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STRONG RATIONAL CONNECTEDNESS OF TORIC VARIETIES

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Abstract. In this paper, we establish that, for any given finitely many distinct points $P_1, \dots, P_r$ and a closed subvariety S of codimension ≥ 2 in a complete toric variety X over an algebraically closed field of characteristic 0, there exists a rational curve $f : \mathbb{P}^1 \to X$ passing through $P_1, \dots, P_r$, disjoint from $S \setminus \{P_1, \dots, P_r\}$ (see Main Theorem). As a corollary we obtain that the smooth loci of complete toric varieties are strongly rationally connected.

Contents

1. Introduction

The concept of rationally connected varieties was independently invented by Kollár–Miyaoka–Mori [1] and Campana [2]. This class of variety has interesting arithmetic and geometric properties.

A class of proper rationally connected varieties comes from the smooth Fano varieties [2], [3] or [4]. Shokurov [5], Zhang [6], Hacon and M$^{\text{c}}$Kernan [7] proved that Fano type (FT) varieties are rationally connected.

An interesting question is whether the smooth locus of a rationally connected variety is rationally connected. In general the answer of the question is NO (For example, see [8], Example 19). However, for the FT (or log del Pezzo) surface case, Keel and M$^{\text{c}}$Kernan gave an affirmative answer, that is, if (S, Δ) is a log del Pezzo surface, then its smooth locus S^{sm} is rationally connected [9], but this does not imply the strong rational connectedness.

The concept of strongly rationally connected varieties (see Definition 2.2) was first introduced by Hassett and Tschinkel [8]. A proper and smooth separably rationally connected variety X over an algebraically closed field is strongly rationally connected (see [1] 2.1 or [4] IV.3.9). Xu [10] proved that the smooth loci of log del Pezzo surfaces are not only rationally connected but also strongly rationally connected, which confirms a conjecture of Hassett and Tschinkel [8], Conjecture 20). It is expected that

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the smooth locus of an FT variety is strongly rationally connected (cf. Example 2.1 and Main Theorem).

Throughout the paper, we work over an algebraically closed field of characteristic 0. It is interesting whether the Main Theorem holds for any algebraically closed field.

Main Theorem. *Let X be a complete toric variety over an algebraically closed field of characteristic 0. Let $p_1, \ldots, p_r \in \mathbb{P}^1$ be r distinct points. Then, for any given distinct points $P_1, \ldots, P_r \in X$ (P_i possibly singular), there is a geometrically free rational curve $f : \mathbb{P}^1 \to X$ over $\{P_i\}, 1 \leq i \leq r$ (see Definition 2.4 and Remark 2.2) with $f(p_i) = P_i$, $1 \leq i \leq r$. Moreover, f can be chosen to be free over $\{P_i\}$ if all points P_i are smooth.*

Main Theorem implies the following:

Let X be a complete toric variety over an algebraically closed field of characteristic 0. Let $p_1, \ldots, p_r \in \mathbb{P}^1$ be r distinct points. Then, for any given distinct points $P_1, \ldots, P_r \in X$ (possibly singular) and any given codimension ≥ 2 subvariety $S \subseteq X$, there is a rational curve $f : \mathbb{P}^1 \to X$ passing through $P_1, \ldots, P_r$ with $f(p_i) = P_i$, $1 \leq i \leq r$, disjoint from $S \setminus \{P_1, \ldots, P_r\}$.

If all points $P_i \in X$ are smooth, then we get the following corollary.

Corollary 1.1. *The smooth locus of a complete toric variety is strongly rationally connected.*

Note that in the Main Theorem, if $\dim X \geq 2$ the curve f can be chosen to be birational (see the proof of Main Theorem). For $\dim X = 1$ and any r, the curve f can be chosen to be a finite morphism. For $\dim X = 0$, $r \leq 1$ and the curve f is a constant morphism.

In the paper, we suppose that $\dim X \geq 1$ and a rational curve is always a nonconstant morphism $f : \mathbb{P}^1 \to X$.

2. Preliminaries

When we say that x is a point of a variety X, we mean that x is a closed point in X.

A normal projective variety X is called *FT* (*Fano Type*) if there exists an effective $\mathbb{Q}$-divisor D, such that (X, D) is klt and $-(K_X + D)$ is ample. See [11] Lemma–Definition 2.6 for other equivalent definitions.

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n. A *toric variety* $X(\Delta)$ is associated to a fan Δ, a finite collection of rational convex cones $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ (see [12] or [13].

Example 2.1. Projective toric varieties are FT. Let K be the canonical divisor of the projective toric variety $X(\Delta)$, T be the torus of X, and $\Sigma = X \setminus T = \sum D_i$ be the complement of T in X. Then K is linearly equivalent to $-\Sigma$. Since X is projective, there is an ample invariant divisor L. Suppose that $L = \sum d_i D_i$. Let the polytope $\Box_L = \{m \in M | \langle m, e_i \rangle + d_i \geq 0, \forall e_i \in \Delta(1)\}$, where M is the dual lattice of N, and $\Delta(1)$ is the set consisting of one-dimensional cones in Δ. Let u be an element in the interior of $\Box_L$. Let χ^u be the corresponding rational function of $u \in M$ (see [12] Section 1.3), and div χ^u be the divisor of χ^u. Then $D = \text{div } \chi^u + L$ is effective and ample and has support Σ. That is, $D = \sum d'_i D_i$ and all $d'_i > 0$.

