OELJEKLAUS–TOMA MANIFOLDS ADMITTING NO COMPLEX SUBVARIETIES

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To Professor Vasile Brînzănescu at his sixty-fifth birthday

ABSTRACT. The Oeljeklaus–Toma (OT)-manifolds are complex manifolds constructed by Oeljeklaus and Toma from certain number fields, and generalizing the Inoue surfaces S_m. On each OT-manifold we construct a holomorphic line bundle with semipositive curvature form ω_0 and trivial Chern class. Using this form, we prove that the OT-manifolds admitting a locally conformally Kähler structure have no non-trivial complex subvarieties. The proof is based on the Strong Approximation theorem for number fields, which implies that any leaf of the null-foliation of ω_0 is Zariski dense.

CONTENTS

1. Introduction

1.1. OT-manifolds and their subvarieties. The Oeljeklaus–Toma (OT)-manifolds are an important class of compact complex manifolds not admitting a Kähler metric. They were discovered by Oeljeklaus and Toma in 2005 [5]. The construction of OT-manifolds uses the Dirichlet unit theorem from number theory (Subsection 1.2; see [7] for additional details of this construction and many related questions). Starting from a degree 3 number field, one obtains a two-dimensional OT-manifold known as Inoue surface S_m (see [2]).

For some number fields, the OT-manifolds are locally conformally Kähler (LCK). An LCK structure on a complex manifold is a Kähler metric on its universal cover $\tilde{M}$, such that the deck transform maps act on $\tilde{M}$ by homotheties. The OT-manifolds serve an important function in the theory of LCK manifolds, providing a counterexample

to a longstanding conjecture of I. Vaisman, [9], who asked whether there exists a compact, non-Kähler LCK-manifold M with all odd Betti numbers even: $b_{2p+1}(M) \equiv 0 \mod 2$. The OT manifolds in dimension 3 are the only known examples of compact LCK-manifolds with even odd Betti numbers, $b_1 = b_5 = 2, b_3 = 0$.

An OT-manifold is LCK if it is constructed from a number field K which has precisely two complex (non-real) embeddings, that is, two distinct homomorphisms $K \xrightarrow{\sigma, \overline{\sigma}} \mathbb{C}$. If the OT manifold has at least four complex embeddings and exactly one real, then it is not LCK. The remaining case is not yet decided.

Inoue surfaces S_m have no curves. We give a generalization of this theorem, proving that an OT-manifold that is LCK has no non-trivial complex subvarieties. In particular, it has no non-constant meromorphic functions (as a meromorphic function without polar set would be holomorphic, and hence constant).

Question 1.1. Is there any OT (non-LCK) manifold that has non-constant meromorphic functions?

The idea of the proof of this result is quite simple. We construct a holomorphic Hermitian line bundle, called *the weight bundle*, on any OT-manifold M. This bundle is topologically trivial, and has semipositive curvature form ω_0. The weight bundle also admits a flat connection, compatible with the holomorphic structure.

To learn about complex subvarieties of an OT-manifold, we study the zero-foliation Σ of ω_0, proving that all its leaves are Zariski dense in M. For an OT-manifold M constructed from a number field K admitting exactly $2t$ distinct complex (non-real) embeddings to $\mathbb{C}$, the leaves of Σ are t-dimensional. When $t = 1$, M is LCK, and Σ is one-dimensional. In this case, we prove that for any positive-dimensional complex subvariety $Z \subset M$, Z contains with each point $z \in Z$ the leaf Σ_z passing through z. Since all leaves of Σ are Zariski dense, the same is true for Z.

