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SEQUENCES OF LCT-POLYTOPES

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Abstract. To r ideals on a germ of smooth variety X one attaches a rational polytope in $\mathbf{R}_+^r$ (the *LCT-polytope*) that generalizes the notion of log canonical threshold in the case of one ideal. We study these polytopes, and prove a strong form of the Ascending Chain Condition in this setting: we show that if a sequence $(P_m)_{m\geq 1}$ of LCT-polytopes in $\mathbf{R}_+^r$ converges to a compact subset Q in the Hausdorff metric, then $Q = \bigcap_{m\geq m_0} P_m$ for some m_0, and Q is an LCT-polytope.

1. Introduction

Let X be a smooth algebraic variety over an algebraically closed field k, of characteristic zero. To a nonzero ideal $\mathfrak{a}$ on X, and to a point x in the zero locus of $\mathfrak{a}$ one associates the local log canonical threshold $\mathrm{lct}_x(\mathfrak{a})$. This positive rational number is an invariant of the singularities of $\mathfrak{a}$ at x that plays a fundamental role in birational geometry (see, e.g., [Kol2] and [EM]).

To r ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ on X, and to a point x that lies in the zero locus of each $\mathfrak{a}_i$ we associate the *LCT-polytope* $\mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$. This is a rational convex polytope in $\mathbf{R}_+^r$ that describes the log canonical thresholds at x of all products $\mathfrak{a}_1^{m_1} \cdots \mathfrak{a}_r^{m_r}$. More precisely, it consists of those $(\lambda_1, \ldots, \lambda_r) \in \mathbf{R}_+^r$ such that the pair $(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r})$ is log canonical at x. In the case $r = 1$, the polytope $\mathrm{LCT}_x(\mathfrak{a})$ is the segment $[0, \mathrm{lct}_x(\mathfrak{a})]$. These polytopes are a special case of the polytopes of quasi-adjunction introduced and studied by the first author in [Lib1] and [Lib2]. Even if one is only interested in the singularities of one ideal $\mathfrak{a}$, studying the LCT-polytopes $\mathrm{LCT}(\mathfrak{a}, \mathfrak{b})$ for various auxiliary ideals $\mathfrak{b}$ gives important information.

Shokurov conjectured in [Sho] that log canonical thresholds in fixed dimension satisfy the Ascending Chain Condition. The conjecture is made in a general setting in which the ambient variety is allowed to have log canonical singularities. Birkar related the general form of the conjecture to the Termination of Flips conjecture (see [Bir] for the precise statement). In the special setting of smooth ambient varieties, Shokurov's conjecture was proved by de Fernex, Ein and the third author in [dFEM], building on ideas and results from [dFM] and [Kol1].

In this note, we consider the Ascending Chain Condition for LCT-polytopes. In particular, we show that given any sequence of LCT-polytopes in $\mathbf{R}^r$ (corresponding to ideals on smooth n-dimensional varieties) $P_1 \subseteq P_2 \subseteq \cdots$, the sequence is eventually stationary. In fact, we prove a much stronger assertion.

We consider the polytopes in $\mathbf{R}^r$ as elements in the space $\mathcal{H}_r$ of all compact subsets of $\mathbf{R}^r$ endowed with the Hausdorff metric. This is a complete metric space, and the subsets lying in a given compact subset $K \subset \mathbf{R}^r$ form a compact subspace of $\mathcal{H}_r$. It is

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easy to see that every LCT-polytope as above is contained in the cube $[0,n]^r \subseteq \mathbf{R}^r$. It follows that every sequence of LCT-polytopes has a convergent subsequence to some compact subset $Q \subseteq [0,n]^r$.

Our main result says that if a sequence of LCT-polytopes $(P_m)_{m\geq 1}$ converges to the compact set Q in the Hausdorff metric, then there is m_0 such that $Q = \cap_{m\geq m_0} P_m$. Furthermore, Q is a rational convex polytope. In fact, there are ideals $\mathfrak{a}_1, \ldots \mathfrak{a}_s \subset K[\![x_1, \ldots, x_n]\!]$ (for some $s \leq r$ and some field extension K of k) such that $Q = \mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_s)$ (under a suitable linear embedding in $\mathbf{R}^r$). If the ground field k has infinite transcendence degree over $\mathbf{Q}$ (for example, if $k = \mathbf{C}$), then we may take $K = k$.

The proof uses the result in [dFEM] about the ACC property of log canonical thresholds on smooth varieties of fixed dimension. In fact, we use in an essential way also the ideas and the constructions in *loc. cit.* We give an introduction to the basic properties of LCT-polytopes in the following section, emphasizing the analogy with the case $r = 1$. The main theorems are proved in the last section.

2. Basics of LCT-polytopes

In this section we present some basic results about LCT-polytopes. We always work over an algebraically closed field k, of characteristic zero. We denote by $\mathbf{R}_+$ the set of nonnegative real numbers, and by $\mathbf{N}$ the nonnegative integers. Our ambient space X is either a smooth variety over k, or $\mathrm{Spec}(k[\![x_1, \ldots, x_n]\!])$. We assume that the reader is familiar with the results about the usual log canonical threshold, for which we refer to [Kol2], Section 8 for the finite type case, and to [dFM] for the case of formal power series.

Let X be a regular scheme, as above, and $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ nonzero ideal sheaves on X. We put

$$\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r) = \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbf{R}_+^r \mid (X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}) \text{ is log canonical}\}.$$

We will mostly be concerned with a local variant of this definition: if $x \in X$ is a closed point, then

$$\mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_r) = \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbf{R}_+^r \mid (X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r}) \text{ is log canonical at } x\}.$$

If the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are principal, with $\mathfrak{a}_i$ generated by f_i, then we simply write $\mathrm{LCT}(f_1, \ldots, f_r)$ and $\mathrm{LCT}_x(f_1, \ldots, f_r)$.

The above sets can be explicitly described in terms of a log resolution, as follows. Suppose that $\pi \colon Y \to X$ is a log resolution of $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$. Recall that this means that Y is nonsingular, π is proper and birational, and we have a simple normal crossings divisor $\sum_{j=1}^N E_j$ on Y such that

$$K_{Y/X} = \sum_{j=1}^N \kappa_j E_j \quad \text{and} \quad \mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y\left(-\sum_{j=1}^N \alpha_{i,j} E_j\right), \quad \text{for } 1 \leq i \leq r.$$

The existence of such a log resolution in the formal power series case is a consequence of the results in [Tem].

It follows from the description of log canonical pairs in terms of a log resolution that $\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ consists precisely of those $\lambda \in \mathbf{R}_+^r$ such that

$$\sum_{i=1}^{r} \alpha_{i,j}\lambda_i \le \kappa_j + 1 \text{ for } 1 \le j \le N. \tag{1}$$

Similarly, $\mathrm{LCT}_x(\mathfrak{a}_1, \ldots \mathfrak{a}_r)$ is cut out by the equations in (1) corresponding to those j such that $x \in \pi(E_j)$.

It follows from the above description that both $\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ and LCT_x $(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ are rational polyhedra (that is, they are cut out in $\mathbf{R}^r$ by finitely many affine linear inequalities, with rational coefficients). We call $\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ and $\mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ the *LCT-polyhedron* of $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$, and respectively, the *LCT-polyhedron at* x of $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$.

Remark 2.1. The above polyhedra are r-dimensional. Indeed, note that they contain the origin, as well as λe_i for $0 < \lambda \ll 1$ (here $e_1, \ldots, e_r$ is the standard basis of $\mathbf{R}^r$).

The following lemma follows immediately from the description of LCT-polyhedra in terms of a log resolution.