Let ϵ be a positive rational number, such that all coefficients of prime divisors in ϵD are strictly less than 1. Then $\Sigma - \epsilon D$ is effective. It is well known that $(X, \Sigma - \epsilon D)$ is klt (see, e.g. [14] Lemma 5.2), and $-(K + \Sigma - \epsilon D) \sim \epsilon D$ is ample. Hence, X is FT.

Definition 2.1. An *isogeny* of toric varieties is a finite surjective toric morphism. Toric varieties X and Y are said to be *isogenous* if there exists an isogeny $X \to Y$. The *isogeny class* of a toric variety X is a set consisting of all toric varieties Y such that X and Y are isogenous.

Theorem 2.1. *Let $f : X \to Y$ be a finite surjective toric morphism. Then there exists a finite surjective toric morphism $g : Y \to X$.*

Proof. Let $f : X \to Y$ be a finite surjective toric morphism of toric varieties and $\varphi : (N', \Delta') \to (N, \Delta)$ be the corresponding map of lattices and fans. Then we can identify N' as a sublattice of N and $\Delta' = \Delta$.

There is an positive integer r such that rN is a sublattice of N'. Let g be the corresponding toric morphism of $(rN, \Delta) \to (N', \Delta)$. Since (rN, Δ) and (N, Δ) induce an isomorphic toric variety, we get $g : Y \to X$ is a finite surjective toric morphism. □

The properties of isogeny:

(1) Isogeny is an equivalence relation.

(2) If a toric variety Y is in the isogeny class of X and $\mu : X \to Y$ is the isogeny, then there is a one-to-one correspondence between the set of orbits $\{O_i^X\}$ of X and the set of orbits $\{O_i^Y = \mu(O_i^X)\}$ of Y. Hence $\dim O_i^X = \dim O_i^Y$ for all i, and the number of orbits is independent of the choice of toric varieties in an isogeny class of X.

A variety X over a characteristic 0 field is rationally connected, if there is a bounded family of rational curves in X, such that any two general points $x_1, x_2 \in X$ can be connected by rational curves in this given family.

Definition 2.2. ([8] Definition 14.) A smooth rationally connected variety Y is *strongly rationally connected* if any of the following conditions hold:

(1) for each point $y \in Y$, there exists a rational curve $f : \mathbb{P}^1 \to Y$ joining y and a generic point in Y;

(2) for each point $y \in Y$, there exists a very free rational curve containing y;

(3) for any finite collection of points $y_1, \ldots, y_m \in Y$, there exists a very free rational curve containing the y_j as smooth points;

(4) for any finite collection of jets

$$\operatorname{Spec} k[\epsilon]/\langle \epsilon^{N+1} \rangle \subset Y, \quad i = 1, \ldots, m$$

supported at distinct points $y_1, \ldots, y_m$, there exists a very free rational curve smooth at $y_1, \ldots, y_m$ and containing the prescribed jets.

Definition 2.3. Let X be a complete normal variety, B be a set of finitely many closed points in $\mathbb{P}^1$, and $g : B \to X$ be a morphism. A rational curve $f : \mathbb{P}^1 \to X$ is called *weakly free* over g if there exists an irreducible family of rational curves T and an evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ such that

(1) $f = f_{t_0} = \mathrm{ev}|_{\mathbb{P}^1 \times t_0}$ for some $t_0 \in T$,

(2) for any $t \in T$, $f_t = \mathrm{ev}|_{\mathbb{P}^1 \times t}$ is a rational curve and $f_t|_B = g$,

(3) the evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ by $\mathrm{ev}(x, t) = f_t(x)$ is dominant.

We say that a *rational curve* $f' : \mathbb{P}^1 \to X$ is a *general deformation* of f, or f' is a *sufficiently general weakly free rational curve*, if $f' = f_t$ is a general member of the family.

We say that a *weakly free rational curve* $h : \mathbb{P}^1 \to X$ is a *general deformation of* f, if there is a weakly free rational curve $f' : \mathbb{P}^1 \to X$, with family T' and the evaluation morphism $\mathrm{ev}' : \mathbb{P}^1 \times T' \to X$, such that T' contains the generic point η of T (that is, a scheme point of T' isomorphic to the generic point of T) with $f'_\eta = \mathrm{ev}'|_{\mathbb{P}^1 \times \eta} = f_\eta$, and $h = f'_{\eta'} = \mathrm{ev}'|_{\mathbb{P}^1 \times \eta'}$, where η' is the general point of T'.

Definition 2.4. Let X be a complete normal variety, B be a set of finitely many closed points in $\mathbb{P}^1$, and $g : B \to X$ be a morphism. A rational curve $f : \mathbb{P}^1 \to X$ is called *geometrically free* over g if there exists an irreducible family of rational curves T and an evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ such that

(1) $f = f_{t_0} = \mathrm{ev}|_{\mathbb{P}^1 \times t_0}$ for some $t_0 \in T$;
(2) for any $t \in T$, $f_t = \mathrm{ev}|_{\mathbb{P}^1 \times t}$ is a rational curve and $f_t|_B = g$;
(3) for any codimension 2 subvariety Z in X, $f_t(\mathbb{P}^1) \cap Z \subseteq g(B)$ for general $t \in T$ (general meaning t belongs to a dense open subset in T, depending on Z).

