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AN IMPROVEMENT TO LAGUTINSKII–PEREIRA INTEGRABILITY THEOREM

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Dedicated to my mother, In memoriam

Abstract. We give an improvement of an integrability theorem due to J. V. Pereira for holomorphic foliations of dimension one on complex manifolds. We give bounds for the degree and number of invariant reduced divisors contained in linear systems of projective manifolds. We construct examples of foliations on projective spaces with the maximum number of invariant hyperplanes. Also, in the appendix of this work we prove a Jouanolou–Ghys type theorem for one-dimensional foliations on compact complex manifolds.

1. Introduction

Darboux presented in [10], a theory on the existence of first integrals for polynomial differential equations based on the existence of sufficiently many invariant algebraic curves. Concomitantly Poincaré, in [22], considered the problem of algebraic integration of polynomial differential equations in the plane. He observed that, in this case, it would be sufficient to bound the degree of algebraic solutions. Nowadays this problem is known as *Poincaré's Problem*. Many authors have been working on these problems and on some of its generalizations, see for instance the papers Cerveau and Lins Neto [7], Carnicer [5], Soares [23], Pereira [20], Brunella and Mendes [3], Esteves and Kleiman [12], Cavalier and Lehmann [6], and Zamora [27].

The improvement and generalization of the Darboux theory of integrability was given by Jouanolou in [17] characterizing the existence of rational first integrals for Pfaff equations on $\mathbb{P}^n_k$, where k is an algebraically closed field of characteristic zero. More generally, Jouanolou proved in [46] that on a complex compact manifold X satisfying certain conditions on its Hodge-to-de Rham spectral sequence, a Pfaff equation $\omega \in \mathbb{P}H^0(M, \Omega^1_M \otimes \mathcal{L})$, where $\mathcal{L}$ is a line bundle, admits a meromorphic first integral if and only if possesses an infinite number of invariant irreducible divisors.

Ghys in [13] showed that the Jouanolou's result is valid for all compact complex manifold M. More precisely, if ω does not admit a meromorphic first integral, then the number of invariant irreducible divisors is at most

$$\dim_{\mathbb{C}}(H^1(M, \Omega^1_f)) + \dim_{\mathbb{C}}(H^0(M, \Omega^2_M \otimes \mathcal{L})/\omega \wedge H^0(M, \Omega^1_f)) + 2,$$

where Ω^1_f denotes the sheaf of closed holomorphic 1-forms on M.

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A discrete dynamical version of Jouanolou–Ghys theorem was recently proved by Cantat. In [4], he proved that if a holomorphic endomorphism $\varphi : M \circlearrowleft$ does not preserve a non-trivial meromorphic fibration, then the number of φ-invariant irreducible hypersurfaces is at most

$$\dim_{\mathbb{C}}(H^1(M,\Omega^1_M)) + \dim(M).$$

In the appendix (see Section 6) of this work, we prove a Jouanolou–Ghys type theorem for one-dimensional foliations. That is, a one-dimensional foliation $\mathcal{F}$ without meromorphic first integral on a compact complex manifold M admits only a finite number of invariant irreducible divisors. Moreover, the number of invariant irreducible divisors is at most

$$\dim_{\mathbb{C}}(H^1(M,\Omega^1_f)) + \dim_{\mathbb{C}}(H^0(M,K_{\mathcal{F}})/i_{X_{\mathcal{F}}}(H^0(M,\Omega^1_f))) + \dim(M),$$

where $K_{\mathcal{F}}$ is the canonical bundle of $\mathcal{F}$ and $i_{X_{\mathcal{F}}}$ denotes the linear map given by contraction of holomorphic 1-forms on M on the direction of vector field $X_{\mathcal{F}}$ defining $\mathcal{F}$.

Now, let $V \subset H^0(M,\mathcal{O}(D))$ a linear system, with $\dim_{\mathbb{C}}(V) > 1$, where D is an effective divisor on M. We can raise the following question:

Given a linear system $V \subset H^0(M,\mathcal{O}(D))$, where D is an effective divisor on M. Let $\mathcal{F}$ be an one-dimensional foliation without meromorphic first integral. What is the maximum number of $\mathcal{F}$-invariant divisors contained in V?

Pereira in [21] showed, using the concept of *extactic section*, that a foliation on $\mathbb{P}^2$, of degree $d > 1$, that does not admit rational first integral of degree $\leq k$, has at most

$$\frac{(d-1)}{k} \cdot \binom{\binom{k+2}{k}}{2} + \binom{k+2}{k}. \tag{1.1}$$

invariant curves of degree k.

The zero locus of extactic section is the inflection locus of linear systems with respect to the vector field inducing the foliation, see Section 2. The extactic section for polynomial differential equations in the plane was already known to Lagutinskii [11] and was rediscovered independently by Pereira [21].

Denote by $\varepsilon(\mathcal{F},V)$ the extactic section of $\mathcal{F}$ with respect to V. We recall that a holomorphic (or meromorphic) first integral for $\mathcal{F}$ is a holomorphic (resp. meromorphic) map $\Theta : M \longrightarrow Y$, where Y is a complex manifold with $\dim(Y) < \dim(M)$, such that the fibers of Θ are invariants by $\mathcal{F}$. Pereira showed in [21] the following theorem:

Theorem 1.1. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex manifold M. If V is a finite-dimensional linear system such that $\varepsilon(\mathcal{F},V)$ vanishes identically, then there exits an open and dense set U, possibly intersecting the singular set of $\mathcal{F}$, where $\mathcal{F}_{|U}$ admits a holomorphic first integral. Moreover, if M is a projective variety, then $\mathcal{F}$ admits a meromorphic first integral.*

In the non-algebraic and non-compact cases Theorem 1.1 does not guarantee that the vanishing of extactic section $\varepsilon(\mathcal{F}, V)$ implies in the existence of a meromorphic first integral for $\mathcal{F}$. We provide the following improvement for Pereira's theorem.

Theorem 1.2. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex manifold M and V a finite-dimensional linear system. If $\boldsymbol{\varepsilon}(\mathcal{F}, V)$ vanishes identically then $\mathcal{F}$ admits a meromorphic first integral.*

Let (M, L) be a polarized projective variety; denote by $\mathscr{N}(\mathcal{F}, V)$ the number of $\mathcal{F}$-invariant reduced divisors contained in the linear system $V \subset H^0(M, \mathcal{O}(D))$. We use Theorem 1.2 to show the following generalization of Pereira bound (1.1) on projective manifolds.

