, we must have the lower bound $D^2\phi > -\mathrm{Id}$ on the Hessian of ϕ. If $\sup_\Omega \phi = \phi(x_{\max})$ then $\nabla\phi(x_{\max}) = 0$, so integrating the bound on the second derivative along straight lines, we get for every y that

$$\phi(y) > \phi(x_{\max}) - |y - x_{\max}|^2.$$

Using that ϕ is periodic and $\phi(0) = 0$, it follows from this that $|\phi| < C$ for some uniform C. It follows that we have a uniform bound on u on the set $[-1,2]^n$, from which the convexity of u implies a bound on $|\nabla u|$ on the smaller set $[0,1]^n$. The bound $|\nabla\phi| < C$ follows from this, using again that ϕ is periodic. □

Now we turn to the proof of Lemma 3.1, using ideas from Trudinger–Wang [TW2]. While it is not strictly necessary, it is convenient to study instead the Legendre transform v of the convex function u. The dual coordinate y is defined by $y = \nabla u$, and then v is defined by the equation

$$v(y) + u(x) = y \cdot x. \tag{3.1}$$

The Legendre transform v is of the form

$$v(y) = \tfrac{1}{2}|y|^2 + \psi(y), \tag{3.2}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is periodic. To see that ψ defined by (3.2) is periodic, note that (3.1) implies

$$\psi(y) + \phi(x) = -\tfrac{1}{2}|x - y|^2. \tag{3.3}$$

Now if we write ξ for one of the periods of ϕ, then the relation $y(x) = x + \nabla\phi(x)$ implies that $y(x+\xi) = y + \xi$. Using this in (3.3) we get

$$\psi(y+\xi) + \phi(x+\xi) = -\tfrac{1}{2}|x - y|^2.$$

Now since $\phi(x+\xi) = \phi(x)$, comparing this with (3.3) we get $\psi(y+\xi) = \psi(y)$. Therefore ψ is also periodic with the same periods as ϕ. In addition, by Lemma 3.2 we have bounds $|\psi|, |\nabla\psi| < C$.

Moreover, v satisfies the equation

$$\begin{aligned} v^{ij} L_{ij} &= \tilde{A}, \\ L &= \log\det(v_{ab}), \end{aligned} \tag{3.4}$$

where $\tilde{A}(y) = A(x)$, so certainly $\sup|\tilde{A}| = \sup|A|$. Since $\det(v_{ab})(y) = \det(u_{ab})^{-1}(x)$, to prove Lemma 3.1 it is enough to bound $\det(v_{ab})$ from above and below by a constant depending on $\sup|\tilde{A}|$.

3.1. Upper bound for $\det(u_{ij})$.

Consider the function

$$f(y) = L + \tfrac{1}{2}|y|^2 + 2\psi,$$

where $L = \log\det(v_{ab})$. Since v_{ij} and ψ are both periodic, the global minimum of f must be achieved at some point $p \in [-1,1]^n$. We can rewrite f as

$$f(y) = L - \tfrac{1}{2}|y|^2 + 2v.$$

At the point p we have

$$0 \leqslant v^{ij} f_{ij} = \tilde{A} - v^{ii} + 2n,$$

So

$$v^{ii}(p) \leqslant \sup|\tilde{A}| + 2n.$$

By the arithmetic–geometric mean inequality, we have

$$\det(v^{ij})(p) \leqslant \left(\frac{\mathrm{Tr}(v^{ij})(p)}{n}\right)^n \leqslant c,$$

where $c = (\frac{1}{n}\sup_\Omega|\tilde{A}| + 2)^n > 0$ is a constant. Since p is the minimum of f, for any $y \in \Omega = [0,1]^n$ we have

$$\log\det(v_{ij})(y) \geqslant \log\det(v_{ij})(p) + \tfrac{1}{2}|p|^2 + 2\psi(p) - \tfrac{1}{2}|y|^2 - 2\psi(y) > c',$$

for some constant c'. The last inequality is given since $\det(v_{ij})(p) \geqslant \frac{1}{c}$ and the rest can be bounded by Lemma 3.2 in the region Ω. Thus we have $\det(v_{ij})(x) > e^{c'} > 0$. By Legendre duality, we have

$$\det(u_{ij}) < e^{-c'},$$

where c' is some constant only depending on $\sup|\tilde{A}| = \sup|A|$.

