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TWO DIMENSIONAL INVISIBILITY CLOAKING FOR HELMHOLTZ EQUATION AND NON-LOCAL BOUNDARY CONDITIONS

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Abstract. Transformation optics constructions have allowed the design of cloaking devices that steer electromagnetic, acoustic and quantum parameters waves around a region without penetrating it, so that this region is hidden from external observations. The material parameters used to describe these devices are anisotropic, and singular at the interface between the cloaked and uncloaked regions, making physical realization a challenge. These singular material parameters correspond to singular coefficient functions in the partial differential equations modeling these constructions and the presence of these singularities causes various mathematical problems and physical effects on the interface surface.

In this paper, we analyze the two dimensional cloaking for Helmholtz equation when there are sources or sinks present inside the cloaked region. In particular, we consider nonsingular approximate invisibility cloaks based on the truncation of the singular transformations. Using such truncation we analyze the limit when the approximate cloaking approaches the ideal cloaking. We show that, surprisingly, a non-local boundary condition appears on the inner cloak interface. This effect in the two dimensional (or cylindrical) invisibility cloaks, which seems to be caused by the infinite phase velocity near the interface between the cloaked and uncloaked regions, is very different to the earlier studied behavior of the solutions in the three dimensional cloaks.

1. Introduction

There has recently been much activity concerning *cloaking*, or rendering objects invisible to detection by electromagnetic, acoustic, or other type of waves or physical fields. Many suggestions to implement cloaking has been based on *transformation optics*, that is, designs of electromagnetic or acoustic devices with customized effects on wave propagation, made possible by taking advantage of the transformation rules for the material properties of optics. All perfect cloaking devices based on transformation optics require anisotropic and singular material parameters, whether the conductivity (electrostatic) [16, 17], index of refraction (Helmholtz) [24], [10], permittivity and permeability (Maxwell) [33], [10], mass tensor (acoustic) [10], [6], [9], or effective mass (Schrödinger) [13, 14, 39]. By *singular* material parameters, we mean that at least one of the eigenvalues or the values of the functions describing the material properties goes to zero or infinity at some points when the material parameters are represented in Euclidean coordinates, typically on the interface between the cloaked and uncloaked regions. Both the anisotropy and singularity present serious challenges in trying to physically realize such theoretical plans using metamaterials. Analogous

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difficulties are encountered in the study of invisibility cloaks base on ray-theory [25] and plasmonic resonances [1, 29].

To justify the invisibility cloaking constructions, one needs to study physically meaningful solutions of the resulting partial differential equations on the whole domain, including the region where material parameters become singular. In [10], the *finite energy solutions* are defined to be at least measurable functions with finite energy in (degenerate) singular weighted Sobolev spaces, and satisfy the equations in the distributional sense.

Due to the presence of singular material parameters, or mathematically speaking, partial differential equations with singular coefficient functions, the question how the waves behave in cloaking devices near the surface where the material parameters are singular is complicated. Indeed, very different kind of behaviors of solutions are possible: In the three dimensional case, it is proved in [10] that the transformation optics construction based on a blow up map allows cloaking with respect to time-harmonic solutions of the Helmholtz equation or Maxwell's equations as long as the object being cloaked is passive. In fact, for the Helmholtz equation, the object can be an active source or sink. Moreover, in [10] it is shown that the finite energy solutions for the Helmholtz equation in the three dimensional case satisfy a hidden boundary condition, namely waves inside the cloaked region satisfy the Neumann boundary conditions. For Maxwell's equations, the finite energy solutions inside the cloaked region need to have vanishing Cauchy data i.e., the hidden boundary conditions are over-determined. This leads to non-existence of finite energy solutions for Maxwell's equations with generic internal currents [10]. Physically, this non-existence results is related to the so-called extraordinary boundary effects on the interface between the cloaked and uncloaked regions [40].