Remark 2.1. If X is smooth projective over an algebraically closed field of characteristic 0, then weak freeness over g is equivalent to usual freeness over g if $|B| \leq 2$. Indeed, let f_t be such a curve that for some point $p \in \mathbb{P}^1 \setminus B$, the tangent map $d_{(p,t)} : T_{(p,t)}\mathbb{P}^1 \times T \to T_{f_t(p)}X$ is surjective. Then the sheaf $f_t^* T_X \otimes I_B$ is free at p, that is, generated by global sections near p, where I_B is the ideal sheaf of B. On the other hand, $f_t^* T_X \otimes I_B = \sum \mathcal{O}_{\mathbb{P}^1}(a_i)$. Thus by the local generatedness all $a_i \geq 0$, $H^1(P^1, f_t^* T_X \otimes I_B) = 0$ and $f_t^* T_X \otimes I_B$ is generated by global sections. So f_t is free over g.

Remark 2.2. In our application, we usually assume that g is one-to-one, and $B = \{p_i\}$ a fixed subset of $\mathbb{P}^1$. Let $P_i = g(p_i)$. Without confusion, we say that f is geometrically free over $\{P_i\}$ (resp. weakly free over $\{P_i\}$) instead of saying that f is geometrically free over g (resp. weakly free over g).

Weak freeness and geometric freeness are generalizations of usual freeness (see [4] II.3.1 Definition) if the curve passes through singularities. To consider weakly free rational curves or geometrically free rational curves, we think of them as general members in a certain family. In particular, we can suppose that the morphism ev is flat.

Example 2.2. Let X be a projective cone over a conic. Let T be the family of all lines through the vertex O. Then $l \in T$ is weakly free and geometrically free over O by the construction.

For a complete normal algebraic variety, geometrical freeness implies weakly freeness (see [15] Proposition 3.3.3). But there exists weakly free rational curves that are not geometrically free. For example, a general rational curve on the projective rational surface in Example 19 [8] is weakly free but not geometrically free.

Let X be a variety and $\pi : \tilde{X} \to X$ be a resolution, $C \subset \tilde{X}$ be a curve, and $D \subset \tilde{X}$ be a proper subvariety. Let $P_1, \ldots, P_r \in X$ be r distinct points. We say that the curve C intersects D over $\{P_i\}$ *only in divisorial points* of D, if

(1) $\pi^{-1}(P_i) \subset D$ is a divisor for each i;
(2) the curve C intersects D properly, and each point of $C \cap \pi^{-1}(P_i)$ lies in a unique irreducible component of $\pi^{-1}(P_i)$ for each i. Note that C is possibly singular.

We need the following resolution in the proof of the Main Lemma and the Main Theorem.

Theorem 2.2. *Let X be a toric variety. Let Σ be the invariant locus of X. Let $P_1, \ldots, P_r \in X$ be r distinct points. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over $P_1, \ldots, P_r$. Then there exists a resolution $\pi : \tilde{X} \to X$, such that*

(1) *$\pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor with simple normal crossing;*
(2) *$\pi^{-1}(P_j) \subseteq \pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor for each point P_j;*
(3) *$\pi : \tilde{X} \to X$ is an isomorphism over $X \setminus (Sing\ X \cup \{P_i\})$;*
(4) *sufficiently general $\tilde{f}(\mathbb{P}^1)$ intersects $\pi^{-1}(\Sigma \cup \{P_i\})$ over each P_i only in divisorial points of $\pi^{-1}(\Sigma \cup \{P_i\})$.*

More generally, let $f_j : \mathbb{P}^1 \to X$, $1 \leq j \leq m$ be finitely many sufficiently general weakly free rational curve over a subset of $\{P_i\}$, where $\{P_i\}$ is a set of finitely many distinct points in X. Then there exists a resolution $\pi : \tilde{X} \to X$ such that

(1′) *$\pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor with simple normal crossing;*
(2′) *$\pi^{-1}(P_i) \subseteq \pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor for each point P_i;*
(3′) *$\pi : \tilde{X} \to X$ is an isomorphism over $X \setminus (Sing\ X \cup \{P_i\})$;*
(4′) *For each j, sufficiently general $\tilde{f}_j(\mathbb{P}^1)$ intersects $\pi^{-1}(\Sigma \cup \{P_i\})$ over each P_i only in divisorial points of $\pi^{-1}(\Sigma \cup \{P_i\})$, where $\tilde{f}_j : \mathbb{P}^1 \to \tilde{X}$ is the proper birational transformation of a general deformation of f_j and is a (weakly) free rational curve.*

Proof. When the ground field is of characteristic 0, (1)–(3) follow from usual facts in the resolution theory, e.g., see [16] Main Theorems I and II. However, in the toric or toroidal case, the same result holds for any field. More precisely, if all P_i are invariant, we can use a toric resolution. If some P_i are not invariant, they can be converted into toroidal invariant points P_i after a toroidalization (for a reference, see Propositoin 3.2 in [17]).