Lemma 2.2. *Given the nonzero ideals* $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$*, there are closed points* $x_1, \ldots, x_m \in X$ *such that*

$$\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r) = \bigcap_{j=1}^{m} \mathrm{LCT}_{x_j}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r).$$

Because of this lemma, from now on we will focus on the local LCT-polyhedra.

Lemma 2.3. *Let* $\mathfrak{a}_1, \ldots \mathfrak{a}_r$ *be nonzero ideals on* X.

(i) If $x \in \mathrm{Supp}(V(\mathfrak{a}_i))$*, then* $\{\lambda_i \mid \lambda \in \mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)\}$ *is bounded.*

(ii) If $x \notin \mathrm{Supp}(V(\mathfrak{a}_r))$*, then* $\mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_r) = \mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_{r-1}) \times \mathbf{R}_+$.

Proof. With the notation in (1), we see that if $x \in \mathrm{Supp}(V(\mathfrak{a}_i))$, then there is j with $\alpha_{i,j} > 0$, and such that $x \in \pi(E_j)$. It follows that if $\lambda \in \mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$, then $\lambda_i \le (\kappa_j + 1)/\alpha_{i,j}$, which gives ($i$). The assertion in (ii) is clear. ☐

In light of this lemma, it is enough to study the sets $\mathrm{LCT}_x(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ for $x \in \bigcap_i \mathrm{Supp}(V(\mathfrak{a}_i))$. In this case, we see that the LCT-polyhedron at x of $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ is bounded, hence it is a polytope. We will henceforth refer to it as the *LCT-polytope at* x of $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$.

Remark 2.4. A related construction, giving polyhedra as invariants of tuples of divisors, was used in [Lib2] and [Lib1]. Consider a collection of germs

$$f_1(x_1, \ldots, x_{n+1}), \ldots, f_r(x_1, \ldots, x_{n+1}),$$

of reduced local equations of divisors $D_i = V(f_i)$ at a point $P \in X = \mathbf{C}^{n+1}$, that we assume to have isolated non-normal crossings (cf. [Lib2]). With each $\varphi \in \mathcal{O}_P$ one associates the top degree form:

$$\begin{aligned} &\omega_\varphi(j_1, \ldots, j_r | m_1, \ldots, m_r) \\ &\quad = f_1^{\frac{j_1 - m_1 + 1}{m_1}} \cdot \ldots \cdot f_r^{\frac{j_r - m_r + 1}{m_r}} \varphi(x_1, \ldots, x_{n+1}) dx_1 \wedge \ldots \wedge dx_{n+1}, \end{aligned} \tag{2}$$

on the unramified covering $X_{m_1,\dots,m_r}$ of $X \smallsetminus \sum_i D_i$ with Galois group $\oplus_i \mathbf{Z}/m_i\mathbf{Z}$. The form ω_φ extends to a holomorphic form on a resolution of singularities of a compactification $\overline{X}_{m_1,\dots,m_r}$ of $X_{m_1,\dots,m_r}$ if and only if $(\frac{j_1+1}{m_1},\dots,\frac{j_r+1}{m_r}) \in \mathbf{R}^r$ satisfies a system of linear inequalities, i.e., it belongs to a polytope $\mathcal{P}(\varphi|f_1,\dots,f_r)$. This system can be described in terms of a log-resolution $\pi\colon Y \to X$ of the principal ideal $(f_1\cdots f_r)$ as above, using the resolution of $\overline{X}_{m_1,\dots,m_r}$ given by a resolution of the quotient singularities of the normalization of $\overline{X}_{m_1,\dots,m_r} \times_X Y$. This leads to the following explicit collection of inequalities describing when $\lambda = (\lambda_1,\dots,\lambda_r) \in \mathcal{P}(\varphi|f_1,\dots,f_r)$ (cf. [Lib1, (4)]):

$$\sum_{i=1}^{r} \alpha_{i,j}(1-\lambda_i) \leq \kappa_j + 1 + e_j(\varphi) \text{ for } 1 \leq j \leq N. \tag{3}$$

Here $\alpha_{i,j}$, κ_j are as in (1), and $e_j(\varphi)$ is the multiplicity of $\pi^*(\varphi)$ along E_j.

Vice versa, for a fixed $(\frac{j_1+1}{m_1},\dots,\frac{j_r+1}{m_r})$ with $0 \leq j_i < m_i$ for all i, the set of $\varphi \in \mathcal{O}_P$ such that the given point lies in $\mathcal{P}(\varphi|f_1,\dots,f_r)$ is an ideal $\mathcal{A}(j_1,\dots,j_r|m_1,\dots,m_r) \subset \mathcal{O}_P$ (an *ideal of quasi-adjunction*).

Allowing φ to run over all elements in $\mathcal{O}_P$ produces a *finite* collection of polytopes in the $[0,1]^r$. We similarly have a finite collection of ideals of quasi-adjunction. Moreover, every ideal of quasi-adjunction $\mathcal{A}$ can be written as $\mathcal{A} = \mathcal{A}(j_1,\dots,j_r|m_1,\dots,m_r)$ for some point $(\frac{j_1+1}{m_1},\dots,\frac{j_r+1}{m_r})$ that can be chosen in the boundary of a polytope (3). The subset of the boundary consisting of those $(\frac{j_1+1}{m_1},\dots,\frac{j_r+1}{m_r})$ defining a particular $\mathcal{A}$ is a polyhedral subset (face of quasi-adjunction). Therefore one has a correspondence between faces $\mathcal{F}$ of the polytopes $\mathcal{P}(\varphi|f_1,\dots,f_r)$ and certain ideals $\mathcal{A}(\mathcal{F})$ in $\mathcal{O}_P$.

The polytope (3) corresponding to $\varphi = 1$ coincides with the image of the LCT-polytope (1) for $\mathfrak{a}_i = (f_i)$ via the affine map $(\lambda_i) \to (1-\lambda_i)$. An ideal of quasi-adjunction $\mathcal{A}(\mathcal{F})$ associated to a point $(\frac{j_1+1}{m_1},\dots,\frac{j_r+1}{m_r}) \in \mathcal{F}$ coincides with the multiplier ideal of the divisor $\sum \mu_i D_i$, where $\mu_i = 1 - \frac{j_i+1}{m_i} - \varepsilon$, with $0 < \varepsilon \ll 1$. Indeed, strict inequality in the conditions (3) is equivalent to φ being a section of $\pi_*(K_{Y/X} - \lfloor \sum_i (1-\lambda_i)\pi^*(D_i)\rfloor)$. In the case $r=1$, each polytope (3) is a segment $[\alpha,1]$, and the face of quasi-adjunction α is a jumping coefficient for the multiplier ideals of $f = f_1$. If the singularity of f at P is isolated, the collection of such α coincides with the subset of the spectrum of the singularity of f in the interval $[0,1]$.

Example 2.5. If $r=1$, then $\mathrm{LCT}(\mathfrak{a}) = [0, \mathrm{lct}(\mathfrak{a})]$, and $\mathrm{LCT}_x(\mathfrak{a}) = [0, \mathrm{lct}_x(\mathfrak{a})]$.