Another point of view in dealing with the singular anisotropic design for cloaking devices is to approximate the ideal cloaking parameters by nonsingular, or even nonsingular and isotropic, parameters [12, 13, 14, 19, 20, 28], which has its advantages in practical fabrication. In the truncation based nonsingular approximate cloaking for three dimensional Helmholtz equation [14], when it approaches the ideal cloaking, one can obtain above Neumann hidden boundary condition for the finite energy solution. Considering limits of the nonsingular and isotropic approximate cloaks in the three dimensional case, one can obtain different types of Robin boundary conditions by varying slightly the way how the approximative cloak in constructed, see [12, 13, 14].

Similarly, for Maxwell's equations it has been studied how the approximate cloak behave in the limit when the approximate cloaks approach the ideal one [28]. We note that for Maxwell's equations there are various suggestions what kind of limiting cloaks are possible in three dimensions. These suggestions are based on constructions where additional layers (e.g. perfectly conducting layer) is attached inside the cloak [10] or where the ideal cloak corresponds to some of the possible self-adjoint extensions of Maxwell's equations [37, 38]. In the two dimensional or cylindrical cloaking construction for Maxwell's equations, the eigenvalues of the permittivity and the permeability of the cloaking medium do not only contain eigenvalues approaching to zero (as in 3D) but also some of the eigenvalues approach infinity at the cloaking interface.

Then, the electric flux density D and magnetic flux density B may blow up even when there are no sources inside the cloak and an incident plane wave is scattered from the cloak, see [11]. However, if a soft-hard (SH)-surface is included inside the cloak, the solutions behave well.

These above examples show the different behaviors the solutions may have, in different type of cloaking devices, near the interface between the cloaked and uncloaked regions.

In this paper, we analyze the two dimensional cloaking for the Helmholtz equation when there are sources present inside the cloaked region. We start with the nonsingular approximate cloaking based on the truncation of the singular transformation. Taking the limit when the approximate cloaks approach the ideal cloak, we show that a non-local boundary condition appears on the inner cloak interface. This type of boundary behavior is very different from that the solutions have in three dimensional case discussed in [14, 40]. The main result is formulated as Theorem 3.2. We note that in the recent preprint [32] of Hoai-Minh Nguyen a different type of formulation, based on a transmission problem, is given for the non-local boundary condition appearing in two-dimensional cloaking. In [32], also cloaking for more general second order equations and quantitative convergence properties of the three and two dimensional approximative cloaks are analyzed.

Physically speaking, the non-local boundary condition is possible due to the fact that the phase velocity of the waves in the invisibility cloak approaches infinity near the interface between the cloaked and uncloaked regions, even though the group velocity stays finite, see [7].

We note that as the most important experimental implementations of invisibility cloaks [34] have been based on cylindrical cloaks, the appearance of such boundary condition could also be studied in the present experimental configurations, at least on micro-wave frequencies. We note that the early experimental implementations of cloaking were actually for the so-called reduced parameter set, which does not have the same singular behavior as the material parameters studied here. However, the nonlocal boundary condition considered in this paper could be studied using numerical simulations or possibly even in experimental tests using the current metamaterials.

We also study the eigenvalues, i.e., resonances inside the ideal cloak corresponding to the non-local boundary condition. As these eigenvalues play an essential role in the study of almost trapped states [13] and in the development of the invisible sensors [2, 15], such resonances can be used to study analogous constructions in cylindrical geometry.

The rest of the paper is organized as following. In Section 2, we present the basics on transformation optics in the electrostatics setting and apply them to the construction of acoustic ideal cloak for the two dimensional Helmholtz equation. Section 3 is devoted to the nonsingular approximate acoustic cloaking construction and analysis of behaviors of acoustic waves as it approaches the ideal cloaking.