To fulfill (4), we need extra resolution over intersections of the divisorial components of $\pi^{-1}(\Sigma \cup \{P_i\})$ through which general $\tilde{f}$ is passing over P_i. Termination of such resolution follows from an estimate by the multiplicities of intersection for $f(\mathbb{P}^1)$ with Σ. The last resolution is independent of the choice of a general rational curve by Lemma 3.1 below. However it depends on the choice of intersections of divisorial components. For more details, see the proof of Lemma 4.3.4 in [15].

For the general statement, we can get (1′)–(3′) in a similar manner as above. To fulfil (4′), we just need extra resolutions over each point P_i. □

We discuss some examples of rational curves on projective spaces and the quotients of projective spaces.

Example 2.3. For any given subvariety S of codimension ≥ 2 in $\mathbb{P}^n$, any points $P_1, \dots, P_r \in \mathbb{P}^n$, and any integer $d \geq r$, there exists a rational curve C of degree d, such that each $P_i \in C$ and $C \cap S = \emptyset$.

Indeed, we can construct a tree T with r branches, such that each P_i is a smooth points on a unique branch and disjoint from S. The tree can be smoothed into a rational curve C passing through $P_1, \dots, P_r$, disjoint from S. The rational curve C has degree r. For $d \geq r$, we can attach $d - r$ rational curves to the tree T, and smooth it.

Applying Example 2.3, we get

Example 2.4. Let $\pi : \mathbb{P}^n \to X$ be a finite morphism, S be a codimension ≥ 2 subvariety in X, and $\{P_i\}_{i=1}^m$ be a set of m points outside S. Then there exists a rational curve C, such that each $P_i \in C$ and $C \cap S = \emptyset$.

In particular, the same result holds if X is a quotient space $\mathbb{P}^n/G$, where G is a finite group, for example, if X is a weighted projective space. It is well known that if X is a complete $\mathbb{Q}$-factorial toric variety with Picard number one, then there exists a weighted projective space Y and a finite toric morphism $\pi : Y \to X$. So the same result holds for rational curves on a complete $\mathbb{Q}$-factorial toric variety with Picard number one. It is a very special case of our Main Theorem.

3. Proof of main theorem

In this section we prove Main Theorem. Let us first prove Main Lemma, which is a special case of Main Theorem.

Main Lemma. *Let X be a complete toric variety. Let $P, Q \in X$ be two distinct points (P, Q possibly singular). Let $S \subseteq X$ be a closed subvariety of codimension ≥ 2. Then there exists a weakly free rational curve on X over P, Q, disjoint from $S \setminus \{P, Q\}$.*

To prove Main Lemma, we need some preliminaries.

Lemma 3.1. *Let f be a weakly free rational curve on X, and $F_1, \dots, F_s \subseteq X$ be s proper irreducible subvarieties in X. Then there exist $s', 0 \leq s' \leq s$, subvarieties among $\{F_j\}$ (after renumbering we assume that they are $F_1, \dots, F_{s'}$) such that a general deformation of f intersects $F_1, \dots, F_{s'}$, and is disjoint from $F_{s'+1}, \dots, F_s$.*

The proof of this Lemma is a standard exercise in incidence relations. See [15] Lemma 4.3.2 for a detailed proof.

Lemma 3.2. *Let X be a complete toric variety. Let $P, Q \in X$ be two points (possibly singular), and S be a closed subvariety of codimension ≥ 2. Let $F_1, \dots, F_s$ be all the irreducible components of Sing X. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over P, Q. Suppose $f(\mathbb{P}^1)$ intersects $F_1 \setminus \{P, Q\}, \dots, F_{s'} \setminus \{P, Q\}$, and is disjoint from $F_{s'+1} \setminus \{P, Q\}, \dots, F_s \setminus \{P, Q\}$. Then there exists a weakly free rational curve f' over $\{P, Q\}$, such that $f'(\mathbb{P}^1)$ is disjoint from $((S \setminus \mathrm{Sing}\ X) \cup F_{s'+1} \cup \cdots \cup F_s) \setminus \{P, Q\}$. Moreover, for any fixed closed subvariety Z of X, if $f(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$, then $f'(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$.*

Proof. Applying Theorem 2.2 to the toric variety X and two points $\{P, Q\}$, we get a resolution $\pi : \tilde{X} \to X$ satisfying 1)-3) in the theorem and a weakly free rational curve $\tilde{f} : \mathbb{P}^1 \to \tilde{X}$ satisfying 4) in the theorem. A general deformation $\tilde{f}'$ of $\tilde{f}$ is weakly free, so $\tilde{f}'$ is free by Remark 2.1 above. Moreover, we can assume that $\tilde{f}'$ is disjoint from $(S \setminus \mathrm{Sing}\ X) \setminus \pi^{-1}\{P, Q\}$ by [4] II.3.7.