Example 2.6. If $\mathfrak{a}_i = (x_1^{q_{i,1}}\cdots x_n^{q_{i,n}}) \subseteq k[x_1,\dots,x_n]$, then

$$\mathrm{LCT}(\mathfrak{a}_1,\dots,\mathfrak{a}_r) = \left\{ \lambda = (\lambda_1,\dots,\lambda_r) \in \mathbf{R}_+^r \;\middle|\; \sum_{i=1}^{r} q_{i,j}\lambda_i \leq 1 \text{ for } 1 \leq j \leq n \right\}.$$

Example 2.7. One can generalize the previous example to the case of arbitrary monomial ideals. This extends Howald's Theorem from [How], which is the case $r=1$. Suppose that $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ are nonzero ideals in $k[x_1,\dots,x_n]$ generated by monomials. Let $P_{\mathfrak{a}_i}$ denote the Newton polyhedron of $\mathfrak{a}_i$, that is, $P_{\mathfrak{a}_i}$ is the convex hull of $\{u \in \mathbf{N}^n \mid x^u \in \mathfrak{a}_i\}$. Here, if $u = (u_1,\dots,u_n) \in \mathbf{N}^n$, we denote by x^u the monomial

$x_1^{u_1}\cdots x_n^{u_n}$. By taking a toric resolution of $\mathfrak{a}_1\cdot\ldots\cdot\mathfrak{a}_r$, it is easy to see that

$$\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)=\mathrm{LCT}_0(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)=\left\{(\lambda_1,\ldots,\lambda_r)\in\mathbf{R}_+^r\;\middle|\; e\in\sum_{i=1}^r\lambda_i P_{\mathfrak{a}_i}\right\},$$

where $e=(1,\ldots,1)\in\mathbf{R}^n$.

Example 2.8. In the case of plane curves, readily available explicit resolutions allow the computation of LCT-polytopes. In terms of the polytopes of quasi-adjunction considered in [Lib1], the LCT-polytope is the image of the polytope "farthest" from the origin along the line $x_1=\ldots=x_r$ under the change of variables $(\lambda_i)\to(1-\lambda_i)$.

a) If $f=x$, $g=x-y^2\in k[x,y]$, then

$$\mathrm{LCT}_0(f,g)=\{(\lambda_1,\lambda_2)\in\mathbf{R}_+^2\mid\lambda_1\le 1,\ \lambda_2\le 1,\ \lambda_1+\lambda_2\le 3/2\}. \tag{4}$$

b) If $f=x^2+y^5$, $g=x^5+y^2\in k[x,y]$, then $\mathrm{LCT}_0(f,g)$ is the intersection of the unit square and of the half planes

$$10\lambda_1+4\lambda_2\le 7,\quad 4\lambda_1+10\lambda_2\le 7. \tag{5}$$

Remark 2.9. Even if one is interested in the singularities of an ideal $\mathfrak{a}$, considering the LCT-polytopes for several ideals gives interesting information. Suppose, for example, that $\mathfrak{a}$ is a nonzero ideal on X, and $x\in X$ is a closed point in $\mathrm{Supp}(V(\mathfrak{a}))$. One defines a function $\varphi\colon\mathbf{R}_+\to\mathbf{R}_+$ by $\varphi(t)=\mathrm{lct}_x(\mathfrak{a}\cdot\mathfrak{m}_x^t)^{-1}$, where $\mathfrak{m}_x$ is the ideal defining x. This is a convex nondecreasing function that encodes useful information about the singularities of $\mathfrak{a}$ at x. For example, one can show that the right derivative $\varphi_r'(0)$ is equal to $\mathrm{lct}_x(\mathfrak{a})^{-1}\cdot\max\frac{\mathrm{ord}_E(\mathfrak{m}_x)}{\mathrm{ord}_E(\mathfrak{a})}$, where the maximum is over all divisors E over X that compute $\mathrm{lct}_x(\mathfrak{a})$.

Note that φ is determined by $P:=\mathrm{LCT}_x(\mathfrak{a},\mathfrak{m}_x)$, and conversely. Indeed, $\varphi(t)=\alpha$ if and only if $\mathrm{lct}(\mathfrak{a}^{1/\alpha}\cdot\mathfrak{m}_x^{t/\alpha})=1$. Therefore $\varphi(t)$ is characterized by the fact that $(1,t)$ lies on the boundary of $\varphi(t)\cdot P$.

We record in the following proposition some general properties of LCT-polytopes. We denote by $e_1,\ldots,e_r$ the standard basis in $\mathbf{R}^r$. For $\lambda=(\lambda_i)$ and $\mu=(\mu_i)$ in $\mathbf{R}_+^r$, we put $\lambda\preceq\mu$ if $\lambda_i\le\mu_i$ for all i. We also put $\lambda\prec\mu$ if $\lambda_i\le\mu_i$ for all i, with strict inequality when $\mu_i>0$.

Proposition 2.10. *Suppose that $\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ are nonzero ideals on X, and $x\in X$ is a closed point such that $x\in\mathrm{Supp}(V(\mathfrak{a}_i))$ for all i.*

(i) If $m_1,\ldots,m_r$ are positive integers, then the polytope $\mathrm{LCT}_x(\mathfrak{a}_1^{m_1},\ldots,\mathfrak{a}_r^{m_r})$ is equal to the image of $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$ by the map $(u_1,\ldots,u_r)\to(u_1/m_1,\ldots,u_r/m_r)$.

(ii) If $\mathfrak{a}_i'\subseteq\mathfrak{a}_i$ for every i, then $\mathrm{LCT}_x(\mathfrak{a}_1',\ldots,\mathfrak{a}_r')\subseteq\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$.

(iii) $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)\subseteq\prod_{i=1}^r[0,\mathrm{lct}_x(\mathfrak{a}_i)]\subseteq[0,n]^r$, where $n=\dim(X)$.

(iv) The simplex

$$\left\{\lambda\in\mathbf{R}_+^r\;\middle|\;\sum_{i=1}^r\frac{1}{\mathrm{lct}_x(\mathfrak{a}_i)}\lambda_i\le 1\right\}$$

is contained in $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$.

(v) If λ, $\lambda'\in\mathbf{R}_+^r$ are such that $\lambda\preceq\lambda'$, and $\lambda'\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$, then $\lambda\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$.

Proof. All assertions immediately follow from definition, and from familiar facts about singularities of pairs, see [Kol2] and [dFM]. The assertion in (iv) follows from the fact that $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$ is convex, and the fact that the origin, as well as each $\mathrm{lct}_x(\mathfrak{a}_i)e_i$ lies in $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$. $\square$

Remark 2.11. Suppose that X is a nonsingular affine algebraic variety. It follows from Proposition 2.10 (iv) that if $f_1,\ldots,f_r\in\mathcal{O}(X)$, then $\mathrm{LCT}(f_1,\ldots,f_r)$ is contained in the cube $[0,1]^r$. On the other hand, if $\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ are ideals on X, and if for every i, $g_i\in\mathfrak{a}_i$ is a general linear combination of some fixed set of generators of $\mathfrak{a}_i$, then an argument based on Bertini's Theorem as in [Laz, Proposition 9.2.28] gives

$$\mathrm{LCT}(g_1,\ldots,g_r)=\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)\cap[0,1]^r.$$

Remark 2.12. If $\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ are ideals on a smooth variety X, and if $x\in X$, then $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)=\mathrm{LCT}(\mathfrak{a}_1\cdot\widehat{\mathcal{O}_{X,x}},\ldots,\mathfrak{a}_r\cdot\widehat{\mathcal{O}_{X,x}})$. This follows easily from [dFM, Proposition 2.7], that treats the case of log canonical thresholds. Since $\widehat{\mathcal{O}_{X,x}}\simeq k[\![x_1,\ldots,x_n]\!]$, it follows that in order to study the possible LCT-polytopes in a given dimension n, we may restrict to the case when $X=\mathrm{Spec}(k[\![x_1,\ldots,x_n]\!])$.