Theorem 1.3. *Let $\mathcal{F}$ be a one-dimensional foliation on a polarized projective manifold (M, L) and $V \subset H^0(M, \mathcal{O}(D))$ a linear system, where D is an effective divisor on M. Suppose that $\mathcal{F}$ does not admit a rational first integral. Then*

$$\mathscr{N}(\mathcal{F}, V) \leq \frac{(\deg_L(\mathcal{F}) - \deg_L(M))}{\deg_L(D)} \cdot \binom{h^0(V)}{2} + h^0(V),$$

where $h^0(V) = \dim_{\mathbb{C}} V$. In particular, if $\mathscr{N}(\mathcal{F}, V) \geq \binom{h^0(V)}{2} + h^0(V)$, then

$$\deg_L(D) \leq \deg_L(\mathcal{F}) - \deg_L(M).$$

Theorem 1.3 implies (1.1). In fact, let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^n$, of degree $d > 1$, without rational first integral. Consider the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(k)|$, with $k \geq 1$. Since $h^0(|\mathcal{O}_{\mathbb{P}^n}(k)|) = \binom{k+n}{k}$, it follows Theorem 1.3 that

$$\mathscr{N}(\mathcal{F}, |\mathcal{O}_{\mathbb{P}^n}(k)|) \leq \frac{(d-1)}{k} \cdot \binom{\binom{k+n}{k}}{2} + \binom{k+n}{k}.$$

In Section 4, we give some optimal examples for hyperplanes invariant by a foliation of dimension one on projective spaces.

A generic one-dimensional singular holomorphic foliation $\mathcal{F}$ on a projective manifold leaves no algebraic set invariant, except for its singular locus [9]. When the extactic divisor $\mathcal{E}(\mathcal{F}, V)$ is irreducible we obtain an obstruction for existence of $\mathcal{F}$-invariant divisors contained in the linear system V. More precisely, if $\mathcal{E}(\mathcal{F}, V)$ is irreducible then $\mathcal{F}$ does not admit invariant divisors contained in the linear system V, see Proposition 3.3.

This paper is organized as follows. First, in order to make this presentation as self-contained as possible, in Section 2, we define the extactic divisors of a foliation with respect to linear systems and their main properties and we prove Theorem 1.2. In Section 3, we define the natural notion of degree of foliations with respect to a polarization on projective manifolds and we prove Theorem 1.3. In Section 4, we give some optimal examples for hyperplanes invariant by a one-dimensional foliation on projective spaces. Finally, in Appendix A of this work we prove a Jouanolou–Ghys type theorem for one-dimensional foliations on compact complex manifolds.

2. The extactic divisor

In this section, we digress on extactic divisors and their main properties developed in [21].

Definition 2.1. Let M be a connected complex manifold. A one-dimensional holomorphic foliation is given by the following data:

(i) an open covering $\mathcal{U} = \{U_\alpha\}$ of M;
(ii) for each U_α an holomorphic vector field X_α;
(iii) for every non-empty intersection, $U_\alpha \cap U_\beta \neq \emptyset$, a holomorphic function

$$f_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta);$$

such that $X_\alpha = f_{\alpha\beta} X_\beta$ in $U_\alpha \cap U_\beta$ and $f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$.

We denote by $K_{\mathcal{F}}$ the line bundle defined by the cocycle $\{f_{\alpha\beta}\} \in H^1(M, \mathcal{O}^*)$. Thus, a one-dimensional holomorphic foliation $\mathcal{F}$ on M induces a global holomorphic section $X_{\mathcal{F}} \in H^0(M, TM \otimes K_{\mathcal{F}})$. The line bundle $K_{\mathcal{F}}$ is called the *canonical bundle* of $\mathcal{F}$.

Two sections $X_{\mathcal{F}}$ and $Y_{\mathcal{F}}$ of $H^0(M, TM \otimes K_{\mathcal{F}})$ are equivalent, if there exits a never vanishing holomorphic function $\varphi \in H^0(M, \mathcal{O}^*)$, such that $X_{\mathcal{F}} = \varphi \cdot Y_{\mathcal{F}}$. It is clear that $X_{\mathcal{F}}$ and $Y_{\mathcal{F}}$ define the same foliation. Thus, a holomorphic foliation $\mathcal{F}$ on M is an equivalence of sections of $TM \otimes K_{\mathcal{F}}$.

Let H be a holomorphic line bundle on M. Consider a finite-dimensional linear system $V \subset H^0(M, H)$ and take an open covering $\{\mathcal{U}_\alpha\}$ of M which trivializes H and $K_{\mathcal{F}}$. In the open set $\mathcal{U}_\alpha$, we can consider the Taylor expansion of a section $s \in V$ with respect to the vector field X_α defining a morphism

$$T_\alpha^{(k)} : V \otimes \mathcal{O}_{\mathcal{U}_\alpha} \to \mathcal{O}_{\mathcal{U}_\alpha}^{\oplus k}$$

given by

$$T_\alpha^{(k)}(s_\alpha) = (s_\alpha, X_\alpha(s_\alpha), X_\alpha^2(s_\alpha), \dots, X_\alpha^{(k-1)}(s_\alpha)),$$

where s_α and X_α are local representations, respectively, of a section $s \in V \subset H^0(M, H)$ and the section $X_{\mathcal{F}} \in H^0(M, TM \otimes K_{\mathcal{F}})$ inducing $\mathcal{F}$. If $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ then $s_\alpha = g_{\alpha\beta} s_\beta$ and $X_\alpha = f_{\alpha\beta} X_\alpha$, where $g_{\alpha\beta}, f_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha)$ are the cocycles which define, respectively, the line bundles H and $K_{\mathcal{F}}$. Using the compatibility described above and Leibniz's rule we obtain

$$s_\alpha = g_{\alpha\beta} s_\beta$$
$$X_\alpha(s_\alpha) = f_{\alpha\beta} X_\beta(g_{\alpha\beta}) \cdot s_\beta + g_{\alpha\beta} f_{\alpha\beta} \cdot X_\beta(s_\beta).$$

Following this process up to order $k = \dim_{\mathbb{C}} V > 1$, we obtain

$$T_\alpha^{(k)}(s_\alpha) = \Theta_{\alpha\beta}(\mathcal{F}, V) \cdot T_\beta^{(k)}(s_\beta),$$

where

$$\Theta_{\alpha\beta}(\mathcal{F},V)=\begin{bmatrix} g_{\alpha\beta} & 0 & 0 & 0 & 0 \\ X_\beta(g_{\alpha\beta})\cdot f_{\alpha\beta} & g_{\alpha\beta}\cdot f_{\alpha\beta} & 0 & 0 & 0 \\ \ddots & \ddots & g_{\alpha\beta}\cdot f_{\alpha\beta}^2 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & g_{\alpha\beta}\cdot f_{\alpha\beta}^{k-1} \end{bmatrix}.$$

is a $k\times k$ matrix such that $\det(\Theta_{\alpha\beta}(\mathcal{F},V))\neq 0$. We see that

$$\begin{cases} \Theta_{\alpha\beta}(\mathcal{F},V)(p)\cdot\Theta_{\beta\alpha}(\mathcal{F},V)(p)=I, \text{ for all } p\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta, \\ \Theta_{\alpha\beta}(\mathcal{F},V)(p)\cdot\Theta_{\beta\gamma}(\mathcal{F},V)(p)\cdot\Theta_{\gamma\alpha}(\mathcal{F},V)(p)=I, \text{ for all } p\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta\cap\mathcal{U}_\gamma. \end{cases}$$

That is, the family of matrices $\{\Theta_{\alpha\beta}(\mathcal{F},V)\}$ defines a cocycle of a vector bundle of rank k on M that we denote by $J_{\mathcal{F}}^{k-1}H$. The vector bundle $J_{\mathcal{F}}^{k-1}H$ can be understood as a Jet bundle, with respect to $\mathcal{F}$, of holomorphic sections of H.