The weight bundle L is quite useful for many other purposes. As it was done in [12], one can take the α-th tensor power of L, denoted by L^α, for any real α; this power is well defined, because L is equipped with a natural C^∞-trivialization. The Gauduchon degree $\deg_g$ of L^α, taken with respect to any Gauduchon metric, satisfies $\frac{1}{\alpha} \deg_g L^\alpha = \deg_g L > 0$, hence M admits a line bundle with any prescribed Gauduchon degree. This implies, in particular, that the connected component of the Picard group $\mathrm{Pic}(M)$ is non-compact. Also, this implies that any vector bundle on M has degree zero after tensoring with an appropriate power of L; this is useful for the study of Hermitian–Einstein bundles on M, providing useful tools for the classification of stable bundles, and, eventually, coherent sheaves on M.

A similar argument was used in [12] to study holomorphic vector bundles and subvarieties on homogeneous elliptic fibrations, such as Calabi–Eckmann manifolds and quasi-regular Vaisman manifolds. We pose two questions, very much unsolved, but quite natural in the context presented by [12] and the present paper. Notice that from their construction it is clear that OT-manifolds are affine flat, that is, equipped with a flat, affine, torsion-free connection.

It is shown in [5, Remark 1.7] that some OT manifolds admit a holomorphic foliation with compact leaves which are again OT manifolds. Hence, it is natural to pose the following:

Question 1.2. Are the ones described in [5, Remark 1.7] the only OT manifolds with compact complex subvarieties? Can we classify these subvarieties? Are they always totally geodesic with respect to the flat affine connection?

Question 1.3. Does there exist a stable holomorphic vector bundle of rank > 1 on any OT-manifold of dimension > 2? Do all holomorphic vector bundles admit a flat connection, compatible with the holomorphic structure?

Remark 1.1. It is well known that generic complex tori have no non-trivial complex subvarieties. In [11], it was shown that all stable bundles on a generic complex torus of dimension > 2 have rank 1, and all holomorphic vector bundles admit flat connections. As for compact complex surface of non-Kählerian type, it is proven in [13] that stable holomorphic two-bundles with $c_1 = 0$ and $c_2 = n$ exist for any $n > 0$.

1.2. Number theory and the construction of OT-manifolds. Let $[K : \mathbb{Q}]$ be a number field, that is, a finite extension of $\mathbb{Q}$, of degree n, with $\sigma_1, \ldots, \sigma_s$ the real embeddings of K into $\mathbb{C}$, and $\sigma_{s+1}, \ldots, \sigma_n$ the complex embeddings. Since the complex embeddings of K into $\mathbb{C}$ occur in pairs of complex conjugate embeddings, the number $n - s$ is even, $n - s = 2t$. Let $\sigma = (\sigma_1, \ldots, \sigma_n) : K \to \mathbb{C}^{s+t}$ be the corresponding group homomorphism.

Let $\mathcal{O}_K$ be the ring of algebraic integers of K, $\mathcal{O}_K^*$ its multiplicative group of units and $\mathcal{O}_K^{*;+}$ the group of units which are positive in all the real embeddings of K.

Denote by $\mathbb{H}$ the upper complex half-plane. Using the Dirichlet's unit theorem, Oeljeklaus and Toma proved that $\mathcal{O}_K \rtimes \mathcal{O}_K^{*;+}$ acts freely on $\mathbb{H}^s \times \mathbb{C}^t$ by

$$T_a(z_i) = (z_i + \sigma_i(a)), \quad i = 1, \ldots, s + 2t, \quad a \in \mathcal{O}_K,$$
$$R_u(z_i) = (\sigma_i(u)z_i), \quad i = 1, \ldots, s + 2t, \quad u \in \mathcal{O}_K^{*;+}.$$

(see [5, 7]). Moreover, an *admissible* subgroup $U \subset \mathcal{O}_K^{*;+}$ can always be found such that the action of $\Gamma := \mathcal{O}_K \rtimes U$ is also properly discontinuous. For $t = 1$, every U of finite index in $\mathcal{O}_K^{*;+}$ has this property.

Definition 1.1. The manifold $M_K := (\mathbb{H}^s \times \mathbb{C}^t)/\Gamma$ is called an **OT manifold**. It is a compact complex manifold of dimension $s + 2t$.