Lemma 2.13. *If $\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ are nonzero ideals on X, and if $\mathfrak{m}_x$ is the ideal defining a closed point $x\in X$, then*

$$\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)=\bigcap_{q\geq 1}\mathrm{LCT}_x(\mathfrak{a}_1+\mathfrak{m}_x^q,\ldots,\mathfrak{a}_r+\mathfrak{m}_x^q).$$

Proof. The inclusion "$\subseteq$" is trivial, so let us suppose that $\lambda=(\lambda_i)$ lies in the above intersection. It is enough to show that every $\lambda'\in\mathbf{Q}_+^r$ with $\lambda'\preceq\lambda$ lies in $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$. Therefore, we may assume that $\lambda\in\mathbf{Q}_+^r$. Choose N such that all $N\lambda_i$ are integers. By assumption, we have $\mathrm{lct}((\mathfrak{a}_1+\mathfrak{m}_x^q)^{N\lambda_1}\cdots(\mathfrak{a}_r+\mathfrak{m}_x^q)^{N\lambda_r})\geq 1/N$.

Let $\tau:=\min\{\lambda_i\mid\lambda_i>0\}$. Since the ideals $\mathfrak{a}_1^{N\lambda_1}\cdots\mathfrak{a}_r^{N\lambda_r}$ and $(\mathfrak{a}_1+\mathfrak{m}_x^q)^{N\lambda_1}\cdots(\mathfrak{a}_r+\mathfrak{m}_x^q)^{N\lambda_r}$ are congruent modulo $\mathfrak{m}_x^{qN\tau}$, it follows that

$$\mathrm{lct}_x((\mathfrak{a}_1+\mathfrak{m}_x^q)^{N\lambda_1}\cdots(\mathfrak{a}_r+\mathfrak{m}_x^q)^{N\lambda_r})-\mathrm{lct}_x(\mathfrak{a}_1^{N\lambda_1}\cdots\mathfrak{a}_r^{N\lambda_r})\leq\frac{n}{qN\tau},$$

where $n=\dim(\mathcal{O}_{X,x})$ (see [dFM, Corollary 2.10]). We conclude that $\mathrm{lct}_x(\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_r^{\lambda_r})\geq 1-\frac{n}{q\tau}$. Letting q go to infinity, this gives $\lambda\in\mathrm{LCT}_x(\mathfrak{a}_1\ldots,\mathfrak{a}_r)$. $\square$

The above lemma and the previous remark can be used to reduce proving results about LCT-polytopes on $\mathrm{Spec}(k[\![x_1,\ldots,x_n]\!])$ to proving the similar results on $\mathbf{A}^n$. In order to illustrate this, we give the following

Proposition 2.14. *If $H\subset X$ is a smooth hypersurface containing x, and if $\mathfrak{a}_i$ are ideals on X such that all $\mathfrak{a}_i\mathcal{O}_H$ are nonzero, then*

$$\mathrm{LCT}_x(\mathfrak{a}_1\mathcal{O}_H,\ldots,\mathfrak{a}_r\mathcal{O}_H)\subseteq\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r).$$

Proof. When X is a nonsingular variety over k, this follows easily from Inversion of Adjunction (see [Kol2, Theorem 7.5]). If $X=\mathrm{Spec}(k[\![x_1,\ldots,x_n]\!])$, after a change of coordinates we may assume that $H=(x_1=0)$. In this case, by Lemma 2.13 it is enough to prove the proposition when we replace $\mathfrak{a}_i$ by $\mathfrak{a}_i+\mathfrak{m}_x^q$. Since there are ideals $\mathfrak{a}_i'$ in $k[x_1,\ldots,x_n]$ such that $\mathfrak{a}_i+\mathfrak{m}_x^q=\mathfrak{a}_i'\cdot k[\![x_1,\ldots,x_n]\!]$, we conclude using the case of ideals in $\mathbf{A}^n$ via Remark 2.12. $\square$

Remark 2.15. If X is a nonsingular variety over k, it is sometimes convenient to phrase the description of $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$ in the language of mixed multiplier ideals, for which we refer to [Laz, Chapter 9]. Recall that the pair $(X,\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_r^{\lambda_r})$ is klt at $x\in X$ if and only if the mixed multiplier ideal $\mathcal{J}(X,\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_r^{\lambda_r})$ is not contained in the ideal $\mathfrak{m}_x$ defining x. We deduce using the definition of the LCT-polytopes that $\lambda\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$ if and only if for every $\mu=(\mu_i)\in\mathbf{R}_+^r$ with $\mu\prec\lambda$, we have $\mathcal{J}(X,\mathfrak{a}_1^{\mu_1}\cdots\mathfrak{a}_r^{\mu_r})\not\subseteq\mathfrak{m}_x$.

The following proposition is the generalization to the case $r>1$ of [Kol2, Proposition 8.19]. As above, we denote by $\mathfrak{m}_x$ the ideal defining the closed point $x\in X$.

Proposition 2.16. *Let $\mathfrak{b},\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ be nonzero ideals on X. If $\lambda=(\lambda_j)$ lies in $\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$, and N is a positive integer such that $\mathfrak{a}_i+\mathfrak{m}_x^N=\mathfrak{b}+\mathfrak{m}_x^N$ for some i, then*

$$\lambda-\min\{n/N,\lambda_i\}e_i\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{b},\ldots,\mathfrak{a}_r),$$

where $\mathfrak{b}$ appears on the i^{th} component and $n=\dim(X)$.

Proof. By Lemma 2.3, we may assume that $\mathfrak{a}_i$ vanishes at x. After replacing $\mathfrak{a}_i$ by $\mathfrak{a}_i+\mathfrak{m}_x^N$, we may also assume that $\mathfrak{a}_i=\mathfrak{b}+\mathfrak{m}_x^N$. Arguing as in the proof of Proposition 2.14, we see that it is enough to prove the statement when X is a smooth variety over k. In this case, it is convenient to use the language of mixed multiplier ideals; see Remark 2.15. Let us consider any $\mu=(\mu_j)\in\mathbf{R}_+^r$, with $\mu\prec\lambda$, so by assumption the mixed multiplier ideal $\mathcal{J}(X,\mathfrak{a}_1^{\mu_1}\cdots\mathfrak{a}_r^{\mu_r})$ is not contained in $\mathfrak{m}_x$.

By the Summation Theorem (for the version that we need, see [JM, Corollary 4.2]), we have

$$\mathcal{J}(X,\mathfrak{a}_1^{\mu_1}\cdots(\mathfrak{b}+\mathfrak{m}_x^N)^{\mu_i}\cdots\mathfrak{a}_r^{\mu_r})=\sum_{\alpha+\beta=\mu_i}\mathcal{J}(X,\mathfrak{a}_1^{\mu_1}\cdots\mathfrak{b}^{\alpha}\mathfrak{m}_x^{N\beta}\cdots\mathfrak{a}_r^{\mu_r}).$$

It follows that for some α, $\beta\geq 0$ with $\alpha+\beta=\mu_i$ we have

$$\mathcal{J}(X,\mathfrak{a}_1^{\mu_1}\cdots\mathfrak{b}^{\alpha}\mathfrak{m}_x^{N\beta}\cdots\mathfrak{a}_r^{\mu_r})\not\subseteq\mathfrak{m}_x.$$

If $\mu_i>\frac{n}{N}$, then using $\mathcal{J}(\mathfrak{m}_x^n)\subseteq\mathfrak{m}_x$ we deduce $N\beta<n$, and therefore

$$\left(\mu_1,\ldots,\mu_i-\frac{n}{N},\ldots,\mu_r\right)\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{b},\ldots,\mathfrak{a}_r).$$

We conclude that $\mu-\min\{n/N,\mu_i\}e_i\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{b},\ldots,\mathfrak{a}_r)$ (note that we have $(\mu_1,\ldots,0,\ldots,\mu_r)\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_i,\ldots\mathfrak{a}_r)$ by hypothesis, hence $(\mu_1,\ldots,0,\ldots,\mu_r)\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{b},\ldots\mathfrak{a}_r)$). Since this holds for every $\mu\prec\lambda$, we get the conclusion of the proposition. ☐

An iterated application of the proposition gives the following result improving Lemma 2.13.