On the other hand, let Σ be the invariant locus of X. Notice that $\mathrm{Sing}\ X \subseteq \Sigma$. Then by Theorem 2.2, $\tilde{f}(\mathbb{P}^1)$ intersects $\pi^{-1}(\Sigma \cup \{P, Q\})$ divisorially over P, Q, and $\tilde{f}(\mathbb{P}^1)$ is disjoint from the closure of $\pi^{-1}(F_{s'+1} \setminus \{P, Q\}), \ldots, \pi^{-1}(F_s \setminus \{P, Q\})$. So the general deformation $\tilde{f}'$ of $\tilde{f}$ intersects open subsets of divisors $\pi^{-1}(P)$ and $\pi^{-1}(Q)$, disjoint from the closure of $((S \setminus \mathrm{Sing}\ X) \setminus \pi^{-1}\{P, Q\}) \cup \pi^{-1}(F_{s'+1} \setminus \{P, Q\}) \cup \cdots \cup \pi^{-1}(F_s \setminus \{P, Q\})$. We apply Lemma 3.3 by replacing f' by $\tilde{f}'$, dominant morphism μ by $\pi : \tilde{X} \to X$, $\{P_i\}$ by $\{P, Q\}$, and S by $(S \setminus \mathrm{Sing}\ X) \cup F_{s'+1} \cup \cdots \cup F_s$. Then we get the weakly free rational curve $f' = \pi \tilde{f}' : \mathbb{P}^1 \to X$ as a general deformation of f (see Definition 2.3), passing through points P, Q and disjoint from $((S \setminus \mathrm{Sing}\ X) \cup F_{s'+1} \cup \cdots \cup F_s) \setminus \{P, Q\}$.

Moreover, we can assume that f' is a weakly free rational curve over P, Q, by a base change of the family to which f' belongs (For details, see the proof of Lemma 4.3.1 in [15]).

The last statement can be proved similarly. $\square$

Lemma 3.3. *Let X, X' be two complete varieties with* $\dim X > 0$. *Let $\mu : X' \to X$ be a dominant morphism. Then the image of a weakly free rational curve on X' is weakly free on X in the following sense:*

Let $P_1, P_2, \ldots, P_r \in \mu(X)$ be r distinct points, and $S \subseteq X$ be a closed subvariety. Let $S' = \mu^{-1}S$, and $P_1', P_2', \ldots, P_r' \in X'$ be points such that $\mu(P_i') = P_i$ for $i = 1, \ldots, r$. If $f' : \mathbb{P}^1 \to X'$ is a weakly free rational curve over $P_1', P_2', \ldots, P_r'$, disjoint from $S' \setminus \{P_1', P_2', \ldots, P_r'\}$, then $f = \mu \circ f''$ is a weakly free rational curve on X over $P_1, P_2, \ldots, P_r$, disjoint from $S \setminus \{P_1, P_2, \ldots, P_r\}$, where f'' is a general deformation of f'.

Proof. Since f' is weakly free, ev: $\mathbb{P}^1 \times T' \to X'$ is dominant, where T' is the family associated to f'. Since $\mu : X' \to X$ is dominant, ev: $\mathbb{P}^1 \times T' \to X' \to X$ is dominant. Hence for general deformation $f'' \in T'$ of f', $f = \mu \circ f''$ is a weakly free rational curve on X. $\square$

Lemma 3.4. *Let X be a $\mathbb{Q}$-factorial toric variety, and O be a singular orbit of X. Then there exists an isogeny $\mu : Y \to X$, such that Y is smooth along $\mu^{-1}(O)$.*

Proof. Let (N, Δ) be the lattice and fan associated to X. Let $U \subset X$ be the affine open subvariety containing the orbit O. Then the orbit O corresponds to a simplicial cone σ in Δ. Let N' be the sublattice generated by the primitive elements of σ. Let Y be the toric variety corresponding to (N', Δ) and μ be the natural finite dominant morphism corresponding to $(N', \Delta) \to (N, \Delta)$. By the construction of μ, $\mu^{-1}(O)$ is smooth. $\square$

Proof of Main Lemma. Step 1. After $\mathbb{Q}$-factorization $q : X' \to X$, we can assume that X is a complete $\mathbb{Q}$-factorial toric variety [14] Corollary 3.6). Indeed, take points $P', Q' \in X'$ such that $q(P') = P$ and $q(Q') = Q$. The inverse image $S' = q^{-1}S$ is a closed subvariety of codimension ≥ 2, because q is small. By Lemma 3.3, a weakly free rational curve $f' : \mathbb{P}^1 \to X'$ over $\{P', Q'\}$, disjoint from $S' \setminus \{P', Q'\}$ gives a weakly free rational curve $f = q \circ f' : \mathbb{P}^1 \to X$ over $\{P, Q\}$, disjoint from $S \setminus \{P, Q\}$.

Step 2. A weakly free rational curve can be moved from any smooth variety of codimension ≥ 2 in the sense of Lemma 3.2. So we can reduce the proof of Main Lemma to the case $S = I(X)$, where $I(X)$ denotes the union of orbits of X of codimension ≥ 2. Since X is a toric variety, $\operatorname{Sing} X \subseteq I(X)$.