For $s = t = 1$, M_K reduces to an Inoue surface S_m (where m is a matrix in $\mathrm{SL}(3, \mathbb{Z})$), see [2]. The corresponding number field K is $\mathbb{Q}[T]/P_m(t)$, where $P_m(t)$ is the characteristic polynomial of the matrix m. It is shown in [5] that the manifolds M_K are never Kähler, but that for $t = 1$, M_K is an LCK manifold (see [1] and the more recent survey [6] for definitions and results in LCK geometry). We briefly explain the construction of this LCK metric.

Clearly, the function $\psi(z) = \prod_{i=1}^s (\mathrm{im}\, z_i) + |z_{s+1}|^2$ is plurisubharmonic on $\mathbb{H}^s \times \mathbb{C}$. It defines the Kähler form $\Omega := \partial\bar{\partial}\,\psi$ on $\mathbb{H}^s \times \mathbb{C}$. The group Γ acts on $(\mathbb{H}^s \times \mathbb{C}, \Omega)$ by homotheties:

$$T_a^*\Omega = \Omega,$$
$$R_u^*\Omega = |\sigma_{s+1}(u)|^2\Omega.$$

Let now $\chi : \Gamma \to \mathbb{R}^{>0}$ be the character $\chi(\gamma) = \frac{\gamma^*\Omega}{\Omega}$. We call *automorphic* any p-form $\eta \in \Lambda^p(\mathbb{H}^s \times \mathbb{C})$ which satisfies $\gamma^*\eta = \chi(\gamma)\eta$. For any automorphic function φ on $\mathbb{H}^s \times \mathbb{C}$, the quotient $\frac{\Omega}{\varphi}$ is Γ-invariant and hence projects to an LCK metric ω on M_K. This form satisfies the equation $d\omega = \theta \wedge \omega$, for the closed 1-form θ (called *the Lee form*), which is the projection on M_K of $\tilde{\theta} = -d\log\varphi$:

$$d\omega = -\frac{d\varphi}{\varphi^2} \wedge \tilde{\omega} = -d(\log\varphi) \wedge \omega.$$

It is easily seen that the function $\varphi = \prod_{i=1}^s (\operatorname{im} z_i)^{-1}$ is automorphic, and hence it produces a LCK metric on M_K as described above. This LCK metric generalizes the one constructed by Tricerri on S_m, [8].

The main result of this paper shows that, just as Inoue surfaces S_m have no complex curves, OT-manifolds have no complex subvarieties:

Theorem 1.1. *Let $[K : \mathbb{Q}]$ be a number field of degree $n = s+2$, with s real embeddings and two complex embeddings, and M_K the corresponding LCK OT-manifold. Then M_K has no non-trivial complex subvarieties.*

Proof. See Theorem 3.1. ∎

Corollary 1.1. *The LCK OT-manifold M_K has no non-constant meromorphic functions.*

2. The weight bundle of an OT-manifold

Definition 2.1. Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2t$, with s real embeddings and $2t$ complex embeddings, and $M_K = \mathbb{H}^s \times \mathbb{C}^t / \Gamma$ the associated OT-manifold. Denote by $z_1, \ldots, z_s$ the standard complex coordinates on $\mathbb{H}^s$, and let $\tilde{\theta} := -d\log\prod_{i=1}^s (\operatorname{im} z_i)$. It is easy to see that the form $\tilde{\theta}$ is Γ-invariant. Therefore it is obtained as a lift of a form θ, called *the Lee form* of the OT-manifold. When $t = 1$, this is the Lee form constructed above.

Let M_K be an OT-manifold, and θ its Lee form. Consider a trivial Hermitian line bundle L with connection $\nabla := \nabla_0 + \sqrt{-1}\,\theta^c$, where $\theta^c := I\theta$, and ∇_0 is the trivial connection on L. Clearly, ∇ is Hermitian, and $\nabla^{0,1} = \overline{\partial} + \theta^{0,1}$, where $\theta^{0,1}$ is the (0,1)-part of θ.