Corollary 2.17. *Let $\mathfrak{a}_i$, $\mathfrak{b}_i$ be ideals on X, for $1\leq i\leq r$, and let N be a positive integer such that $\mathfrak{a}_i+\mathfrak{m}_x^N=\mathfrak{b}_i+\mathfrak{m}_x^N$ for all i. If $\lambda=(\lambda_i)\in\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$, then $\lambda'=(\lambda'_i)\in\mathrm{LCT}_x(\mathfrak{b}_1,\ldots,\mathfrak{b}_r)$, where $\lambda'_i=\max\left\{\lambda_i-\frac{n}{N},0\right\}$ for all i.*

Recall that on the space $\mathcal{H}_r$ of all nonempty compact subsets in $\mathbf{R}^r$ we have the Hausdorff metric, defined as follows. If $K\subset\mathbf{R}^r$ is an arbitrary compact set, for every $x\in\mathbf{R}^r$ we put $d(x,K)=\min_{y\in K}d(x,y)$, where $d(x,y)$ denotes the Euclidean distance between x and y. The Hausdorff distance between two compact sets K_1 and K_2 is defined by

$$\delta(K_1,K_2):=\max\left\{\max_{x\in K_1} d(x,K_2),\max_{x\in K_2} d(x,K_1)\right\}.$$

The set of all nonempty compact subsets of $\mathbf{R}^r$ thus becomes a complete metric space. Furthermore, the subspace of $\mathcal{H}_r$ consisting of all compact subsets of a fixed compact set K in $\mathbf{R}^r$ is compact. For some basic facts about the Hausdorff metric, see [Mun, p.281]. Using this notion, we deduce from Corollary 2.17 the next

Corollary 2.18. *Suppose that $\mathfrak{a}_i$, $\mathfrak{b}_i$ are ideals on X, and $x\in X$ lies in $\bigcap_i \operatorname{Supp}(V(\mathfrak{a}_i))$. If N is a positive integer such that $\mathfrak{a}_i+\mathfrak{m}_x^N=\mathfrak{b}_i+\mathfrak{m}_x^N$ for all i, then*

$$\delta(\mathrm{LCT}_x(\mathfrak{a}_1,\ldots,\mathfrak{a}_r),\mathrm{LCT}_x(\mathfrak{b}_1,\ldots,\mathfrak{b}_r))\leq\frac{n\sqrt{r}}{N}.$$

Example 2.19. Let $\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ be proper nonzero ideals on $X=\operatorname{Spec}(k[\![x_1,\ldots,x_n]\!])$. If $\mathfrak{b}_1,\ldots,\mathfrak{b}_r$ are the inverse images of these ideals on $X'=\operatorname{Spec}(k[\![x_1,\ldots,x_n,y]\!])$ via the canonical projection, then $\mathrm{LCT}(\mathfrak{b}_1+(y^d),\mathfrak{b}_2,\ldots,\mathfrak{b}_r)$ is equal to

$$\{(\lambda_1+t,\lambda_2,\ldots,\lambda_r)\mid(\lambda_1,\ldots,\lambda_r)\in\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_r),0\leq t\leq 1/d\}. \tag{6}$$

Indeed, note first that by Lemma 2.13 (or Corollary 2.17), it is enough to prove the above assertion when we replace each $\mathfrak{a}_i$ by $\mathfrak{a}_i+(x_1,\ldots,x_n)^\ell$, for all $\ell\geq 1$. It follows from Remark 2.12 that it is enough to prove the similar equality when the $\mathfrak{a}_i$ are nonzero ideals on $\operatorname{Spec}(k[x_1,\ldots,x_n])$ vanishing at the origin, we have $\mathfrak{b}_i=\mathfrak{a}_i\cdot k[x_1,\ldots,x_n,y]$, and we compute the LCT-polytopes at the origin. In this case it is again convenient to use the language of mixed multiplier ideal sheaves. Recall that by Remark 2.15, we have $\lambda\in\mathrm{LCT}_0(\mathfrak{a}_1,\ldots,\mathfrak{a}_r)$ if and only if for every $\mu=(\mu_i)\in\mathbf{R}_+^r$ with $\mu\prec\lambda$, we have $\mathcal{J}(\mathbf{A}^n,\mathfrak{a}_1^{\mu_1}\cdots\mathfrak{a}_r^{\mu_r})\not\subseteq(x_1,\ldots,x_n)$. It follows from the Summation Theorem (see [JM, Corollary 4.2]) that for every $\mu_1,\ldots,\mu_r\in\mathbf{R}_+$, we have

$$\begin{aligned}\mathcal{J}(\mathbf{A}^{n+1},(\mathfrak{b}_1+(y^d))^{\mu_1}\mathfrak{b}_2^{\mu_2}\cdots\mathfrak{b}_r^{\mu_r}) &= \sum_{\alpha+\beta=\mu_1}\mathcal{J}(\mathbf{A}^{n+1},\mathfrak{b}_1^{\alpha}y^{d\beta}\mathfrak{b}_2^{\mu_2}\cdots\mathfrak{b}_r^{\mu_r})\\ &= \sum_{\alpha+\beta=\mu_1}(y^{\lfloor d\beta\rfloor})\cdot\mathcal{J}(\mathbf{A}^n,\mathfrak{b}_1^{\alpha}\mathfrak{b}_2^{\mu_2}\cdots\mathfrak{b}_r^{\mu_r}),\end{aligned}$$

where the second equality follows from [Laz, Remark 9.5.23]. Therefore, this ideal is not contained in $(x_1,\ldots,x_n,y)$ if and only there is $\beta\in\mathbf{R}_+$ with $\beta_1<1/d$ such that $\mathcal{J}(\mathbf{A}^n,\mathfrak{b}_1^{\mu_1-\beta}\mathfrak{b}_2^{\mu_2}\cdots\mathfrak{b}_r^{\mu_r})$ is not contained in $(x_1,\ldots,x_n)$. The description in (6) easily follows.

3. Limits of LCT-polytopes

Recall that by Remark 2.12, in order to study the possible LCT-polytopes in a given dimension n, we may restrict to the case when $X=\operatorname{Spec}(k[\![x_1,\ldots,x_n]\!])$. Of course, in this case it is not necessary to include the closed point in the notation.

Remark 3.1. Note that if $k\subset K$ is a field extension of algebraically closed fields, and if $\mathfrak{a}_1,\ldots,\mathfrak{a}_r$ are nonzero proper ideals in $k[\![x_1,\ldots,x_n]\!]$, and if we put

$\mathfrak{a}_i' = \mathfrak{a}_i \cdot K[\![x_1, \ldots, x_n]\!]$, then $\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r) = \mathrm{LCT}(\mathfrak{a}_1', \ldots, \mathfrak{a}_r')$. Indeed, by Lemma 2.13 it is enough to show that for all $N \geq 1$ we have

$$\mathrm{LCT}(\mathfrak{a}_1 + \mathfrak{m}^N, \ldots, \mathfrak{a}_r + \mathfrak{m}^N) = \mathrm{LCT}(\mathfrak{a}_1' + (\mathfrak{m}')^N, \ldots, \mathfrak{a}_r' + (\mathfrak{m}')^N), \tag{7}$$

where $\mathfrak{m}$ and $\mathfrak{m}'$ are the maximal ideals in $k[\![x_1, \ldots, x_n]\!]$ and $K[\![x_1, \ldots, x_n]\!]$. Let us fix N. There are ideals $\mathfrak{b}_i$ in $k[x_1, \ldots, x_n]$ such that $\mathfrak{b}_i \cdot k[\![x_1, \ldots, x_n]\!] = \mathfrak{a}_i + \mathfrak{m}^N$ for every i. If $\mathfrak{b}_i' = \mathfrak{b}_i \cdot K[x_1, \ldots, x_n]$, then $\mathfrak{b}_i' \cdot K[\![x_1, \ldots, x_n]\!] = \mathfrak{a}_i'$. It is easy to see that $\mathrm{LCT}_0(\mathfrak{b}_1, \ldots, \mathfrak{b}_r) = \mathrm{LCT}_0(\mathfrak{b}_1', \ldots \mathfrak{b}_r')$, using a log resolution of $\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_r$ to compute the left-hand side of the equality, and the base-extension of this log resolution to $\mathrm{Spec}(K)$ to compute the right-hand side (see for example [dFM, Proposition 2.9] for the case of one ideal). The assertion in (7) is now a consequence of Remark 2.12. Therefore every LCT-polytope of ideals in $k[\![x_1, \ldots, x_n]\!]$ is an LCT-polytope of ideals in $K[\![x_1, \ldots, x_n]\!]$.

