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AN EXAMPLE OF NON-HOMEOMORPHIC CONJUGATE VARIETIES

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Abstract. We give examples of smooth projective varieties over complex numbers, in the context of connected Shimura varieties, which are not homeomorphic to a conjugate of itself by an automorphism of the complex numbers.

1. Introduction

Let X be a quasi-projective variety defined over $\mathbb{C}$. Suppose σ is an automorphism of $\mathbb{C}$. Denote by $X^\sigma := X \times_\sigma \mathbb{C}$, the conjugate of X by the automorphism σ of $\mathbb{C}$, obtained by applying the automorphism σ to the coefficients of the polynomials defining X. It is known that the varieties X and X^σ have the same Betti numbers. In [10], Serre gave an example where the topological spaces $X(\mathbb{C})$ and $X^\sigma(\mathbb{C})$ are not homeomorphic.

Recently, Milne and Suh [6] gave further examples in the context of connected Shimura varieties. Their method is to find a conjugate such that the reductive group underlying the Shimura datum is different, and then apply the super-rigidity results of Margulis.

Our examples are in the same context as that of Milne and Suh, but we work with Shimura's construction of canonical models [12]. Shimura's construction allows us to identify the adelic congruence subgroup defining the conjugate variety as a conjugate by an element of the adjoint group. We then appeal to Mostow rigidity and the failure of strong approximation (or non-triviality of class number) for the adjoint group to get at the desired examples. In our example, the congruent lattices defining the variety and its conjugate are commensurable. Earlier in [9], we observed using Shimura's construction coupled with the theorems of Labesse and Langlands on the mulitplicity of cusp forms for $SL(1, D)$, that a Galois twist of these spaces attached to $SL(1, D)$ over the reflex field preserves the spectrum of the Laplacian; this provides examples of locally symmetric spaces attached to a quaternion division algebra over a number field, which are isospectral but not isometric.

Apart from the work of Milne and Suh, many other constructions of non-homeomorphic conjugate varieties have been constructed by different methods. We refer to [1, 2, 4, 5, 11] and references cited in these papers for different facets of this problem. For example, in [2, 5], the construction of such examples is motivated by the problem of knowing whether the Galois group acts faithfully on the components of the moduli space of surfaces.

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2. The example

Let F be a totally real number field of degree at least two over $\mathbb{Q}$. Let D be an indefinite quaternion division algebra defined over F. We assume that D is split at exactly one real place, say τ_1 of F. This assumption allows us to assume that the reflex field of (F,τ_1) to be F itself. Let V be a vector space of rank $n \geq 2$ over D, equipped with a hermitian inner product with respect to the standard involution on D. We assume that the inner product is definite on the spaces $V \otimes_\tau \mathbb{R}$, for all real embeddings τ of F different from τ_1. In particular, since we have assumed that the degree of F is at least two, the form h is anisotropic. Let G be the group of unitary similitudes of h. We consider G as an algebraic group defined over F, and let G_d be the derived group of G. Denote by PG the adjoint group attached to G, the group obtained by taking the quotient of G modulo its centre. For an algebraic group H defined over a number field F, we let $H_\infty = H(F \otimes \mathbb{R})$ be the real Lie group consisting of the real points of H. Under our assumptions, it follows that

$$G_{d,\infty} \simeq Sp(2n,\mathbb{R}) \times \text{ a compact group}, \quad n \geq 2.$$

Let K_∞ be a maximal compact subgroup of $G_{d,\infty}$ and let $X = G_{d,\infty}/K_\infty$ be the non-compact symmetric space associated to G_d. By our assumptions, X is isomorphic to the Siegel upper half-space $\mathbb{H}_n$ of dimension n. We also have a natural action of G_∞ on X. Denote by $\mathbf{A}$ the adele ring of F, and by $\mathbf{A}_f$ the subring of finite adeles. Let K be a compact open subgroup of $G(\mathbf{A}_f)$, and let $K_d = K \cap G_d(\mathbf{A}_f)$. Denote by

$$\Gamma_K = G_\infty K \cap G(F) \quad \text{and} \quad \Gamma_{d,K} = G_{d,\infty} K_d \cap G_d(F),$$

the corresponding arithmetic lattices in $G(F \otimes \mathbb{R})$ and $G_d(F \otimes \mathbb{R})$, respectively. We assume that K is such that $\Gamma_{d,K}$ is torsion-free, and the natural inclusion $\Gamma_{d,K} \subset \Gamma_K$ is an isomorphism modulo the centre of Γ_K. By a congruence subgroup we will either mean a compact open subgroup contained in the group of finite adele points of the algebraic group, or the corresponding arithmetic lattice contained in the real points of the algebraic group.