Now, using the trivializations $\{\Theta_{\alpha\beta}(\mathcal{F},V)\}$ we get the morphisms

$$T^{(k)}: V\otimes\mathcal{O}_M\to J_{\mathcal{F}}^{k-1}H.$$

Taking the determinant of $T^{(k)}$ we have the morphism

$$\det(T^{(k)}):\bigwedge^k V\otimes\mathcal{O}_M\to\bigwedge^k J_{\mathcal{F}}^{k-1}H,$$

and tensorizing by $(\bigwedge^k V)^*$ we obtain a global section of $\bigwedge^k J_{\mathcal{F}}^{k-1}H\otimes(\bigwedge^k V)^*$ given by

$$\boldsymbol{\varepsilon}(\mathcal{F},V):\mathcal{O}_M\to\bigwedge^k J_{\mathcal{F}}^{k-1}H\otimes\left(\bigwedge^k V\right)^*.$$

Definition 2.2. The section $\boldsymbol{\varepsilon}(\mathcal{F},V)$ is called the extactic section of $\mathcal{F}$ with respect to V and $X_{\mathcal{F}}$. If $\boldsymbol{\varepsilon}(\mathcal{F},V)$ is not identically zero, the extactic divisor of $\mathcal{F}$ with respect to V is the divisor $\mathcal{E}(\mathcal{F},V)=(\boldsymbol{\varepsilon}(\mathcal{F},V))$ given by the zeros of the extactic section.

Remark 1. The extactic section $\varepsilon(\mathcal{F},V)$ depends on the choice of the section

$$X_{\mathcal{F}}\in H^0(M,TM\otimes K_{\mathcal{F}}),$$

but the extactic divisor $\mathcal{E}(\mathcal{F},V)$ does not depend. In fact, suppose that $X_{\mathcal{F}}=\varphi\cdot Y_{\mathcal{F}}$, with $\varphi\in H^0(M,\mathcal{O}^*)$ and $Y_{\mathcal{F}}\in H^0(M,TM\otimes K_{\mathcal{F}})$. A straightforward calculation shows that

$$\varepsilon(X_{\mathcal{F}},V)=\varphi^{\binom{h^0(V)}{2}}\cdot\varepsilon(Y_{\mathcal{F}},V).$$

Then $(\boldsymbol{\varepsilon}(X_{\mathcal{F}},V))=(\boldsymbol{\varepsilon}(Y_{\mathcal{F}},V))$.

Remark 2. Note that the cocycle of $\bigwedge^k J_{\mathcal{F}}^{k-1}H$ is given by

$$\det(\Theta_{\alpha\beta}(\mathcal{F},V))=g_{\alpha\beta}^k\cdot f_{\alpha\beta}^{\binom{k}{2}},$$

where $g_{\alpha\beta}$ and $f_{\alpha\beta}$ are, respectively, the trivializations of H and $K_{\mathcal{F}}$. Therefore, we obtain the isomorphism $\bigwedge^k J_{\mathcal{F}}^{k-1}H\simeq H^{\otimes k}\otimes(K_{\mathcal{F}})^{\otimes\binom{k}{2}}$.

Pereira [21] obtained the following result, which elucidate the role of the extactic divisor:

Proposition 2.1 ([21, Proposition 5]). *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex manifold M. If V is a finite-dimensional linear system, then every $\mathcal{F}$-invariant divisor of V must be contained in the extactic divisor $\mathcal{E}(\mathcal{F}, V)$.*

Proof. Let $\{s_1, \ldots, s_k\}$ be a basis for $V \subset H^0(M, H)$. On the open $\mathcal{U}_\alpha$ the extactic section is given by

$$\varepsilon(\mathcal{F}, V)_\alpha = \det \begin{pmatrix} s_1^\alpha & s_2^\alpha & \cdots & s_k^\alpha \\ X_\alpha(s_1^\alpha) & X_\alpha(s_2^\alpha) & \cdots & X_\alpha(s_k^\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ X_\alpha^{k-1}(s_1^\alpha) & X_\alpha^{k-1}(s_2) & \cdots & X_\alpha^{k-1}(s_k^\alpha) \end{pmatrix}, \tag{2.1}$$

where X_α is a vector field that induces $\mathcal{F}$ on $\mathcal{U}_\alpha$ and s_i^α is local representation of the section s_i, $i = 1, \ldots, k$. Let f_α be the local equation defining an element on V and suppose that $(f_\alpha = 0)$ is $\mathcal{F}$-invariant. Change basis so that V is generated by $f_\alpha, v_2, \ldots, v_\ell$. It follows from the $\mathcal{F}$-invariance of $(f_\alpha = 0)$ that there exist analytic functions $h_\alpha^1, \ldots, h_\alpha^{k-1}$ on $\mathcal{U}_\alpha$ such that

$$X_\alpha^j(f_\alpha) = h_\alpha^j f_\alpha. \tag{2.2}$$

Substituting (2.2) on (2.1) we conclude that f_α is a factor of $\varepsilon(\mathcal{F}, V)_\alpha$, thus $(f_\alpha = 0) \subset \mathcal{E}(\mathcal{F}, V)_{|\mathcal{U}_\alpha}$. □

2.1. Proof of Theorem 1.2.

Proof. Suppose that the foliation $\mathcal{F}$ is given by the collections

$$(\{\mathcal{U}_\alpha\}, \{X_\alpha\}, \{f_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\}).$$

We will show the existence of a local meromorphic first integral on each open $\mathcal{U}_\alpha$. That is, there exists a meromorphic function θ^α such that $X_\alpha(\theta^\alpha) = 0$, where X_α is the vector field defining $\mathcal{F}$ on $\mathcal{U}_\alpha$. After this, we must prove that $\theta^\alpha = \theta^\beta$ on $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, thus we shall obtain a global meromorphic function defining a first integral for $\mathcal{F}$. For the existence of θ^β on $\mathcal{U}_\alpha$, we will use similar arguments given in the proof of Theorem 4.3 of [8] for the case of polynomial vector fields on $\mathbb{C}^2$.

Let $\{s_1, \ldots, s_k\}$ be a $\mathbb{C}$-base for V. Suppose that $\varepsilon(V, \mathcal{F})$ vanishes identically. Then on the open $\mathcal{U}_\alpha$ we have that

$$\varepsilon(\mathcal{F}, V)_\alpha = \det \begin{pmatrix} s_1^\alpha & s_2^\alpha & \cdots & s_k^\alpha \\ X_\alpha(s_1^\alpha) & X_\alpha(s_2^\alpha) & \cdots & X_\alpha(s_k^\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ X_\alpha^{k-1}(s_1^\alpha) & X_\alpha^{k-1}(s_2) & \cdots & X_\alpha^{k-1}(s_k^\alpha) \end{pmatrix} \equiv 0,$$

where X_α is a vector field that induces $\mathcal{F}$ on $\mathcal{U}_\alpha$ and s_i^α is the local representation of the section s_i, $i = 1, \ldots, k$.