Remark 3.2. If k is an algebraically closed field having infinite transcendence degree over $\mathbf{Q}$ (for example, $k = \mathbf{C}$), then every LCT-polytope of r ideals in some $K[\![x_1, \ldots, x_n]\!]$, where K is an algebraically closed field extension of k, can be realized as the LCT-polytope of r ideals in $k[\![x_1, \ldots, x_n]\!]$. Indeed, suppose that $P = \mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$, with $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ proper nonzero ideals in $K[\![x_1, \ldots, x_n]\!]$. Since each $\mathfrak{a}_i$ is finitely generated, we can find an algebraically closed subfield $L \subset K$ of countable transcendence degree over $\mathbf{Q}$, and ideals $\mathfrak{b}_i$ in $L[\![x_1, \ldots, x_n]\!]$ such that $\mathfrak{a}_i = \mathfrak{b}_i \cdot K[\![x_1, \ldots, x_n]\!]$ for every i. Using the fact that k has infinite transcendence degree over $\mathbf{Q}$, we can find an embedding $L \hookrightarrow k$. If $\mathfrak{b}_i' = \mathfrak{b}_i \cdot k[\![x_1, \ldots, x_n]\!]$, we deduce from the previous remark that $\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \mathrm{LCT}(\mathfrak{b}_1', \ldots, \mathfrak{b}_n')$.

By Proposition 2.10 (iv), all LCT-polytopes corresponding to r proper nonzero ideals in the ring $k[\![x_1, \ldots, x_n]\!]$ are contained in the compact set $[0, n]^r$. Therefore, every sequence of LCT-polytopes has a convergent subsequence (in the Hausdorff metric). Our goal is to show that the limit is again an LCT-polytope, corresponding to possibly fewer than r ideals. Furthermore, we prove that in this case, the limit is equal to the intersection of all but finitely many of the given LCT-polytopes.

Theorem 3.3. *If $P_m = \mathrm{LCT}(\mathfrak{a}_1^{(m)}, \ldots, \mathfrak{a}_r^{(m)})$ for $m \geq 1$, where the $\mathfrak{a}_i^{(m)}$ are proper nonzero ideals in $k[\![x_1, \ldots, x_n]\!]$, and if the P_m converge in the Hausdorff metric to a compact set $Q \subseteq \mathbf{R}^r$, then Q is again an LCT-polytope. More precisely, if I is the set of those $i \leq r$ such that $Q \not\subseteq (x_i = 0)$, then we can find proper nonzero ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ in $K[\![x_1, \ldots, x_n]\!]$, with $s = \#I$ and K an algebraically closed field extension of k, such that $Q = j_I(\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_s))$, where $j_I \colon \mathbf{R}^s \hookrightarrow \mathbf{R}^r$ is the inclusion corresponding to the coordinates in I.*

Remark 3.4. We make the convention that the LCT-polytope of an empty set of ideals consists of $\{0\}$. In the context of Theorem 3.3, it can happen that $s = 0$, in which case Q consists of the origin in $\mathbf{R}^r$.

Remark 3.5. It follows from Remark 3.2 that if the transcendence degree of k over $\mathbf{Q}$ is infinite, then in Theorem 3.3 we may take $K = k$.

Theorem 3.6. *If $(P_m)_{m \geq 1}$ and Q are as in Theorem 3.3, then there is m_0 such that $Q = \bigcap_{m \geq m_0} P_m$.*

This result can be considered as a strong form of the Ascending Chain Condition for LCT-polytopes. In fact, it immediately gives

Corollary 3.7. *If $P_m = \mathrm{LCT}(\mathfrak{a}_1^{(m)}, \ldots, \mathfrak{a}_r^{(m)})$ for $m \geq 1$, where the $\mathfrak{a}_i^{(m)}$ are proper nonzero ideals in $k[\![x_1, \ldots, x_n]\!]$, and if $P_1 \subseteq P_2 \subseteq \cdots$, then this sequence is eventually stationary.*

Proof. It is enough to find a subsequence that is eventually stationary. Since $P_m \subseteq [0, n]^r$ for all m, we deduce that after passing to a subsequence, we may assume that the P_m converge to some Q in the Hausdorff metric. Theorem 3.6 implies that there is m_0 such that $Q = \bigcap_{m \geq m_0} P_m$. On the other hand, it is easy to see that in our case $\bigcup_{m \geq 1} P_m \subseteq Q$ (see, e.g., Lemma 3.8(iii) below). This gives $P_m = Q$ for every $m \geq m_0$. $\square$

For the proof of Theorems 3.3 and 3.6, we will need a couple of lemmas. The first one gives some easy properties of Hausdorff convergence that we will need. We denote by $d(\cdot, \cdot)$ the Euclidean distance in $\mathbf{R}^r$, and by $\delta(\cdot, \cdot)$ the Hausdorff metric on the space $\mathcal{H}_r$ of all nonempty compact subsets of $\mathbf{R}^r$.

Lemma 3.8. *Let $(K_m)_{m \geq 1}$ be a sequence of compact subsets in $\mathbf{R}^r$, converging in the Hausdorff metric to the compact subset K.*

(i) If $C \subseteq \mathbf{R}^r$ is closed, and $K_m \subseteq C$ for all m, then $K \subseteq C$.
(ii) If $u_m \in K_m$, and $(u_m)_{m \geq 1}$ converges to $u \in \mathbf{R}^r$, then $u \in K$.
(iii) $\bigcap_m K_m \subseteq K$.

Proof. The assertion in (i) follows easily from definition. For (ii), note that if $u \notin K$, then there is a ball $B(u, \varepsilon)$ centered at u, and of radius $\varepsilon > 0$ that does not intersect K. By assumption, there is m_0 such that $\delta(K_m, K) < \varepsilon/2$ for all $m \geq m_0$. For such m, since $u_m \in K_m$, we have $d(u_m, K) < \varepsilon/2$, hence we can find $w_m \in K$ such that $d(u_m, w_m) < \varepsilon/2$. On the other hand, after possibly enlarging m_0, we may assume that $d(u_m, u) < \varepsilon/2$ for $m \geq m_0$. Therefore,

$$d(u, w_m) \leq d(u, u_m) + d(u_m, w_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

contradicting the fact that $B(u, \varepsilon) \cap K = \emptyset$. This proves (ii), and the assertion in (iii) is a special case. $\square$

For a proper nonzero ideal $\mathfrak{a}$ in $k[\![x_1, \ldots, x_n]\!]$, its order $\mathrm{ord}(\mathfrak{a})$ is the largest nonnegative integer d such that $\mathfrak{a}$ is contained in the d^{th} power of the maximal ideal $\mathfrak{m}$. Recall the following estimates for the log canonical threshold in terms of the order:

$$\frac{1}{\mathrm{ord}(\mathfrak{a})} \leq \mathrm{lct}(\mathfrak{a}) \leq \frac{n}{\mathrm{ord}(\mathfrak{a})} \tag{8}$$

(the first inequality reduces to the case $n = 1$ via Proposition 2.14, whereas the second inequality follows from $\mathrm{lct}(\mathfrak{a}) \leq \mathrm{lct}(\mathfrak{m}^{\mathrm{ord}(\mathfrak{a})}) = n/\mathrm{ord}(\mathfrak{a})$).