By a theorem of Baily–Borel, the quotient space $X_K = \Gamma_K \backslash X$ is a connected, smooth, projective variety. The fundamental group $\overline{\Gamma}_K$ of the variety X_K can be identified with the projection of Γ_K to PG_∞, and also with the lattice $\Gamma_{d,K}$ contained in $G_{d,\infty}$.

For an element $x \in G(\mathbf{A}_f)$, denote by K^x the conjugate lattice $x^{-1}Kx$, and by $\overline{x}$ its image in $PG(\mathbf{A}_f)$ (a bar on top will indicate the image in the adjoint group PG). Further, let $N(\overline{K_d})$ denote the normalizer of $\overline{K_d}$ in $PG(\mathbf{A}_f)$, where $\overline{K_d}$ is the image of K_d in $PG(\mathbf{A}_f)$. The desired example is provided by the following theorem.

Theorem 2.1. *With notation and assumptions as above, suppose $\overline{x}$ does not belong to the set $N(\overline{K_d})PG(F)$. Then X_K and X_{K^x} are conjugate by an automorphism σ of $\mathbb{C}$, but the respective fundamental groups $\overline{\Gamma}_K$ and $\overline{\Gamma}_{K^x}$ are not isomorphic. In particular, X_K and X_{K^x} are not homeomorphic.*

Proof. We first show that the varieties X_K and X_{K^x} are conjugate by an automorphism of $\mathbb{C}$. For this, we recall Shimura's theory of canonical models [12]. Let $\nu : G \to \mathbf{G}_m$ be the reduced norm. By class field theory, the subgroup $F^*\nu(K)$ of the idele group $\mathbf{A}^*$ defines an abelian extension F_K of F. The reciprocity morphism of class field theory,

$$\mathrm{rec} : \mathbf{A}^*/F^* \to \mathrm{Gal}(F^{ab}/F),$$

defines an element $\sigma(x) \in \mathrm{Gal}(F^{ab}/F)$ by the prescription

$$\sigma(x) = \mathrm{rec}(\nu(x)^{-1}).$$

As a consequence of the main theorem of canonical models in [12, Theorem 2.5, p. 159, Section 2.6], the variety X_K has a model defined over the field F_K, and

$$X_K^{\sigma(x)} \simeq X_{K^x}. \tag{2.1}$$

Thus, the varieties X_K and X_{K^x} are conjugate.

Suppose on the contrary, that X_K and X_{K^x} have isomorphic fundamental groups. Since these spaces are Eilenberg–Maclane spaces, there exists a homotopy equivalence

$$\phi : X_K \to X_{K^x}.$$

Since the lattices are irreducible in PG_∞ and the real rank of PG is at least two, by Mostow rigidity [7], the spaces X_K and X_{K^x} are isometric.

Hence, there exists $\overline{g} \in PG_\infty$ such that

$$\overline{g}^{-1}\overline{\Gamma}_{K^x}\overline{g} = \overline{\Gamma}_K.$$

Since the lattices $\overline{\Gamma}_K$ and $\overline{\Gamma}_{K^x}$ are arithmetic and commensurable, it follows by a theorem of Borel [3], that $\overline{g} \in PG(F)$. Hence, there is an element $g \in G(F)$ satisfying,

$$g^{-1}\Gamma_{d,K^x}g = \Gamma_{d,K}.$$

Consider now $G_d(F)$ embedded diagonally in $G_d(\mathbf{A}_f)$. By the strong approximation theorem for G_d, the closure of $\Gamma_{d,K}$ in $G_d(\mathbf{A}_f)$ can be identified with K_d. Further, the closure of Γ_{d,K^x} in $G_d(\mathbf{A}_f)$ can be identified with $g^{-1}K_d^x g$, where we now consider $g \in G(F)$ as diagonally embedded in $G(\mathbf{A}_f)$. Hence, we have

$$g^{-1}K_d^x g = g^{-1}x^{-1}K_d\ x\ g = K_d.$$

Projecting to PG, we obtain

$$\overline{g}^{-1}\overline{x}^{-1}\overline{K_d}\ \overline{x}\ \overline{g} = \overline{K_d},$$

where $\overline{K_d}$ denotes the image of K_d in $PG(\mathbf{A}_f)$. This implies that $\overline{x} \in N(\overline{K_d})PG(F)$, contradicting our choice of $\overline{x}$. □