Indeed, for any subvariety $S \subseteq X$ of codimension ≥ 2, suppose there is a sufficiently general weakly free rational curve $f : \mathbb{P}^1 \to X$ over $P, Q \in X$, disjoint from $I(X) \setminus \{P, Q\}$. Apply Lemma 3.2 to the subvariety S, and the weakly free rational curve f. Since $\operatorname{Sing} X \subseteq I(X)$, $s' = 0$ in Lemma 3.2, that is, $f(\mathbb{P}^1)$ is disjoint from $F_1 \setminus \{P, Q\}, \ldots, F_s \setminus \{P, Q\}$. Then there exists a weakly free rational curve f', such that $f'(\mathbb{P}^1)$ is disjoint from $((S \setminus \operatorname{Sing} X) \cup F_1 \cup \cdots \cup F_s) \setminus \{P, Q\} = ((S \setminus \operatorname{Sing} X) \cup \operatorname{Sing} X) \setminus \{P, Q\} = S \setminus \{P, Q\}$.

Step 3. Suppose that $I(X)$ consists of $\tilde{s}$ distinct orbits $O_1, \ldots, O_{\tilde{s}}$. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over P, Q. By Lemma 3.1, we can assume that $f(\mathbb{P}^1)$ intersects with $O_1 \setminus \{P, Q\}, \ldots, O_{s'} \setminus \{P, Q\}$, and is disjoint from $O_{s'+1} \setminus \{P, Q\}, \ldots, O_{\tilde{s}} \setminus \{P, Q\}$ for some s'.

Notice that s' depends on the points P, Q and the variety X. However, since s' is bounded by $\tilde{s}$, and $\tilde{s}$ is independent of the choice of X in an isogeny class, there exists an $\bar{s}$ such that for any toric variety Y in the isogeny class of X, and two distinct points $P', Q' \in Y$, there exists a weakly free rational curve $f'_{\bar{s}} : \mathbb{P}^1 \to Y$ over P', Q', such that for any $1 \leq i \leq \tilde{s}$, if $f'_{\bar{s}}(\mathbb{P}^1)$ intersects $O_i^Y \setminus \{P', Q'\}$, then $1 \leq i \leq \bar{s}$, where O_i^Y are orbits of Y of codimension ≥ 2. Furthermore, we can assume that $\dim O_1^Y \geq \dim O_2^Y \geq \cdots \geq \dim O_{s'}^Y \geq \dim O_{s'+1}^Y \geq \cdots \geq \dim O_{\tilde{s}}^Y$. This order is good for us, because $\cup_{j \geq s} O_j^Y$ is closed for any s.

We fix a complete toric variety X, two points P, Q and a weakly free rational curve $f_{\bar{s}}$ over P, Q. By Lemmas 3.3 and 3.4, we can suppose that the orbit $O_{\bar{s}}$ is smooth. Indeed, by Lemma 3.4, there is an isogeny $\mu : Y \to X$ such that $O_{\bar{s}}^Y = \mu^{-1}(O_{\bar{s}})$ is smooth. Let $P', Q' \in Y$ such that $\mu(P') = P, \mu(Q') = Q$. Then the existence of a weakly free rational curve $f' : \mathbb{P}^1 \to Y$ over P', Q', disjoint from $O_{\bar{s}}^Y \cup \cdots \cup O_{\tilde{s}}^Y$, implies the existence of a weakly free rational curve $f : \mathbb{P}^1 \to X$ over P, Q, disjoint from $O_{\bar{s}} \cup \cdots \cup O_{\tilde{s}}$, by Lemma 3.3 with $X' = Y, \{P_i\} = \{P, Q\}$ and $S = O_{\bar{s}}^Y \cup O_{\bar{s}+1}^Y \cup \cdots \cup O_{\tilde{s}}^Y$.

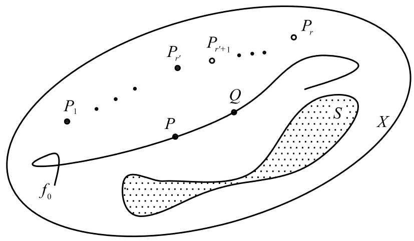
Step 4. Now, we prove that there is a weakly free rational curve $f_{\bar{s}-1}$ over P, Q, such that for any $1 \leq i \leq \tilde{s}$, if $f_{\bar{s}-1}(\mathbb{P}^1)$ intersects $O_i \setminus \{P, Q\}$, then $1 \leq i \leq \bar{s} - 1$. Indeed, we have the following two cases:

(1) If $f_{\bar{s}}(\mathbb{P}^1)$ is disjoint from $O_{\bar{s}} \setminus \{P, Q\}$, then let $f_{\bar{s}-1} = f_{\bar{s}}$.
(2) If $f_{\bar{s}}(\mathbb{P}^1)$ intersects $O_{\bar{s}} \setminus \{P, Q\}$, we apply Lemma 3.2 with $Z = O_{\bar{s}+1} \cup \cdots \cup O_{\tilde{s}}$ and $S = O_{\bar{s}} \cup Z$. Notice that S and Z are closed subvarieties of X of codimension ≥ 2, and $O_{\bar{s}}$ is smooth. In particular, $S \setminus \operatorname{Sing} X \supseteq O_{\bar{s}}$. By assumption, $f_{\bar{s}}(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$. Therefore, by the Lemma, there exists a weakly free rational curve $f_{\bar{s}-1}$ on X, such that for any $1 \leq i \leq \tilde{s}$, if $f_{\bar{s}-1}(\mathbb{P}^1)$ intersects $O_i \setminus \{P, Q\}$, then $1 \leq i \leq \bar{s} - 1$, and $f_{\bar{s}-1}(\mathbb{P}^1)$ is disjoint from $(O_{\bar{s}} \cup Z) \setminus \{P, Q\} = (O_{\bar{s}} \setminus \{P, Q\}) \cup (O_{\bar{s}+1} \setminus \{P, Q\}) \cup \cdots \cup (O_{\tilde{s}} \setminus \{P, Q\})$.