Lemma 3.9. *With the notation in Theorem 3.3, the following are equivalent:*

(i) $Q \subseteq (x_i = 0)$.
(ii) $\lim_{m \to \infty} \mathrm{ord}(\mathfrak{a}_i^{(m)}) = \infty$.
(iii) The set $\{\mathrm{ord}(\mathfrak{a}_i^{(m)}) \mid m \geq 1\}$ is unbounded.

Proof. Suppose first that $Q \subseteq (x_i = 0)$. For every m we have $\mathrm{lct}(\mathfrak{a}_i^{(m)}) \cdot e_i \in P_m$, where $e_1, \ldots, e_r$ is the standard basis of $\mathbf{R}^r$. It follows from Lemma 3.8 (ii) that every limit point of the sequence $\left(\mathrm{lct}(\mathfrak{a}_i^{(m)}) \cdot e_i\right)_{m \geq 1}$ lies in Q. Therefore, $\lim_{m\to\infty} \mathrm{lct}(\mathfrak{a}_i^{(m)}) = 0$, and (ii) follows from the first inequality in (8).

Since the implication (ii)⇒(iii) is trivial, in order to finish the proof of the lemma it is enough to prove (iii)⇒(i). Suppose that $\lambda = (\lambda_1, \ldots, \lambda_r) \in Q$, and $\lambda_i > 0$. We can find m_0 such that $\delta(P_m, Q) < \lambda_i/2$ for all $m \geq m_0$. For every such m, we can find $w^{(m)} = (w_1^{(m)}, \ldots, w_r^{(m)}) \in P_m$ such that $d(w^{(m)}, \lambda) < \lambda_i/2$. In particular, $w_i^{(m)} > \lambda_i/2$. Since $w^{(m)} \in P_m$, we see using the second inequality in (8) that for all $m \geq m_0$

$$\frac{\lambda_i}{2} < w_i^{(m)} \leq \mathrm{lct}(\mathfrak{a}_i^{(m)}) \leq \frac{n}{\mathrm{ord}(\mathfrak{a}_i^{(m)})}.$$

This contradicts (iii). □

The main ingredient in the proof of Theorems 3.3 and 3.6 is the generic limit construction from [Kol1] and [dFEM]. Let $(\mathfrak{a}_1^{(m)})_m, \ldots, (\mathfrak{a}_r^{(m)})_m$ be sequences as in Theorem 3.3. In order to simplify the notation, let us relabel the sequences such that the set I in the theorem is equal to $\{1, \ldots, s\}$. Associated to the s sequences $(\mathfrak{a}_i^{(m)})_{m\geq 1}$, with $1 \leq i \leq s$, we get s *generic limits* $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$. These are ideals in $K[\![x_1, \ldots, x_n]\!]$, where K is a suitable algebraically closed field extension of k. It follows from Lemma 4.3 in [dFEM] and the above Lemma 3.9 that all $\mathfrak{a}_i$ are nonzero. Furthermore, since every $\mathfrak{a}_i^{(m)}$ is contained in the maximal ideal, the same holds for the ideals $\mathfrak{a}_i$. The fundamental property of the generic limit construction is that there is a strictly increasing sequence $(m_\ell)_\ell$ such that for every nonnegative rational numbers $w_1, \ldots, w_s$ we have

$$\lim_{\ell\to\infty} \mathrm{lct}((\mathfrak{a}_1^{(m_\ell)})^{w_1} \cdots (\mathfrak{a}_s^{(m_\ell)})^{w_s}) = \mathrm{lct}(\mathfrak{a}_1^{w_1} \cdots \mathfrak{a}_s^{w_s}) \tag{9}$$

(see [dFEM, Corollary 4.5]).

Remark 3.10. The construction in [dFEM] deals with only two sequences of ideals, but as pointed out in *loc. cit.*, everything generalizes in an obvious way to any finite number of sequences. We also note that the field K given in *loc. cit.* is not algebraically closed, but since we are only interested in (9), we can simply extend the generic limit ideals to an algebraic closure. Equation (9) is stated in *loc. cit.* only for integers $w_1, \ldots, w_s$. On the other hand, if the w_i are rational numbers, and if N is a positive integer such that all $Nw_i \in \mathbf{Z}$, the formula for $(Nw_1, \ldots, Nw_s)$ implies the one for $(w_1, \ldots, w_s)$ by rescaling.

We isolate in the following lemma the key argument needed for the proofs of Theorems 3.3 and 3.6. We use the notation in those theorems, as well the notation for the generic limit ideals introduced above.

Lemma 3.11. *If $\lambda \in \mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_s) \cap \mathbf{Q}^s$, then there are infinitely many m such that $j_I(\lambda) \in P_m$.*

Proof. Write $\lambda=(\lambda_1,\ldots,\lambda_s)$, hence by assumption $\mathrm{lct}(\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_s^{\lambda_s})\geq 1$. Fix a positive integer N such that $N\lambda_i\in\mathbf{Z}$ for every i. Consider the set

$$\Gamma:=\{\mathrm{lct}((\mathfrak{a}_1^{(m)})^{N\lambda_1}\cdots(\mathfrak{a}_s^{(m)})^{N\lambda_s})\mid m\in\mathbf{Z}_{>0}\}.$$

Since the elements of Γ are log canonical thresholds of ideals on $\mathrm{Spec}(k[\![x_1,\ldots,x_n]\!])$, it follows from [dFEM, Theorem 5.1] that Γ satisfies ACC, that is, it contains no infinite strictly increasing sequences. On the other hand, (9) shows that $\frac{1}{N}\,\mathrm{lct}(\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_s^{\lambda_s})$ lies in the closure of Γ. We deduce that there are infinitely many m such that

$$\mathrm{lct}((\mathfrak{a}_1^{(m)})^{N\lambda_1}\cdots(\mathfrak{a}_s^{(m)})^{N\lambda_s})\geq\frac{1}{N}\,\mathrm{lct}(\mathfrak{a}_1^{\lambda_1}\cdots\mathfrak{a}_s^{\lambda_s})\geq\frac{1}{N}.$$

Therefore, $j_I(\lambda)\in P_m$ for all such m. □

We can now give the proofs of our main results.

Proof of Theorem 3.3. With the above notation, it is enough to show that $Q=j_I(\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_s))$ (of course, we may assume that $s\geq 1$, as otherwise there is nothing to prove). Note first that Lemma 3.11 gives the inclusion $j_I(\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_s))\subseteq Q$. Indeed, since $\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_s)\cap\mathbf{Q}^s$ is dense in $\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_s)$, and Q is closed, it is enough to prove the inclusion $j_I(\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_s)\cap\mathbf{Q}^s)\subseteq Q$, and this follows from the lemma (note that by Lemma 3.8 (iii), the intersection of infinitely many of the P_m is contained in Q).