2.1. Congruence subgroups with small normalizers. One way of producing congruence lattices K and an element $x \in G(\mathbf{A}_f)$ satisfying the hypothesis of the Theorem, is to impose an additional arithmetical condition on the field F. Let S be a finite set of places of F containing the archimedean places and the finite places of F at which D is ramified. Let S_f denote the subset of S which are non-Archimedean, and S' the complement of S in the collection of places of F.

We assume that the group $C_{F,S}/C_{F,S}^{2n}$ is non-trivial, where $C_{F,S}$ is the S-class group of F obtained by considering ideals without any S-component. Here $C_{F,S}^{2n}$ is the subgroup consisting of the $2n$-multiples of elements in $C_{F,S}$. In particular, for example, if S-class number of F is divisible by 2, then the above condition holds.

Since we are working with groups of type C_n, for v not in S, the group $G_v := G \times_{\mathrm{Spec}\ F} \mathrm{Spec}\ F_v$ is split. We can assume that the groups G_d, PG extend to Chevalley group schemes over the local ring $\mathcal{O}_v$ for $v \in S'$. Thus, we have an exact sequence over $\mathcal{O}_v$, $v \in S'$ of group schemes,

$$1 \to \mathbf{G}_m \to G \to PG \to 1.$$

By Hilbert Theorem 90 for étale cohomology, we have a surjection $G(\mathcal{O}_v) \to PG(\mathcal{O}_v)$ for $v \in S'$. Further, from Bruhat–Tits theory, we know that the groups $PG(\mathcal{O}_v)$ are maximal compact and also maximal subgroups of $PG(F_v)$. We can also assume that the adele groups associated to G and PG are formed with respect to these classes of compact open subgroups.

Now let $K_m^S = \prod_{v \in S'} G(\mathcal{O}_v)$ be a maximal compact subgroup of the group of S-adeles $G(\mathbf{A}^S)$, the subgroup of the adele group $G(\mathbf{A})$ having no S-component. Here by $\mathbf{A}^S$, we mean the subgroup of adeles of F having no S-component. We choose a compact open subgroup of the form $K = K_S K^S$ satisfying the following:

- The group K_S is a compact open subgroup in $G_{S_f} := \prod_{v \in S_f} G(F_v)$.
- The group K^S is a compact open subgroup of K_m^S.
- The subgroup $K_d^S := K^S \cap G_d(\mathbf{A}^S)$ is normal in K_m^S.
- The arithmetic lattice $\Gamma_{d,K}$ is torsion-free.

This can be achieved by considering principal congruence subgroups at a finite collection of places not in S of F.

Since $\overline{K_m^S} = \prod_{v \in S'} PG(\mathcal{O}_v)$ is maximal in $PG(\mathbf{A}^S)$, it follows from the maximality of $PG(\mathcal{O}_v)$ and the fact that the groups $PG(F_v)$ are simple, that the normalizer $N(\overline{K_d^S})$ of $\overline{K_d}$ in $PG(\mathbf{A}^S)$ is precisely $\overline{K_m^S}$.

Thus, to produce an element $x \in G(\mathbf{A}_f)$ such that $\overline{x}$ does not belong to the double coset $N(\overline{K_d})PG(F)$, it is enough to show that x does not belong to the set $Z(\mathbf{A}_f)G_{S_f}K_m^S G(F)$. For this, it is enough to work with the S-adele component $x^S \in G(\mathbf{A}^S)$ of x.