To say that $\varepsilon(\mathcal{F}, V)_\alpha \equiv 0$ means that the columns of the above matrix are dependent over the field of meromorphic functions $\mathscr{M}(\mathcal{U}_\alpha)$. Hence, there are meromorphic functions $\theta_1^\alpha, \dots, \theta_k^\alpha$ on $\mathcal{U}_\alpha$, such that

$$M_i^\alpha := \sum_{j=1}^{k} \theta_j^\alpha X_\alpha^i(s_j^\alpha) = 0, \quad 0 \le i \le k-1. \tag{2.3}$$

Now, let $r(\alpha)$ be the smallest integer with the property that there exist meromorphic functions $\theta_1^\alpha, \dots, \theta_{r(\alpha)}^\alpha$ and $s_1^\alpha, \dots, s_{r(\alpha)}^\alpha \in V$, linearly independent over $\mathbb{C}$, such that (2.3) holds. We clearly have $1 < r(\alpha) \le k$ and we may assume $\theta_{r(\alpha)}^\alpha = 1$. Applying the derivation X_α to both sides of (2.3) we obtain

(2.4)

$$X_\alpha(\theta_1^\alpha) X_\alpha^i(s_1^\alpha) + \theta_1^\alpha X_\alpha^{i+1}(s_1^\alpha) + \cdots + \underbrace{X_\alpha(\theta_{r(\alpha)})}_{0} X_\alpha^i(s_{r(\alpha)}^\alpha) + \underbrace{\theta_{r(\alpha)}^\alpha}_{1} X_\alpha^{i+1}(s_{r(\alpha)}^\alpha) = 0,$$

for all $0 \le i \le r(\alpha) - 2$. Subtracting (2.4) from M_{i+1}^α we obtain

$$X_\alpha(\theta_1^\alpha) X_\alpha^i(s_1^\alpha) + \cdots + X_\alpha(\theta_{r(\alpha)-1}^\alpha) X_\alpha^i(s_{r(\alpha)-1}) = 0, \ 0 \le i \le r(\alpha) - 2.$$

By the minimality of $r(\alpha)$ we must have $X_\alpha(\theta_1^\alpha) = \cdots = X_\alpha(\theta_{r(\alpha)-1}^\alpha) = 0$ and hence, provided these are not all constants, we have a first integral for X_α on $\mathcal{U}_\alpha$. This in fact occurs because, since M_0^α is

$$M_0^\alpha = \theta_1^\alpha s_1^\alpha + \cdots + \theta_{r(\alpha)-1}^\alpha s_{r(\alpha)-1}^\alpha + s_{r(\alpha)}^\alpha = 0,$$

we conclude that not all the θ^α's could be constant since $s_1^\alpha, \dots, s_{r(\alpha)}^\alpha \in V$ are linearly independent over $\mathbb{C}$.

Now we will show that $r = r(\alpha) = r(\beta)$, for all $\alpha, \beta \in \Lambda$. Suppose that $r(\alpha) < r(\beta)$. In $\mathcal{U}_{\alpha\beta} \neq \emptyset$ we have that $s_i^\alpha = g_{\alpha\beta} s_i^\beta$, $i = 1, \dots, r(\alpha)$, and $X_\alpha = f_{\alpha\beta} X_\beta$, with $f_{\alpha\beta}, g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$. Using this we conclude that $X_\beta(\theta_i^\alpha) = 0$ on $\mathcal{U}_{\alpha\beta}$, for all $i = 1, \dots, r(\alpha) - 1$, and

$$\theta_1^\alpha s_1^\beta + \cdots + \theta_{r(\alpha)-1}^\alpha s_{r(\alpha)-1}^\beta + \theta_{r(\alpha)}^\alpha s_{r(\alpha)}^\beta = 0.$$

Applying the derivation X_β in this equation and using that $X_\beta(\theta_i^\alpha) = 0$, for all $i = 1, \dots, r(\alpha)$, we obtain

$$\sum_{j=1}^{r(\alpha)<r(\beta)} \theta_j^\alpha X_\beta^i(s_j^\beta) = 0, \quad 0 \le i \le r(\alpha) - 1,$$

by the minimality of $r(\beta)$ we can conclude that $\theta_1^\alpha = \cdots = \theta_{r(\alpha)}^\alpha = 0$, but this is a contradiction, since $\theta_{r(\alpha)}^\alpha = 1$. The case $r(\beta) < r(\alpha)$ is similar.

Consider the equations

$$\theta_1^\alpha s_1^\alpha + \cdots + \theta_{r-1}^\alpha s_{r-1}^\alpha + s_r^\alpha = 0,$$
$$\theta_1^\beta s_1^\alpha + \cdots + \theta_{r-1}^\beta s_{r-1}^\alpha + s_r^\alpha = 0.$$

Subtracting these equations we obtain

$$(\theta_1^\alpha - \theta_1^\beta) s_1^\alpha + \cdots + (\theta_{r-1}^\alpha - \theta_{r-1}^\beta) s_{r-1}^\alpha = 0.$$

Define $h_i^{\alpha\beta} = (\theta_i^\alpha - \theta_i^\beta) \in \mathscr{M}(\mathcal{U}_{\alpha\beta})$, $i = 1, \dots, r-1$. Applying the derivation X_α to the last equation and using $X_\alpha(\theta_i^\alpha) = X_\alpha(\theta_i^\beta) = 0$ we obtain

$$\sum_{j=1}^{r-1} h_j^{\alpha\beta} X_\alpha^i(s_j^\alpha) = 0, \quad 0 \leq i \leq r-2.$$

Again, by the minimality of r we have that $h_1^{\alpha\beta} = \cdots = h_{r-1}^{\alpha\beta} = 0$, i.e, $\theta_i^\alpha = \theta_i^\beta$ on $\mathcal{U}_{\alpha\beta}$, for all , $i = 1, \dots, r-1$. Therefore, we obtain a meromorphic first integral Θ^i locally given by $\Theta^i_{|\mathcal{U}_\alpha} = \theta_i^\alpha$, for some $i = 1, \dots, r-1$. □

3. The degree of foliations with respect to a polarization

Let (M, L) be a n-dimensional polarized projective manifold, i.e., M is connected and L is a very ample line bundle on M. The degree of a holomorphic vector bundle E on M related to the polarization L is defined by

$$\deg_L(E) = \int_M c_1(E) \cdot L^{n-1},$$

where $\int_M$ denote the degree of cycle of dimension n.