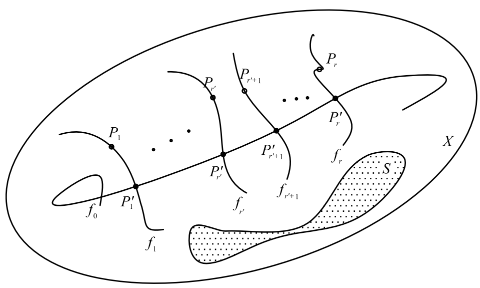
Step 5. By induction on $\bar{s}$, there is a weakly free rational curve f_0 over P, Q, disjoint from $I(X) \setminus \{P, Q\}$. $\square$

Proof of Main Theorem. Step 1. First, let us consider $S = \operatorname{Sing} X$.

There is a free rational curve $f_0 : C_0 \cong \mathbb{P}^1 \to X$ over $\{P, Q\}$, and it is disjoint from $\{P_i\} \cup S$ for $\dim X \geq 2$, where $P, Q \notin \{P_i\} \cup S$ are any two distinct smooth points on X. Indeed, we can apply Main Lemma to the subvariety $\{P_i\} \cup S$ and two smooth points P, Q. Since $f_0(\mathbb{P}^1)$ is in the smooth locus of X, f_0 is free over $\{P, Q\}$, and it is disjoint from $\{P_i\} \cup S$ for $\dim X \geq 2$. We can also suppose that the curve f_0 is birational onto its image (see [4] Theorem II.3.14).



I. Assume that $P_1, \ldots, P_{r'}$ are all the smooth points among $\{P_i\}$ of X, for some r', $1 \leq r' \leq r$. Since f_0 is birational onto its image, we can choose an isomorphism $\phi : \mathbb{P}^1 \to C_0$, such that $P'_i = f_0(\phi(p_i))$ and $P_i \in X$ are all distinct. Thus, we identify $p_i \in \mathbb{P}^1$ with $\phi(p_i) \in C_0$ and $C_0 = \mathbb{P}^1$ under the isomorphism. For each j, applying the Main Lemma to $S = \operatorname{Sing} X \cup \{P_i\}$ and points $P = P_j, Q = P'_j$, there is a weakly free rational curve $f_j : C_j \cong \mathbb{P}^1 \to X$ over $\{P_j, P'_j\}$ with $f_j(0_j) = P_j$ and $f_j(\infty_j) = P'_j$, where $0_j, \infty_j \in C_j$ for each $1 \leq j \leq r$, disjoint from $S \setminus \{P_j, P'_j\}$.



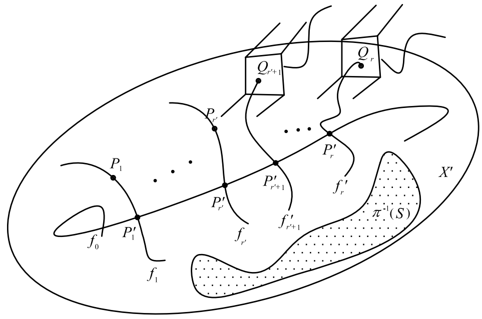
Applying the general statement of Theorem 2.2 to weakly free rational curves $f_0, f_1, \ldots, f_r$, and the set $\{P_i\}$ in Theorem 2.2 to the set $\{P_{r'+1}, \ldots, P_r\}$ here, we get a resolution $\pi : X' \to X$.

We construct a comb of smooth rational curves C and a morphism $f : C \to X'$ as follows.

For each $1 \leq i \leq r'$, since P_i and P'_i are smooth points, $f_i(\mathbb{P}^1)$ is contained in the smooth locus of X. Therefore f_i is free for each $1 \leq i \leq r'$ by [4] II.3.11. We identify the curve $f_i : C_i \cong \mathbb{P}^1 \to X$ birationally with a free rational curve $f_i : C_i \cong \mathbb{P}^1 \to X'$. We also identify $P_i \in X$ with $P_i \in X'$ for $1 \leq i \leq r'$, and $P'_i \in X$ with $P'_i \in X'$ for $1 \leq i \leq r$. More precisely, $f_i(0_i) = P_i$, where $0_i \in C_i, 1 \leq i \leq r'$, and $f_i(\infty_i) = P'_i$ where $\infty_i \in C_i, 1 \leq i \leq r$.