Claim 2.1. *In these assumptions, the curvature ω_0 of ∇ is $-\sqrt{-1}\,d\theta^c$. Moreover, this form is of type (1,1).*

Proof. A simple computation shows that in the standard coordinates $z_1, \ldots z_s$, $z_{s+1}, \ldots z_{s+t}$, ω_0 can be written as follows:

$$\omega_0 = \sqrt{-1}\,\partial\overline{\partial}\log\varphi = \sqrt{-1}\,\sum_{i=1}^s \frac{dz_i \wedge d\overline{z}_i}{|\operatorname{im} z_i|^2},$$

∎

Definition 2.2. Let M_K be an OT-manifold, and L the holomorphic Hermitian bundle defined above. Then L is called *the weight bundle* of M_K.

We restate Claim 2.1 as

Theorem 2.1. *Let M_K be an OT-manifold, and L its weight bundle with the holomorphic Hermitian structure and the Chern connection ∇ defined above. Consider the form $\omega_0 := \sqrt{-1}\,\nabla^2$. Then ω_0 is a semi-positive form, which can be written in the standard coordinates $z_1, \dots z_s, z_{s+1}, \dots, z_{s+t}$ as follows:*

$$\omega_0 = \sqrt{-1}\,\partial\bar{\partial}\log\varphi = \sqrt{-1}\sum_{i=1}^{s}\frac{dz_i \wedge d\bar{z}_i}{|\operatorname{im} z_i|^2}$$

Remark 2.1. The Vaisman manifolds are, by definition, LCK manifolds (M, I, g) satisfying the additional condition $\nabla^g\theta = 0$, where ∇^g is the Levi–Civita connection of an LCK metric g. For all Vaisman manifolds, the two-form $\omega_0 = d\theta^c$ is semi-positive, being zero only on the direction of $\theta^\sharp - I\theta^\sharp$. This is a general fact, proven in [10], independent of the particular form of θ. OT-manifolds are far from being Vaisman (they never admit any Vaisman metric), but the particular expression of their Lee form gives ω_0 the same property as for Vaisman manifold. This is what inspired our construction.

Remark 2.2. An object of interest in conformal geometry and, in particular, LCK geometry is the *weight bundle*. It is the real line bundle $L \longrightarrow M$ associated to the representation $\mathrm{GL}(2n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}$ (see [6]). Then L can be complexified and endowed with the Chern connection $\nabla_0 + \sqrt{-1}\,\theta^c$ (where ∇_0 is the trivial connection). It can be verified that $\omega_0 = \sqrt{-1}\nabla^2$, and hence ω_0 can be seen as the curvature form of this Chern connection. When $t = 1$ and M_K is the corresponding LCK OT-manifold, this construction gives the weight bundle introduced in Definition 2.2.

Remark 2.3. For any OT-manifold M, in addition to the Chern connection $\nabla = \nabla_0 + \sqrt{-1}\,\theta^c$, the weight bundle L also admits the connection $\nabla_0 + \theta$, which is flat because $d\theta = 0$. It is clear that the $(0,1)$-part of ∇ coincides with the $(0,1)$-part of this flat connection.

The following claim is obvious from the explicit form of ω_0 (Theorem 2.1).

Claim 2.2. In the assumptions of Theorem 2.1, let $\tilde{\Sigma}$ be the holomorphic foliation on the covering $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$ generated by the vector fields $\frac{\partial}{\partial z_{s+1}}, \dots, \frac{\partial}{\partial z_{s+t}}$. Then:

    (i) The foliation $\tilde{\Sigma}$ is Γ-invariant, hence it is obtained as the pullback of a holomorphic foliation Σ on $M_K = \tilde{M}_K/\Gamma$.