Let D be an analytic hypersurface on M defined locally by functions $\{s_\alpha \in \mathcal{O}(\mathcal{U}_\alpha)\}$, where $\{\mathcal{U}_\alpha\}$ is an open covering of M. If $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, then there exist $g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha)$, such that $s_\alpha = g_{\alpha\beta} s_\beta$. Let $\mathcal{F}$ be a holomorphic foliation on M given by collections

$$(\{X_\alpha\}; \{\mathcal{U}_\alpha\}; \{f_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha)\}),$$

where $f_{\alpha\beta}$ is the cocycle inducing $K_\mathcal{F}$. Consider the following functions:

$$\zeta_\alpha^{(\mathcal{F},D)} = X_\alpha(s_\alpha)_{|D} \in \mathcal{O}(\mathcal{U}_\alpha \cap D).$$

If $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap D \neq \emptyset$, using Leibniz's rule we get $\zeta_\alpha^{(\mathcal{F},D)} = f_{\alpha\beta} g_{\alpha\beta} \zeta_\beta^{(\mathcal{F},D)}$. With this we obtain a global section $\zeta^{(\mathcal{F},D)}$ of the line bundle $(K_\mathcal{F} \otimes \mathcal{O}(D))_{|D}$. The *tangency variety* (see [2]) of $\mathcal{F}$ with D is given by

$$\mathcal{T}(\mathcal{F}, D) = \{p \in D; \zeta^{(\mathcal{F},D)}(p) = 0\}.$$

Since $\zeta^{(\mathcal{F},D)} \in H^0(D, (K_\mathcal{F} \otimes \mathcal{O}(D))_{|D})$, we have the following adjunction formula:

$$\mathcal{T}(\mathcal{F}, D) = (K_\mathcal{F} + D)_{|D}. \tag{3.1}$$

Definition 3.1. Let (M, L) be a polarized variety and $\mathcal{F}$ a one-dimensional foliation on M. The degree of $\mathcal{F}$ with respect to the polarization L is the intersection number

$$\deg_L(\mathcal{F}) := \int_L \mathcal{T}(\mathcal{F}, L) \cdot L^{n-2}.$$

Proposition 3.1. *Let $\mathcal{F}$ be a foliation on a polarized variety (M, L). Then*

$$\deg_L(\mathcal{F}) = \deg_L(K_\mathcal{F}) + \deg_L(M),$$

where $\deg_L(M) = \deg_L(L)$ is the degree of M with respect to L.

Proof. It follows from Definition 3.1 and adjunction formula (3.1) that

$$\begin{aligned}\deg_L(\mathcal{F}) &= \int_L \mathcal{T}(\mathcal{F}, L)\cdot L^{n-2} = \int_L (K_{\mathcal{F}} + L)\cdot L^{n-2}\\ &= \int_M K_{\mathcal{F}}\cdot L^{n-1} + \int_M L^n\\ &= \deg_L(K_{\mathcal{F}}) + \deg_L(M).\end{aligned}$$

□

Throughout this paper we shall assume $\deg_L(K_{\mathcal{F}}) = \deg_L(\mathcal{F}) - \deg_L(M) \geq 0$.

Example 1. Let $\mathcal{F}$ be a foliation on a projective manifold M with $\mathrm{Pic}(M) \simeq \mathbb{Z}$. We can take a ample line bundle L to be a positive generator of $\mathrm{Pic}(M)$, so we denote by $\mathcal{O}_M(r) := L^{\otimes r}$ the rth tensor power of L. If we write $K_{\mathcal{F}} = \mathcal{O}_M(d_{\mathcal{F}} - 1)$, then $\deg_L(K_{\mathcal{F}}) = (d_{\mathcal{F}} - 1)\deg_L(M)$. Hence

$$\deg_L(\mathcal{F}) = \deg_L(K_{\mathcal{F}}) + \deg_L(M) = d_{\mathcal{F}}\cdot \deg_L(M).$$

In the case where $M = \mathbb{P}^n$ we will have, as is known, that $\deg(\mathcal{F}) = d_{\mathcal{F}}$.

The unique projective manifold that admits a one-dimensional foliation of degree zero is the projective space.

Proposition 3.2. *Let $\mathcal{F}$ be a foliation on a polarized variety (M, L) of $\deg_L(\mathcal{F}) = 0$, then $M = \mathbb{P}^n$.*

Proof. If follows Proposition 3.1 that $\deg_L(K_{\mathcal{F}} \otimes L) = \deg_L(\mathcal{F}) = 0$. This implies that $K_{\mathcal{F}} \otimes L = \mathcal{O}(M)$, i.e., $K_{\mathcal{F}} = L^{-1}$. Thus, the foliation is given by a non-zero holomorphi section of $TM \otimes L^{-1}$. The result follows from Whal's theorem of characterization of projective spaces [24], since L is very ample. □

3.1. Proof of Theorem 1.3. It follows from Theorem 1.2 that if $\mathcal{F}$ does not have a rational first integral, then $\boldsymbol{\varepsilon}(\mathcal{F}, V) \neq 0$. Thus, the extactic section $\boldsymbol{\varepsilon}(\mathcal{F}, V)$ defines an effective divisor $\mathcal{E}(\mathcal{F}, V)$ whose associated line bundle is $\bigwedge^k J_{\mathcal{F}}^{k-1}\mathcal{O}(D) \otimes (\bigwedge^k V)^*$, where $k = \dim_{\mathbb{C}} V \leq h^0(D)$. Let $\mathscr{N}(\mathcal{F}, V)$ be the number of divisors of $V \subset H^0(M, \mathcal{O}(D))$ invariant by $\mathcal{F}$. It follows from Proposition 2.1 that every divisor $C \in V$ invariant by $\mathcal{F}$ is contained in the extactic divisor $\mathcal{E}(\mathcal{F}, V)$. Using this fact we can claim that

$$\deg_L(D)\mathscr{N}(\mathcal{F}, V) \leq \deg_L(\mathcal{E}(\mathcal{F}, V)).$$

Indeed, it is enough to group the $\mathcal{F}$-invariant divisors of the following form:

$$\mathcal{E}(\mathcal{F}, V) = \sum_{j=1}^{\mathscr{N}(\mathcal{F},V)} C_j + R,$$

where $C_i \in V$ is a divisor invariant by $\mathcal{F}$ and R is a divisor without $\mathcal{F}$-invariant divisor contained in V. Since $\deg_L(C_j) = \deg_L(D)$, for all $j = 1, \ldots, \mathscr{N}(\mathcal{F}, V)$, we obtain

$$\deg_L(D)\mathscr{N}(\mathcal{F}, V) = \sum_{j=1}^{\mathscr{N}(\mathcal{F},V)} \deg_L(C_j) \leq \deg_L(\mathcal{E}(\mathcal{F}, V)).$$

This shows the claim above. However, the line bundle associated with the extactic divisor $\mathcal{E}(\mathcal{F},V)$ is given by $\bigwedge^k J^{k-1}_{\mathcal{F}}\mathcal{O}(D)\otimes(\bigwedge^k V)^*$. This implies that

$$\mathcal{O}(\mathcal{E}(\mathcal{F},V)) = \bigwedge^k J^{k-1}_{\mathcal{F}}\mathcal{O}(D)\otimes\left(\bigwedge^k V\right)^*.$$

It follows from Remark 2 that $\bigwedge^k J^{k-1}_{\mathcal{F}}\mathcal{O}(D)\simeq \mathcal{O}(D)^{\otimes k}\otimes (K_{\mathcal{F}})^{\otimes\binom{k}{2}}$, thus

$$\mathcal{O}(\mathcal{E}(\mathcal{F},V)) = \mathcal{O}(D)^{\otimes k}\otimes (K_{\mathcal{F}})^{\otimes\binom{k}{2}}\otimes\left(\bigwedge^k V\right)^*.$$

Since $\bigwedge^k V^*$ is a trivial vector bundle, then $\deg_L\left(\bigwedge^k V^*\right)=0$. Now, calculating the degree $\deg_L(\mathcal{E}(\mathcal{F},V))$, we obtain

$$\begin{aligned}\deg_L(\mathcal{E}(\mathcal{F},V)) &= \deg_L\left(\mathcal{O}(D)^{\otimes k}\otimes (K_{\mathcal{F}})^{\otimes\binom{k}{2}}\right) + \underbrace{\deg_L\left(\bigwedge^k V^*\right)}_{=0}\\ &= k\deg_L(D)+\deg_L(K_{\mathcal{F}})\binom{k}{2}.\end{aligned}$$

Finally, the result follows from $\deg_L(D)\cdot\mathscr{N}(\mathcal{F},V)\leq \deg_L(\mathcal{E}(\mathcal{F},V))$ and Proposition 3.1. This proves Theorem 1.3.