We now prove the reverse inclusion: suppose that $u=(u_1,\ldots,u_r)\in Q$ (hence $u_i=0$ for $i>s$), and let us show that $(u_1,\ldots,u_s)\in\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_s)$. Note first that by Lemma 3.8 (i), we have $Q\subseteq\mathbf{R}_+^r$. Fix $\varepsilon>0$, and let us choose $w=(w_1,\ldots,w_s)\in\mathbf{Q}_+^s$ such that $w_i\leq u_i$ for all i, with strict inequality if $u_i>0$, and such that $(u_i-w_i)<\varepsilon$ for all i. We will show that in this case $\mathrm{lct}(\mathfrak{a}_1^{w_1}\cdots\mathfrak{a}_s^{w_s})\geq 1$. Since this holds for every $\varepsilon>0$, we get $\mathrm{lct}(\mathfrak{a}_1^{u_1}\cdots\mathfrak{a}_s^{u_s})\geq 1$, that is, $u\in j_I(\mathrm{LCT}(\mathfrak{a}_1,\ldots,\mathfrak{a}_s))$.

Let (m_ℓ) be a strictly increasing sequence such that (9) holds. We can choose q such that for all $m\geq q$ we have $\delta(P_m,Q)<\min\{u_i-w_i\mid u_i>0\}$. For every such m, let us choose $v_m\in P_m$ with $d(v_m,u)<\min\{u_i-w_i\mid u_i>0\}$. We may assume that $v_m\in\mathbf{Q}^r$. Since $v_m=(v_{m,1},\ldots,v_{m,r})\in P_m$, we have $\mathrm{lct}((\mathfrak{a}_1^{(m)})^{v_{m,1}}\cdots(\mathfrak{a}_r^{(m)})^{v_{m,r}})\geq 1$. On the other hand, by construction $w_i\leq v_{m,i}$ for every $i\leq s$, hence we have $\mathrm{lct}((\mathfrak{a}_1^{(m)})^{w_1}\cdots(\mathfrak{a}_s^{(m)})^{w_s})\geq 1$ for all $m\geq q$. Therefore, (9) implies $\mathrm{lct}(\mathfrak{a}_1^{w_1}\cdots\mathfrak{a}_s^{w_s})\geq 1$, completing the proof. □

Proof of Theorem 3.6. It is enough to show that there is m_0 such that $Q\subseteq P_m$ for all $m\geq m_0$. Indeed, in this case $Q\subseteq\bigcap_{m\geq m_0}P_m\subseteq Q$, where the second inclusion follows from Lemma 3.8 (iii).

Let us assume that this is not the case. After possibly replacing the sequence $(P_m)_{m\geq 1}$ by a subsequence, we may assume that $Q\not\subseteq P_m$ for any m. Note that by Theorem 3.3, Q is a rational polytope, so it is the convex hull of its vertices, which lie in $\mathbf{Q}^r$. Furthermore, by the above proof, each such vertex lies in $j_I(P(\mathfrak{a}_1,\ldots,\mathfrak{a}_s))$; hence by Lemma 3.11, it lies in infinitely many P_m. After replacing the sequence $(P_m)_{m\geq 1}$ by a subsequence, and after doing this for all vertices of Q, we conclude that all vertices of Q lie in P_m for all m. Therefore $Q\subseteq P_m$ for all m, a contradiction. This concludes the proof of the theorem. □

Example 3.12. It follows from Example 2.19 that if $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are proper nonzero ideals in $k[\![x_1, \ldots, x_n]\!]$, then $\mathrm{LCT}(\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ is the intersection of a sequence $P_1 \supset P_2 \supset \ldots$ that is *not* eventually stationary, where each P_i is the LCT-polytope of r proper nonzero ideals in $k[\![x_1, \ldots, x_n, y]\!]$.

Remark 3.13. If in Theorem 3.3 we have $P_m = \mathrm{LCT}(f_1^{(m)}, \ldots, f_r^{(m)})$ with the $f_i^{(m)}$ nonzero elements in the maximal ideal of $k[\![x_1, \ldots, x_n]\!]$, then one can obtain Q as (the linear embedding of) $\mathrm{LCT}(f_1, \ldots, f_s)$, with f_i nonzero elements in the maximal ideal of some $K[\![x_1, \ldots, x_n]\!]$. Indeed, one can modify the construction in [dFEM] by replacing the Hilbert schemes parametrizing all ideals in quotient rings $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^d$ with parameter spaces for principal ideals in these rings (when $r=1$, this is done in [Kol1]).

Since the set of all log canonical thresholds $\mathrm{lct}(f)$, with $f \in k[\![x_1, \ldots, x_n]\!]$ satisfies ACC, it follows that there is a largest such invariant that is < 1. Finding this value for arbitrary n is an open problem. For example, it is well-known that this value is equal to $\frac{5}{6}$ if $n = 2$. Indeed, if $f \in k[\![x, y]\!]$ has order ≥ 3, then we have $\mathrm{lct}(f) \leq \frac{2}{3}$ by (8). On the other hand, if the multiplicity of f at 0 is two, then f is formally equivalent to $x^2 + y^m$, for some $m \geq 2$, and $\mathrm{lct}_0(x^2 + y^m) = \frac{1}{2} + \frac{1}{m}$ (see [Laz, Section 9.3.C]). As the following example shows, one can get similar results for $r \geq 2$.

Example 3.14. We know that if $f, g \in k[\![x, y]\!]$ are nonzero elements in the maximal ideal of $k[\![x, y]\!]$, then $\mathrm{LCT}(f, g) \subseteq [0, 1]^2$. In fact, we have $\mathrm{LCT}(f, g) = [0, 1]^2$ if and only if after a change of variables $(f, g) = (x, y)$, and otherwise

$$\mathrm{LCT}(f, g) \subseteq \{(\lambda_1, \lambda_2) \in [0, 1]^2 \mid \lambda_1 + \lambda_2 \leq 3/2\}.$$

Indeed, it follows from Example 2.6 that $\mathrm{LCT}(x, y) = [0, 1]^2$. If there is no change of variable such that $(f, g) = (x, y)$, then there is a line in the tangent space at the origin to $X = \mathrm{Spec}(k[\![x, y]\!])$ that is contained in $T_0(V(f)) \cap T_0(V(g))$. This corresponds to a point p on the exceptional divisor E in the blow-up $B = \mathrm{Bl}_0(X) \xrightarrow{\pi} X$, and the condition says that $\mathrm{ord}_p(\pi^*(f)), \mathrm{ord}_p(\pi^*(g)) \geq 2$. It follows that if F is the exceptional divisor on the blow-up of B at p, then for every $(\lambda_1, \lambda_2) \in \mathrm{LCT}(f, g)$ we have

$$2\lambda_1 + 2\lambda_2 \leq \lambda_1 \cdot \mathrm{ord}_F(f) + \lambda_2 \cdot \mathrm{ord}_F(g) \leq \mathrm{ord}_F(K_{-/X}) + 1 = 3.$$

Example 2.8(a) shows that there are f and g such that $\mathrm{LCT}(f, g) = \{(\lambda_1, \lambda_2) \in [0, 1]^2 \mid \lambda_1 + \lambda_2 \leq 3/2\}$.

We note that if $r \geq 3$, then

$$\mathrm{LCT}(f_1, \ldots, f_r) \subseteq \{(\lambda_1, \ldots, \lambda_r) \in [0, 1]^r \mid \lambda_1 + \cdots + \lambda_r \leq 2\}, \tag{10}$$

for every nonzero $f_1, \ldots, f_r \in (x, y)$. Indeed, we see by considering the exceptional divisor E on B above that if $\mathrm{lct}(f_1^{\lambda_1} \cdots f_r^{\lambda_r}) \geq 1$, then $\sum_i \lambda_i \leq \sum_i \lambda_i \cdot \mathrm{ord}_E(f_i) \leq 2$. We also observe that if $f_1, \ldots, f_r$ are general linear forms, then $\pi \colon B \to X$ gives a log resolution of $(X, (f_1 \cdots f_r))$, and we see that in this case we have equality in (10).

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