3.2. Lower bound for $\det(u_{ij})$. Now consider the function

$$g(y) = -L - \beta|\nabla v|^2 + v = -L + \psi - \beta|\nabla v|^2 + \tfrac{1}{2}|y|^2.$$

Choose β sufficiently small such that on $\mathbb{R}^n$, we have

$$\beta|\nabla v|^2 \leqslant \tfrac{1}{4}|y|^2 + 1.$$

Such β exist since $\nabla v = y + \nabla\psi$ and $\sup|\nabla\psi| < c$. This implies that

$$\tfrac{1}{4}|y|^2 - 1 \leqslant -\beta|\nabla v|^2 + \tfrac{1}{2}|y|^2 \leqslant \tfrac{1}{2}|y|^2,$$

so that

$$\sup_{B(2)}\left(-\beta|\nabla v|^2 + \frac{1}{2}|y|^2\right) < \inf_{\mathbb{R}^n\setminus B(4)}\left(-\beta|\nabla v|^2 + \frac{1}{2}|y|^2\right), \tag{3.5}$$

where $B(R)$ denotes the ball of radius R around the origin. Since $-L+\psi$ is periodic and the fundamental domain $\Omega \subset B(2)$, this inequality shows that the global minimum of $g(y)$ is achieved at a point $q \in B(4)$. Indeed if $y \notin B(4)$ and $y' \in \Omega$ denotes the corresponding point in the fundamental domain, then the inequality (3.5) implies that $g(y') < g(y)$, so the minimum cannot be outside of $B(4)$.

At q, we have

$$0 = g_i = -L_i - 2\beta v_k v_{ki} + v_i \tag{3.6}$$

and

$$0 \leqslant v^{ij}g_{ij} = -\tilde{A} - 2\beta v^{ij}(v_{ki}v_{kj} + v_k v_{ijk}) + n = -\tilde{A} - 2\beta v_{kk} - 2\beta v_k L_k + n,$$

where we used that $v^{ij}v_{ijk} = L_k$. Using equation (3.6) we then get that at the point q,

$$\begin{aligned} 0 &\leqslant -\tilde{A} - 2\beta v_{kk} - 2\beta v_k L_k + n \\ &= -\tilde{A} + n - 2\beta v_{kk} - 2\beta|\nabla v|^2 + 4\beta^2 v_k v_i v_{ik} \\ &\leqslant \sup|\tilde{A}| + n - 2\beta v_{kk} + 4\beta^2|\nabla v|^2 v_{kk}. \end{aligned}$$

Because of Lemma 3.2, we already have a bound for $|\nabla v|$ on $B(4)$. So we can choose β sufficiently small such that $4\beta^2|\nabla v|^2(q) \leqslant \beta$, which implies

$$\beta v_{kk}(q) \leqslant \sup|\tilde{A}| + n.$$

Thus we have at q, by arithmetic–geometric mean inequality again,

$$\log\det(v_{ij})(q) \leq c$$

for a constant c depending only on $\sup|\tilde{A}| = \sup|A|$. Since q is the minimum of g and we can bound $|\nabla v|$ and $|v|$ uniformly on $\Omega = [0,1]^n$ by Lemma 3.2, we have for any $y \in \Omega$

$$L(y) \leqslant L(q) + \beta\left(|\nabla v|^2(q) - |\nabla v|^2(y)\right) + v(y) - v(q) < c'',$$

where c'' is a finite constant only depending on $\sup|A|$. Thus by Legendre duality again, we have

$$\det(u_{ij}) = \det(v_{ij})^{-1} > e^{-c''}$$

on Ω, but $\det(u_{ij})$ is periodic, so this lower bound holds everywhere. This completes the proof of Lemma 3.1.

4. Higher order estimates

Given the determinant bound of Lemma 3.1, we can obtain estimates for all derivatives of the solution in terms of the "modulus of convexity", using the results of Caffarelli–Gutierrez [CG] on solutions of the linearized Monge–Ampère equation and Caffarelli [C] on the Monge–Ampère equation. This is roughly identical to the discussion in Section 5.1 of Donaldson [D2], which in turn is based on the work of Trudinger and Wang [TW1, TW2].

Lemma 4.1. *Suppose that $\phi : \mathbb{R}^n \to \mathbb{R}$ is a periodic smooth function and $u(x) = \frac{1}{2}|x|^2 + \phi(x)$ satisfies Abreu's* equation (1.1). Then there exist constants c_k depending on A such that

$$c_1 I < (u_{ij}) < c_2 I,$$
$$|\nabla^k \phi| < c_k.$$

Proof. To prove this, we rewrite the equation in the form

$$\begin{aligned} U^{ij} w_{ij} &= A, \\ w &= (\det(u_{ab}))^{-1}, \end{aligned} \tag{4.1}$$

where U^{ij} is the cofactor matrix of the Hessian u_{ij}. This form follows from the original form of the equation because $(U^{ij})_i = 0$.

Now define the section $S(h) = \{x \in \mathbb{R}^n \mid u(x) < h\}$. There exists a linear transformation T which normalizes this section, which means

$$B(0, \alpha_n) \subset T(S(h)) \subset B(0, 1), \tag{4.2}$$

where $\alpha_n > 0$ is a constant depending on n only. The bound on $\det(u_{ab})$ from Lemma 3.1 and the results in [CG] imply that we have a Hölder bound

$$\|w\|_{C^\alpha(S(h/2))} \leqslant C_0,$$

but the constants α, C_0 depend on the norms of T and T^{-1} (in addition to depending on $\sup w, \sup|A|$, but those are already controlled). In [CG] this is shown for the homogeneous equation where $A = 0$, but the argument can be extended to the non-homogeneous case. Namely, the Harnack inequality (Theorem 5 in [CG]) extends as explained in Trudinger–Wang [TW3] after which the arguments are identical.