The irreducibility of extactic divisor $\mathcal{E}(\mathcal{F},V)$ gives an obstruction for the existence of $\mathcal{F}$-invariant divisors contained on V.

Proposition 3.3. *Let $\mathcal{F}$ be a foliation without rational first integral. If $\mathcal{E}(\mathcal{F},V)$ is irreducible then $\mathcal{F}$ does not admit invariant divisors contained in the linear system $V\subset|\mathcal{O}(D)|$.*

Proof. Suppose that $\mathcal{F}$ possesses an invariant divisor $C\in V$. Since all divisors $C\in V$ invariant by $\mathcal{F}$ are contained in the extactic divisor (Proposition 3.1) and by hypothesis $\mathcal{E}(\mathcal{F},V)$ is irreducible, we have that $C=\mathcal{E}(\mathcal{F},V)$. But

$$\deg_L(C)=\deg_L(D)<k\cdot\deg_L(D)+\deg_L(K_{\mathcal{F}})\binom{k}{2}=\deg_L(\mathcal{E}(\mathcal{F},V)),$$

which is absurd, since $k>1$. □

4. Bounding invariant hyperplane sections

Let M^n be a projective manifold embedded on a projective space $\mathbb{P}^N$. Let $|\mathcal{O}_M(1)|$ be the complete linear system of hyperplane sections of M^n on $\mathbb{P}^N$. In the next result, we will use the Zak's number defined by

$$\mathrm{Zak}(n,N) := \left[\frac{(4n-N+3)^2}{8(2n-N+1)}\right],$$

where $[x]$ denote the largest integer not exceeding x.

Corollary 4.1. *Let $\mathcal{F}$ be a one-dimensional foliation on a smooth algebraic variety $M^n \subset \mathbb{P}^N$. Suppose that $\mathcal{F}$ does not admit a rational first integral, then the number of $\mathcal{F}$-invariant hyperplane sections is at most*

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1\right) \cdot \binom{\mathrm{Zak}(n,N)}{2} + \mathrm{Zak}(n,N).$$

Proof. It follows from Theorem 1.3, and the fact that $\deg_{\mathcal{O}_{\mathcal{M}}(1)}(\mathcal{O}_M(1)) = \deg(M)$, that the number of $\mathcal{F}$-invariant hyperplane sections is at most

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1\right) \cdot \binom{h^0(M, \mathcal{O}_M(1))}{2} + h^0(M, \mathcal{O}_M(1)).$$

Now, the result follows from

$$h^0(M, \mathcal{O}_{_M}(1)) \leq \mathrm{Zak}(n,N),$$

see [25, p. 117, Theorem 2.10]. □

Example 2. Let $\mathcal{F}$ be a one-dimensional foliation on a projective manifold $M^n \subset \mathbb{P}^N$. Suppose that $\mathcal{F}$ does not admit a rational first integral. Then, if $N \leq 2n$, the number of $\mathcal{F}$-invariant hyperplane sections is at most

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1\right) \cdot \binom{\binom{n+2}{2}}{2} + \binom{n+2}{2}.$$

This is a consequence of Corollary 4.1 and of the following result (see [25, Corollary 2.9]): if $N \leq 2n$, then $h^0(M, \mathcal{O}_{_M}(1)) \leq \binom{n+2}{2}$.

Example 3. We recall that a non-singular algebraic variety $M^n \subset \mathbb{P}^N$ is called *linearly normal* if $h^0(M, \mathcal{O}_M(1)) = N+1$. Zak's Linear Normality theorem say that if $N < \frac{3}{2}n + 1$ then M^n is linearly normal, see [26]. Let $\mathcal{F}$ be a one-dimensional foliation on a linearly normal projective manifold $M \subset \mathbb{P}^N$. Suppose that $\mathcal{F}$ does not admit a rational first integral. Then it follows from Corollary 4.1 that the number of $\mathcal{F}$-invariant hyperplane sections is at most

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1\right) \cdot \binom{N+1}{2} + N + 1.$$

In particular, if $\deg(\mathcal{F}) = \deg(M)$ the number of $\mathcal{F}$-invariant hyperplane sections is at most $\frac{3}{2}n + 1$.

4.1. Optimal examples on projective spaces. In this section, we consider foliations on $\mathbb{P}^n$. We construct some examples of foliations with the maximum number of invariant hyperplanes. Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n$, of degree $d > 0$, and suppose that $\mathcal{F}$ does not admit a rational first integral. It follows from Example 3 that the number of $\mathcal{F}$-invariant hyperplanes is bounded by

$$\binom{n+1}{2}(d-1) + n + 1.$$

The next result gives us the number of invariant hyperplanes by a foliation on $\mathbb{P}^n$ which contain a fixed ℓ-plane, particularly the number of invariant hyperplanes through a point and the number of invariant hyperplanes containing an invariant line.

Proposition 4.1. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n$ of degree $d > 0$ and suppose that $\mathcal{F}$ does not admit a rational first integral. Then, the number of $\mathcal{F}$-invariant hyperplanes which contain a fixed ℓ-plane, $0 \le \ell \le n-2$, is bounded by*

$$\binom{n-\ell}{2}(d-1)+n-\ell.$$

Proof. We may assume the ℓ-plane $\mathbb{L}^\ell$ is the base locus of the linear subsystem $V_{n-\ell} \subset |\mathcal{O}_{\mathbb{P}^n}(1)|$ generated by $z_{\ell+1},\ldots,z_n$. Any hyperplane containing $\mathbb{L}^\ell$ belongs to $V_{n-\ell}$. The result follows by observing that $h^0(V_{n-\ell}) = n-\ell$. □

The next result says us that all the $\mathcal{F}$-invariant linear subspaces are contained in the linear extactic $\mathcal{E}(|\mathcal{O}_{\mathbb{P}^n}(1)|,\mathcal{F})$.