    (ii) The foliation Σ is the null-space of the form ω_0 constructed above.

Claim 2.3. Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2$, with s real embeddings and two complex embeddings, M_K the corresponding LCK OT-manifold, and $\Sigma \subset$

TM_K the holomorphic foliation defined in Claim 2.2. Consider a complex closed subvariety $Z \subset M_K$. Then Σ is tangent to Z at any point of Z:

$$(2.1) \qquad\qquad \forall z \in Z, \quad \Sigma\big|_z \subset T_z Z.$$

Proof. The form ω_0 has $(n-1)$ positive eigenvalues, where $n = \dim_{\mathbb{C}} M_K$, and its zero eigenspace at z is $\Sigma\big|_z$. Unless (2.1) holds at $z \in Z$, the restriction $\omega_0\big|_z$ has $m = \dim Z$ positive eigenvalues at z. Then $\int_Z \omega_0^m > 0$. This is impossible, because ω_0 is exact. $\qquad\square$

Corollary 2.1. *In assumption of Claim 2.3, let Σ_z be a leaf of Σ passing through $z \in Z$. Then $\Sigma_z \subset Z$.*

3. Complex subvarieties in LCK OT-manifold

Using Corollary 2.1, we can easily prove the main result of this paper.

Theorem 3.1. *Let $[K : \mathbb{Q}]$ be a number field of degree $n = s+2$, with s real embeddings and two complex embeddings, and let M_K be the corresponding OT-manifold. Then M_K has no non-trivial complex subvarieties.*

Proof. Theorem 3.1 follows from Corollary 2.1 and the following more general proposition. $\qquad\square$

Proposition 3.1. *Let $[K : \mathbb{Q}]$ be a number field of degree $n = s+2t$, $t > 0$, with s real embeddings and $2t$ complex embeddings, and let $M_K = \mathbb{H}^s \times \mathbb{C}^t / \Gamma$ be the associated (non-Kähler) OT-manifold. Let $\Sigma \subset TM_K$ be the foliation defined in Claim 2.2. Consider a leaf of Σ, and let Z be its closure. Then*

  (i) *The preimage $\pi^{-1}(Z)$ of Z in $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$ contains the set*

  $$Z_{\alpha_1, \ldots, \alpha_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid \operatorname{im} z_i = \alpha_i\}$$

   for some positive numbers $(\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$.
  (ii) *Any complex subvariety of M_K containing Z must coincide with M_K.*

Proof. The implication (i) $\Rightarrow$ (ii) is clear, because any complex manifold containing $Z_{\alpha_1, \ldots, \alpha_s}$ must have the same dimension as M_K. The proof of (i) is a bit more elaborate.

Let $\mathcal{O}$ be the ring of integers in K. By construction, the group $\Gamma = \pi_1(M_K)$ is a cross-product of the additive group $\mathcal{O}^+$ of $\mathcal{O}$ with a subgroup of the multiplicative group $\mathcal{O}^*$. Let $\tilde{\Sigma}$ be the pullback of the foliation Σ to $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$. A leaf of $\tilde{\Sigma}$ is given as

$$T_{t_1, \ldots, t_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid z_i = t_i\},$$

for some $(t_1, \ldots, t_s) \in \mathbb{H}^s$. Let $\tilde{Z} := \pi^{-1}(Z)$ be the preimage of the corresponding closure of a leaf of Σ. Clearly, $\tilde{Z}$ is the closure of $\Gamma(T_{t_1, \ldots, t_s})$. Therefore, to prove Proposition 3.1 (i) it is sufficient to show that the closure of $\Gamma(T_{t_1, \ldots, t_s})$ contains $Z_{\alpha_1, \ldots, \alpha_s}$. In fact, even the smaller group $\mathcal{O}^+ \subset \Gamma$ will suffice, as seen from the following lemma, which proves Proposition 3.1.