The reduced norm map $\nu : G \to \mathbf{G}_m$ induces a surjection $\nu : G(\mathbf{A}_f) \to \mathbf{A}_f^*$. Hence, we need to show that the image group

$$\nu\left(Z(\mathbf{A}_f)G_{S_f}K_m^S G(F) \cap G(\mathbf{A}^S)\right)$$

is a proper subgroup of $(\mathbf{A}^S)^*$. Since the image lands in the subgroup $(\mathbf{A}^S)^{*2n}\nu(K_m^S)$ F^*, we need to know that the group

$$(\mathbf{A}^S)^*/(\mathbf{A}^S)^{*2n}F^*\nu(K_m^S)$$

is non-trivial. Its image in the S-class group $C_{F,S}$ of F lies in the subgroup $C_{F,S}^{2n}$. Hence, if $C_{F,S}/C_{F,S}^{2n}$ is non-trivial, we can find an element $x \in G(\mathbf{A}_f)$ such that $\overline{x}$ does not lie in the coset $N(\overline{K_d})G(F)$.

Remark. The above construction is similar to and can be compared with the construction given by Vignéras in [13], where she works with fields with class numbers divisible by 2.

Remark. It is of interest to produce examples of congruence subgroups K such that $N(\overline{K_d})PG(F)$ is not equal to $PG(\mathbf{A}_f)$ over an arbitrary field F without imposing any arithmetical restrictions on F. This certainly seems plausible. For instance, if we are working with an arithmetical Riemann surface associated to an adelic congruence subgroup $K \subset SL_1(D)(\mathbf{A}_f)$, where D is a quaternion division algebra defined over F, then $N(\overline{K})/\overline{K}$ gives raise to automorphisms of the Riemann surface. As K varies, generically one expects that this automorphism group is small. However, it seems to be quite delicate to produce such examples.

Let S be a sufficiently large set of places of F containing the archimedean places and also the places at which the group $G_d(F_v)$ is compact. The failure of strong approximation for the adjoint group PG [8, Proposition 7.13] implies that the set $PG_S PG(F)$ is not dense in $PG(\mathbf{A})$, where $PG_S = \prod_{v \in S} PG(\mathbb{F}_v)$. Thus, there exists a sufficiently small compact open subgroup $M \subset PG(\mathbf{A}^S)$ such that $MPG_S PG(F)$ is not equal to the full adele group $PG(\mathbf{A})$, where $PG(\mathbf{A}^S)$ is the adele group associated to PG without a S-component.

The problem can thus be reduced to the following question: given a non-Archimedean local field F, a reductive, non-anisotropic group G defined over F, and a congruence subgroup $L \subset G(F)$, to show that there exists a congruence subgroup K such that its normalizer in $G(F)$ is contained in L. D. Prasad has shown me how to construct such candidates of congruence subgroups with small normalizers: the details, however, are not only somewhat complicated but also have to be worked out.

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References

[1] H. Abelson, *Topologically distinct conjugate varieties with finite fundamental group*, Topology **13** (1974), 161–176.

[2] E. Artal Bartolo, J. Carmona Ruber and J. I. Cogolludo Agustín, *Effective invariants of braid monodromy*, Trans. Amer. Math. Soc. **359**(1) (2007), 165–183 (electronic).

[3] A. Borel, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. **224** (1966), 78–89.

[4] F. Charles, *Conjugate varieties with distinct real cohomology algebras*, J. Reine Angew. Math. **630** (2009), 125–139.

[5] R. W. Easton and R. Vakil, *Absolute Galois acts faithfully on the components of the moduli space of surfaces: a Belyi-type theorem in higher dimension*, Int. Math. Res. Not. IMRN (2007), no. 20, Art. ID rnm080, 10pp.

[6] J. S. Milne and J. Suh, *Nonhomeomorphic conjugates of connected Shimura varieties*, Amer. J. Math. **132**(3) (2010), 731–750.

[7] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, NJ, 1973. Annals of Mathematics Studies, No. 78.

[8] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory, Vol. 139 of Pure and Applied Mathematics*, Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[9] C. S. Rajan, *On isospectral arithmetical spaces*, Amer. J. Math. **129**(3) (2007), 791–806.

[10] J.-P. Serre, *Exemples de variétés projectives conjuguées non homéomorphes*, C. R. Acad. Sci. Paris **258** (1964), 4194–4196.

[11] I. Shimada, *Non-homeomorphic conjugate complex varieties*, in 'Singularities-Niigata–Toyama 2007', **56**, Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, 2009, 285–301.
[12] G. Shimura, *On canonical models of arithmetic quotients of bounded symmetric domains*, Ann. Math. **91**(2) (1970), 144–222.
[13] M.-F. Vignéras, *Variétés riemanniennes isospectrales et non isométriques*, Ann. Math. (2) **112**(1) (1980), 21–32.

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