Proposition 4.2. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n$ that does not admit a rational first integral. Then all the $\mathcal{F}$-invariant linear subspaces are contained in the linear extactic $\mathcal{E}\left(|\mathcal{O}_{\mathbb{P}^n}(1)|,\mathcal{F}\right)$, where X is a vector field that induces $\mathcal{F}$ in homogeneous coordinates.*

Proof. In fact, if $\mathcal{F}$ admits no rational first integral then $\boldsymbol{\varepsilon}(\mathcal{F},|\mathcal{O}_{\mathbb{P}^n}(1)|) \neq 0$. Every linear k-codimensional subspace on $\mathbb{P}^n$ is the intersection of the zeros of k homogeneous polynomials of degree one, linearly independent, let us say $f_1,\ldots,f_k \in |\mathcal{O}_{\mathbb{P}^n}(1)|$. Then we can take

$$\{f_1,\ldots,f_k,h_{k+1},\ldots,h_{n+1}\}$$

to form a basis for $|\mathcal{O}_{\mathbb{P}^n}(1)|$. Now, if the linear space $W := \{f_1 = \cdots = f_k = 0\}$ is $\mathcal{F}$-invariant, it follows that $X(f_i) \in \mathcal{I}(f_1,\ldots,f_k)$, for all $i = 1,\ldots,k$, and so we obtain $X^j(f_i) \in \mathcal{I}(f_1,\ldots,f_k)$. Expanding the determinant

$$\boldsymbol{\varepsilon}(\mathcal{F},|\mathcal{O}_{\mathbb{P}^n}(1)|) = \det\begin{pmatrix} f_1 & \cdots & f_k & \cdots & h_{n+1} \\ X(f_1) & \cdots & X(f_k) & \cdots & X(h_{n+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X^n(f_1) & \cdots & X^n(f_k) & \cdots & X^n(h_{n+1}) \end{pmatrix}$$

in any of the kth first columns we see that $\boldsymbol{\varepsilon}(\mathcal{F},|\mathcal{O}_{\mathbb{P}^n}(1)|) \in \mathcal{I}(f_1,\ldots,f_k)$. Therefore

$$\mathcal{E}(\mathcal{F},|\mathcal{O}_{\mathbb{P}^n}(1)|) \supset W.$$

□

Consider the vector fields, defined in affine coordinates $z_0 = 1$, by

$$X_d^0 = \sum_{i=1}^{n}(z_i^{d-1}-1)z_i\frac{\partial}{\partial z_i},$$

$$X_d^1 = \frac{\partial}{\partial z_1} + \sum_{i=2}^{n}(z_i^{d-1}-1)z_i\frac{\partial}{\partial z_i},$$

$$X_d^\ell = \sum_{i=1}^{\ell}(z_1^d+\cdots+\widehat{z_i^d}+\cdots+z_\ell^d)\frac{\partial}{\partial z_i} + \sum_{i=\ell+1}^{n}(z_i^{d-1}-1)z_i\frac{\partial}{\partial z_i}, \quad 2 \le \ell \le n-2.$$

Remark that all the foliations $\mathcal{F}_{X_d^\ell}$ on $\mathbb{P}^n$ induced by X_d^ℓ, $0 \le \ell \le n-2$, leave the hyperplane at infinity invariant.

X_d^0 is an n-dimensional version of a member of the so called "family of degree four" in $\mathbb{P}^2$, one of the examples given by Lins Neto in [18]. A straightforward calculation shows that the $\binom{n+1}{2}(d-1)+n+1$ hyperplanes given by equation

$$(z_0 \dots z_n) \prod_{0\le i,j\le n} (z_i^{d-1} - z_j^{d-1}) = 0$$

are invariant by $\mathcal{F}_{X_d^0}$.

For each $\ell = 1, \dots, n-2$ consider the ℓ-plane $\mathbb{L}_\ell = \{z_{\ell+1} = \cdots = z_n = 0\}$, which is the base locus of the linear system $\sum_{j=\ell+1}^{n} \lambda_i z_i$. The foliation $\mathcal{F}_{X_d^\ell}$ leaves invariant the $\binom{n-\ell}{2}(d-1)+n-\ell$ hyperplanes, all containing $\mathbb{L}_\ell$, whose equation is

$$(z_{\ell+1} \dots z_n) \prod_{\ell+1\le i,j\le n} (z_i^{d-1} - z_j^{d-1}) = 0.$$

Moreover, the $(n-\ell)$-plane $\mathbb{L}_\ell^\perp = \{z_1 = \cdots = z_\ell = 0\}$ is also $\mathcal{F}_{X_d^\ell}$-invariant, whereas the hyperplane $\{z_i = 0\}$, $1 \le i \le \ell$ are not.

Remark 3. The foliation $\mathcal{F}_{X_d^0}$ on $\mathbb{P}^n$ induced by the vector field X_d^0 is the unique foliation of degree d that leaves invariant the following arrangement of hyperplanes:

$$\mathscr{A}_d = \left\{ (z_0 \dots z_n) \prod_{0\le i,j\le n} (z_i^{d-1} - z_j^{d-1}) = 0 \right\}.$$

Indeed, the singular set $\mathrm{Sing}(\mathcal{F})$ of $\mathcal{F}$ is isolated and non-degenerated. On the other hand, we can see that $\mathrm{Sing}(\mathcal{F})$ is determined by intersection of the hyperplanes of $\mathscr{A}_d$. It follows from [16] that $\mathcal{F}$ is unique.

5. Holomorphic foliation with all leaves compact

The results of this work give conditions for a one-dimensional holomorphic foliation to be tangent to the fibers of a meromorphic function. In [15], Gomez–Mont provides an extension of a theorem due to Edwards, Millett and Sullivan concerning foliations with all leaves compact. He proved that if a singular holomorphic foliation $\mathcal{F}$, of codimension q, on a projective manifold have all leaves algebraic, then $\mathcal{F}$ is tangent to the fibers if a rational map. Pereira proved in [21] the Gomez–Mont's result using extactic divisors.

See the following examples of one-dimensional holomorphic foliations with leaves compact:

Example 4. Let E be an elliptic curve and $M = (E \times E) \times E$. Consider on M a foliation $\mathcal{F}$ by lines on $(E \times E)_z$ with a slop that depends holomorphicaly on $z \in E$.

Example 5. Consider a discrete subgroup Γ of $SL(2,\mathbb{C})$ such that $X = SL(2,\mathbb{C})/\Gamma$ is compact. The action over X of the diagonal subgroup of $SL(2,\mathbb{C})$ induces a one-dimensional holomorphic foliation $\mathcal{F}$ on X with infinity leaves compact. Moreover, the union these leaves compact is dense in X, see [13, Example 6; 14].