Lemma 3.1. *Let* $[K : \mathbb{Q}]$ *be a number field of degree* $n = s + 2t$, $t > 0$ *with* s *real embeddings and* $2t$ *complex embeddings, and* $\tilde{M}_K := \mathbb{H}^s \times \mathbb{C}^t$, *equipped with the action of* $\mathcal{O}^+$ *as in Subsection 1.2. Consider the subset*

$$T_{t_1,\ldots,t_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid z_i = t_i\},$$

in $\tilde{M}_K$. *Then the closure of* $\mathcal{O}^+(T_{t_1,\ldots,t_s})$ *coincides with*

$$Z_{\alpha_1,\ldots,\alpha_s} := \{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid \operatorname{im} z_i = \alpha_i,\},$$

with $\alpha_i := \operatorname{im} t_i$.

Proof. Equivalently, we may state that the closure of an orbit of the standard action of $\mathcal{O}^+$ in $\mathbb{H}^s$ is the set $\{(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{s+t}) \mid \operatorname{im} z_i = \alpha_i\}$. This in turn is equivalent to the following. $\qquad\square$

Lemma 3.2. (*cf.* [5, Claim following Lemma 2.4]) *Let* $[K : \mathbb{Q}]$ *be a number field of degree* $n = s + 2t$, $t > 0$ *with* s *real embeddings* $\sigma_1, \ldots, \sigma_s$ *and* $2t$ *complex embeddings. Consider the additive group* $\mathcal{O}^+$ *of the corresponding ring of integers. Let* $\sigma : \mathcal{O}^+ \longrightarrow \mathbb{R}^s$ *map* ξ *to* $\sigma_1(\xi), \ldots, \sigma_s(\xi)$. *Then the image of* $\mathcal{O}^+$ *is dense in* $\mathbb{R}^s$.

Proof.[1] Let K be a number field, $\mathcal{O}_K$ its ring of integers, $\mathfrak{P}$ the set of all prime ideals of $\mathcal{O}_K$, V the product of all archimedean completions of K, and V_1 the product of some, but not all, archimedean completions. Denote by $\mathcal{O}_\nu$ the completion of $\mathcal{O}_K$ at $\nu \in \mathfrak{P}$, and let K_ν be the corresponding local field. Consider the adele space $\mathfrak{A}$, obtained as a subset of the product $V \times \prod_{\nu \in \mathfrak{P}} K_\nu$, where all components, except finitely many, belong to $\mathcal{O}_\nu$, and let $\mathfrak{A}_1$ be the image of projection of $\mathfrak{A}$ to $V_1 \times \prod_{\nu \in \mathfrak{P}} K_\nu$. Denote by $\tau : K \longrightarrow \mathfrak{A}_1$ the natural homomorphism, which is tautological componentwise.

From the Strong Approximation theorem (see [3] or [4, Theorem 20.4.4][2]) it follows that the image $\tau(K)$ of K is dense in $\mathfrak{A}_1$. Let

$$\mathcal{O}_{\mathfrak{A}_1} := \mathfrak{A}_1 \cap \left(V_1 \times \prod_{\nu \in \mathfrak{P}} \mathcal{O}_\nu \right)$$

be the set of points of $\mathfrak{A}$, corresponding to the integer adeles. Clearly, $\mathcal{O}_{\mathfrak{A}_1}$ is open in $\mathfrak{A}_1$. Therefore, the intersection $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1}$ is dense in $\mathcal{O}_{\mathfrak{A}_1}$. On the other hand, $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1}$ consists of those elements of the number field which are integer at all non-archimedean places. This gives $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1} = \tau(\mathcal{O}_K)$. Therefore, the image of $\mathcal{O}_K$ to V_1 is dense. $\qquad\square$

Remark 3.1. The above argument actually proves that the image of $\mathcal{O}_K$ in the product V_1 of all archimedean completions of K except one is dense in V_1.

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[2]http://modular.fas.harvard.edu/papers/ant/html/node84.html

Acknowledgments

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