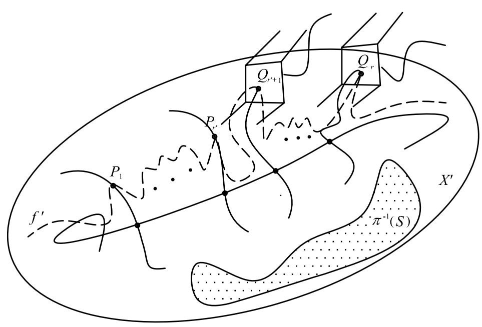
For each $r' + 1 \leq j \leq r$, P_j is singular. Let $f'_j : C_j \cong \mathbb{P}^1 \to X'$ be the proper birational transformation of a sufficiently general deformation of f_j. Since $\pi : X' \to X$ is a resolution in Theorem 2.2, $f'_j(C_j)$ intersects $\pi^{-1}P_j$ over P_j only in divisorial points for $r' + 1 \leq j \leq r$, and is disjoint from the closure of $\pi^{-1}(S \setminus \{P_i\})$. Let Q_j be the point in $f'_j(C_j) \cap \pi^{-1}P_j$ over P_j for $r'+1 \le j \le r$. We can suppose that f_i is very free for $1 \le i \le r'$ and f'_j is very free for $r'+1 \le j \le r$ by [3] 1.1. or [4] II.3.11.

By the construction of $f_i, 1 \le i \le r'$ and $f'_j, r'+1 \le j \le r$, $f_i(C_i)$ and $f'_j(C_j)$ are disjoint from the closure of $\pi^{-1}(S \setminus \{P_1, \ldots, P_r\}) = \pi^{-1}(S \setminus \{P_{r'+1}, \ldots, P_r\})$.



II. Gluing $\cup_{i=0}^r C_i$, we get a comb of smooth rational curves $C = \sum_{i=0}^r C_i$ and a morphism $f : C \to X'$. Indeed, we identify points $\infty_i \in C_i$ with $p_i \in C_0$ for each $1 \le i \le r$. Then we have a comb of smooth rational curves $C = \sum_{i=0}^r C_i$ and a morphism $f : C \to X'$ because $f_0(p_i) = f_i(\infty_i) = P'_i$. Notice that $f(C)$ is disjoint from the closure of $\pi^{-1}(S \setminus \{P_1, \ldots, P_r\})$.

In the end, $f : C \to X'$ can be smoothed into a rational curve $f' : \mathbb{P}^1 \to X'$ such that f' is free over $\{P_i\}, 1 \le i \le r'$ and $\{Q_j\}, r'+1 \le j \le r$ with $f'(p_i) = P_i, 1 \le i \le r'$ and $f'(p_j) = Q_j, r'+1 \le j \le r$, and is disjoint from the closure of $\pi^{-1}(S \setminus \{P_1, \ldots, P_r\})$ (See [4] II.7.6[1]). By construction, for $\dim X \ge 2$ the smoothing f' is birational onto its image, and for $\dim X = 1$, f' is finite onto its image.



Step 2. Now we consider any closed subvariety S of codimension ≥ 2.

By Step 1, there is a free rational curve $f' : \mathbb{P}^1 \to X'$ over $\{P_1, \ldots, P_{r'}, Q_{r'+1}, \ldots, Q_r\}$, disjoint from the closure of $\pi^{-1}(\mathrm{Sing}\, X \setminus \{P_1, \ldots, P_r\})$, where $\pi : X' \to X$ is the resolution in Step 1. On the other hand, $\pi^{-1}((S \setminus \mathrm{Sing}\, X) \setminus \{P_1, \ldots, P_r\})$ is a codimension ≥ 2 subvariety on X' by Theorem 2.2 3'). So a general deformation f'' of f' is free over $\{P_1, \ldots, P_{r'}, Q_{r'+1}, \ldots, Q_r\}$ with $f''(p_i) = P_i, 1 \le i \le r'$ and $f''(p_j) = Q_j, r'+1 \le j \le r$, disjoint from $\pi^{-1}((S \setminus \mathrm{Sing}\, X) \setminus \{P_1, \ldots, P_r\})$ by [4] II.3.7. Since f' is disjoint from the closure of $\pi^{-1}(\mathrm{Sing}\, X \setminus \{P_1, \ldots, P_r\})$, f'' is

[1]Theorem II.7.6 proves the existence of smoothing keeping points fixed. For combs, the same arguments of the proof gives more: a general smoothing of a comb is free over B, where B is the set of intersection points of the handle and each teeth of the comb.

disjoint from $\pi^{-1}(\mathrm{Sing}\ X \setminus \{P_1, \ldots, P_r\})$. Hence f'' is disjoint from $\pi^{-1}(\mathrm{Sing}\ X \setminus \{P_1, \ldots, P_r\}) \cup \pi^{-1}((S \setminus \mathrm{Sing}\ X) \setminus \{P_1, \ldots, P_r\}) = \pi^{-1}(S \setminus \{P_1, \ldots, P_r\})$. Therefore, $\pi f''$ is a general deformation of $\pi f'$ over $\{P_1, \ldots, P_r\}$ with $\pi f''(p_i) = P_i, 1 \leq i \leq r$, disjoint from $S \setminus \{P_1, \ldots, P_r\}$, and thus $\pi f'$ is a geometrically free rational curve over $\{P_1, \ldots, P_r\}$ on X with $\pi f'(p_i) = P_i, 1 \leq i \leq r$. $\square$

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