So we simply need to control T and T^{-1}. Note that from Lemma 3.2 we have

$$\tfrac{1}{2}|x|^2 - C \leqslant u(x) \leqslant \tfrac{1}{2}|x|^2 + C$$

for some C, which implies that if $u(x) = h$, then

$$\sqrt{2(h-C)} \leqslant |x| \leqslant \sqrt{2(h+C)}.$$

In particular, for sufficiently large h, we have

$$B(\sqrt{h}) \subset S(h) \subset B(\sqrt{3h}). \tag{4.3}$$

Let us choose h large enough so that $[0,1]^n \subset S(h/2)$. Then (4.3) together with (4.2) gives bounds on T and T^{-1} for this choice of h. We then have a C^α bound on w in $S(h/2)$, and in particular in $[0,1]^n$. Since w is periodic, this gives a C^α bound on w everywhere.

Now Caffarelli's Schauder estimate [C] gives $C^{2,\alpha}$ bounds on u in $[0,1]^n$. From this standard Schauder estimates [GT] applied to equations (4.1) give bounds on the higher order derivatives of u. □

5. Abelian varieties

Theorem 1.1 can be interpreted as prescribing the scalar curvature of a torus invariant metric in the Kähler class of a flat metric over an Abelian variety. In this section we briefly explain this.

Let V be a n-dimensional complex vector space and $\Lambda \cong \mathbb{Z}^{2n}$ a maximal lattice in V such that the quotient $M = V/\Lambda$ is an Abelian variety, i.e., a complex torus that can be holomorphically embedded in projective space. For brevity, we only consider the Abelian varieties $M = \mathbb{C}^n/\Lambda$ where $\Lambda = \mathbb{Z}^n + i\mathbb{Z}^n$. The case of general lattices can be reduced to this case by an affine transformation.

We write $z \in M$ as $z = x + iy$, where x and $y \in \mathbb{R}^n$ and can be viewed as the periodic coordinates of M. Let

$$\omega_0 = \frac{\sqrt{-1}}{2}\sum_{\alpha=1}^n dz_\alpha \wedge d\bar{z}_\alpha = \sum_{\alpha=1}^n dx_\alpha \wedge dy_\alpha$$

be the standard flat metric with associated local Kähler potential $\frac{1}{2}|z|^2$. The group $T^n = (S^1)^n$ acts on M via translations in the Langrangian subspace $i\mathbb{R}^n \subset \mathbb{C}^n$, thus we can consider the following space of torus invariant Kähler metrics in the fixed class $[\omega_0]$:

$$\mathcal{H}_{T^n} = \left\{\psi \in C^\infty_{T^n}(M) : \omega_\psi = \omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\psi > 0\right\}.$$

Functions invariant under the translation of T^n are independent of y, so they are smooth function on $M/T^n \cong T^n$. In other words, they are smooth $\mathbb{Z}^n$-periodic functions in the variable $x \in \mathbb{R}^n$.

Let us write the Kähler potential in complex coordinates as

$$f(z) = \tfrac{1}{2}|z|^2 + 4\psi(x),$$

where ψ is a periodic function of x. Then it is easy to see that $f_{i\bar{j}} = \delta_{ij} + \psi_{ij}$, where ψ_{ij} is the real Hessian of ψ. Let us also define the function $v : \mathbb{R}^n \to \mathbb{R}$ given by

$$v(x) = \tfrac{1}{2}|x|^2 + \psi(x).$$

Note that then $f_{i\bar{j}} = v_{ij}$ so in particular v is strictly convex. It follows that the scalar curvature of the metric is given by

$$S = -\sum_{i,\bar{j}} f^{i\bar{j}} \left[\log\det(f_{a\bar{b}})\right]_{i\bar{j}} = -\frac{1}{4}\sum_{i,j} v^{ij} \left[\log\det(v_{ab})\right]_{ij}. \tag{5.1}$$

Let us now take the Legendre transform of v, with dual coordinate $t = \nabla v(x)$. The transformed function u is defined by

$$u(t) + v(x) = t \cdot x.$$

A calculation (see [A]) then gives

$$S(x) = -\frac{1}{4}\sum_{i,j} \frac{\partial^2 u^{ij}(t)}{\partial t_i \partial t_j}, \tag{5.2}$$

i.e., the scalar curvature equation is equivalent to Abreu's equation, where the equivalence is given by the Legendre transform.

Thus Theorem 1.1 implies that we can prescribe the scalar curvature of torus invariant metrics on an Abelian variety, as long as we work in the Legendre transformed "symplectic" coordinates instead of the complex coordinates. For example in complex dimension 1, working in complex coordinates essentially amounts to working in a fixed conformal class. In this case Kazdan–Warner [KW] have given necessary and sufficient conditions for a function to be the scalar curvature of a metric conformal to a fixed metric g. Their conditions are that either $S \equiv 0$, or the average of S with respect to g is negative and S changes sign. When working in symplectic coordinates (and only S^1-invariant metrics) then we have seen that the condition is that S has zero mean.

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