Appendix A. Jouanolou–Ghys theorem for one-dimensional foliations

Let M be a compact complex manifold. Let Ω^1_M(resp. $\mathscr{M}^1_M$) be the sheaf of holomorphic (resp. meromorphic) 1-forms on M; we have a short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_M \longrightarrow \mathscr{M}^1_M \longrightarrow \mathscr{Q}^1_M \longrightarrow 0,$$

where $\mathscr{Q}^1_M$ denotes the quotient sheaf $\mathscr{M}^1_M/\Omega^1_M$. Let $\{f_\alpha = 0\}$ be an irreducible analytic hypersurface of M. The meromorphic form

$$\frac{df_\alpha}{f_\alpha}$$

is a well-defined global section of $\mathscr{Q}^1_M$. In fact, if $f_\alpha = g_{\alpha\beta} f_\beta$, with $g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\alpha)$ then

$$\frac{df_\alpha}{f_\alpha} - \frac{df_\beta}{f_\beta} = \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} \in \Omega^1_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}.$$

Denote by $\mathrm{Div}(M, \mathcal{F})$ the abelian group of divisors on M invariant by $\mathcal{F}$. Now, consider the following $\mathbb{C}$-linear maps:

$$\begin{array}{ccccc} \mathrm{Div}(M,\mathcal{F}) \otimes \mathbb{C} & \longrightarrow & H^1(M, \mathcal{O}^*) & \longrightarrow & H^1(M, \Omega^1_f) \\ (f_\alpha) & \longmapsto & \left[\dfrac{f_\alpha}{f_\beta} = g_{\alpha\beta}\right] & \longmapsto & \left[\dfrac{dg_{\alpha\beta}}{g_{\alpha\beta}}\right], \end{array}$$

where Ω^1_f denotes the sheaf of closed holomorphic 1-forms on M. Since M is compact $\dim_{\mathbb{C}} H^1(M, \Omega^1_f) < \infty$, see [13]. We will denote the composed of these linear maps by

$$\zeta : \mathrm{Div}(M, \mathcal{F}) \otimes \mathbb{C} \longrightarrow H^1(M, \Omega^1_f).$$

Let $\mathrm{Div}_0(M, \mathcal{F})$ be the kernel of ζ. If $(f_\alpha) \in \mathrm{Div}_0(M, \mathcal{F})$ then there are closed 1-forms ν_α on $\mathcal{U}_\alpha$ such that

$$\nu_\alpha - \nu_\beta = \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} = \frac{df_\alpha}{f_\alpha} - \frac{df_\beta}{f_\beta}$$

on $\mathcal{U}_\alpha \cap \mathcal{U}_\alpha \neq \emptyset$. Thus, we have a well-defined global closed meromorphic 1-form η given on $\mathcal{U}_\alpha$ by

$$\eta_{|\mathcal{U}_\alpha} = \frac{df_\alpha}{f_\alpha} + \nu_\alpha.$$

Contacting by X_α we have a linear map $i_{X_{\mathcal{F}}} : H^0(M, \Omega^1_f) \to H^0(M, K_{\mathcal{F}})$. Finally, we get the following $\mathbb{C}$-linear map:

$$\begin{array}{ccc} \xi : \mathrm{Div}_0(M, \mathcal{F}) & \longrightarrow & H^0(M, K_{\mathcal{F}})/i_{X_{\mathcal{F}}}(H^0(M, \Omega^1_f)) \\ (f_\alpha) & \longmapsto & \overline{\dfrac{df_\alpha}{f_\alpha}(X_\alpha) + \nu_\alpha(X_\alpha)}. \end{array}$$

Theorem A.1. *Let $\mathcal{F}$ be a one-dimensional foliation on a compact complex manifold M. If $\mathcal{F}$ admits*

$$\dim_{\mathbb{C}}(H^1(M, \Omega^1_f)) + \dim_{\mathbb{C}}(H^0(M, K_{\mathcal{F}})/i_{X_{\mathcal{F}}}(H^0(M, \Omega^1_f))) + \dim(M)$$

invariants irreducible analytic hypersurfaces, then $\mathcal{F}$ admit a meromorphic first integral.

Proof. Suppose that $\mathcal{F}$ admits at least

$$\dim_{\mathbb{C}}(H^1(M,\Omega^1_f)) + \dim_{\mathbb{C}}(H^0(M,K_{\mathcal{F}})/i_{X_{\mathcal{F}}}(H^0(M,\Omega^1_f))) + \dim(M)$$

invariants irreducible analytic hypersurfaces. Under this assumption, we conclude that

$$\dim_{\mathbb{C}} \mathrm{Div}_0(M,\mathcal{F}) \geq \dim_{\mathbb{C}}(H^0(M,K_{\mathcal{F}})/i_{X_{\mathcal{F}}}(H^0(M,\Omega^1_f))) + \dim(M)$$

and $\dim_{\mathbb{C}} \ker(\xi) \geq \dim(M)$. Therefore, we have closed meromorphic 1-forms $\eta_1,\ldots,\eta_n$, where $n = \dim(M)$, with different set of poles and such that $\eta_i(X_\alpha) \equiv 0$, for all α and $i = 1,\ldots,n$.

We claim that $\eta_1,\ldots,\eta_n$ are linearly dependent over the field of meromorphic functions $\mathscr{M}(M)$. Otherwise, on the open $\mathcal{U}_\alpha$ there exists a meromorphic function $R^\alpha \neq 0$ such that

$$\eta_1 \wedge \cdots \wedge \eta_n|_{\mathcal{U}_\alpha} = R^\alpha dz_1^\alpha \wedge \cdots \wedge dz_n^\alpha.$$

Contracting by $X_\alpha = \sum_{i=1}^n P_i^\alpha \frac{\partial}{\partial z_i^\alpha}$ we have $R^\alpha \cdot (dz_1 \wedge \cdots \wedge dz_n)(X_\alpha) = 0$, since $\eta_i(X_\alpha) = 0$, for all $i = 1,\ldots,n$. But $R^\alpha \neq 0$, thus

$$0 = (dz_1^\alpha \wedge \cdots \wedge dz_n^\alpha)(X_\alpha) = \sum_{i=1}^n (-1)^{i+1} P_i^\alpha dz_1^\alpha \wedge \cdots \wedge \widehat{dz_i^\alpha} \wedge \cdots \wedge dz_n^\alpha.$$

This implies that $P_1^\alpha = \cdots = P_n^\alpha = 0$, i.e, $X_\alpha \equiv 0$, an absurd. Let W be the $\mathscr{M}(M)$-linear space generated by $\{\eta_1,\ldots,\eta_n\}$, suppose that $\dim_{\mathscr{M}(M)} W = k$ and

$$W = \langle \eta_1,\ldots,\eta_k \rangle_{\mathscr{M}(M)}.$$

There exist meromorphic functions $R_1,\ldots,R_k, \in \mathscr{M}(M)$, such that

$$\eta_{k+1} = R_1\eta_1 + \cdots + R_k\eta_k. \tag{A.1}$$

Sine η_i is closed, $i = 1,\ldots,k$, by differentiation we obtain

$$0 = dR_1 \wedge \eta_1 + \cdots + dR_k \wedge \eta_k.$$

Now, contracting by X_α results

$$0 = X_\alpha(R_1)\eta_1 + \cdots + X_\alpha(R_k)\eta_k.$$

Thus $X_\alpha(R_1) = \cdots = X_\alpha(R_k) = 0$, for all α. That is, the meromorphic function R_i, $i = 1,\ldots,k$, is either a first integral for the foliation $\mathcal{F}$ or is constant. It remains to observe that at least one rational function R_i is not constant. Indeed, this follows from relation (A.1) and that the poles set of $\eta_1,\ldots,\eta_k$ are different. □

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