2. Perfect acoustic cloaking

2.1. Background: electrostatics. Our analysis is closely related to the inverse problem for electrostatics, or Calderón's conductivity problem [3, 5, 30, 31, 36]. Let $\Omega \subset \mathbb{R}^d$ be a domain, at the boundary of which electrostatic measurements are to be made, and denote by $\sigma(x)$ the anisotropic conductivity within. In the absence of sources, an electrostatic potential u satisfies a divergence form equation,

$$\nabla \cdot \sigma \nabla u = 0 \tag{1}$$

on Ω. To uniquely fix the solution u it is enough to give its value, f, on the boundary. In the idealized case, one measures, for all voltage distributions $u|_{\partial\Omega} = f$ on the boundary the corresponding current fluxes, $\nu\cdot \sigma\nabla u$, where ν is the exterior unit normal to $\partial\Omega$. Mathematically this amounts to the knowledge of the Dirichlet–Neumann (DN) map, Λ_σ. corresponding to σ, i.e., the map taking the Dirichlet boundary values of the solution to (1) to the corresponding Neumann boundary values,

$$\Lambda_\sigma : \quad u|_{\partial\Omega} \mapsto \nu\cdot \sigma\nabla u|_{\partial\Omega}.$$

If $F : \Omega \to \Omega, \quad F = (F^1, \dots, F^d)$, is a diffeomorphism with $F|_{\partial\Omega} = \text{Identity}$, then by making the change of variables $y = F(x)$ and setting $u = v \circ F^{-1}$, we obtain

$$\nabla \cdot \widetilde{\sigma} \nabla v = 0, \tag{2}$$

where $\widetilde{\sigma} = F_*\sigma$ is the push forward of σ in F,

$$(F_*\sigma)^{jk}(x) = \frac{1}{\det[\frac{\partial F}{\partial y}(y)]} \sum_{p,q=1}^{d} \frac{\partial F^j}{\partial y^p}(y) \frac{\partial F^k}{\partial y^q}(y) \sigma^{pq}(y)\Big|_{y=F^{-1}(x)}. \tag{3}$$

This can be used to show that

$$\Lambda_{F_*\sigma} = \Lambda_\sigma.$$

Thus, there is a large (infinite-dimensional) family of conductivities which all give rise to the same electrostatic measurements at the boundary. This observation is due to Luc Tartar (see [21] for an account.) Calderón's inverse problem for anisotropic conductivities is then the question of whether two conductivities with the same DN operator must be push-forwards of each other. There are a number of positive results in this direction in two dimensions [4, 22, 23, 26, 35] , but it was shown in [16, 17] in three dimensions and in [20] two dimensions that, if one allows singular maps, then in fact there are counterexamples, i.e., conductivities that are undetectable to electrostatic measurements at the boundary.

From now on, we will restrict ourselves to the two dimensional case. For each $R > 0$, let $B_R = \{|x| \le R\}$ and $\Sigma_R = \{|x| = R\}$ be the central ball and sphere of radius R, resp., in $\mathbb{R}^2$, and let $O = (0, 0, 0)$ denote the origin. To construct an invisibility cloak, for simplicity we use the specific singular coordinate transformation $F : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus B_1$, given by

$$x = F(y) := \begin{cases} y, & \text{for } |y| > 2, \\ \left(1 + \frac{|y|}{2}\right) \frac{y}{|y|}, & \text{for } 0 < |y| \le 2. \end{cases} \tag{4}$$

Letting $\sigma_0 = 1$ be the homogeneous isotropic conductivity on $\mathbb{R}^2$, F then defines a conductivity σ on $\mathbb{R}^2\backslash B_1$ by the formula

$$\sigma^{jk}(x) := (F_*\sigma_0)^{jk}(x), \tag{5}$$

cf. (3). More explicitly, the matrix $\sigma = [\sigma^{jk}]_{j,k=1}^2$ is of the form

$$\sigma(x) = \frac{|x|-1}{|x|}\Pi(x) + \frac{|x|}{|x|-1}(I - \Pi(x)), \quad 1 < |x| < 2,$$

where $\Pi(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection to the radial direction, defined by

$$\Pi(x)\, v = \left(v \cdot \frac{x}{|x|}\right)\frac{x}{|x|}, \tag{6}$$

i.e., $\Pi(x)$ is represented by the matrix $|x|^{-2}xx^t$, cf. [20].

One sees that $\sigma(x)$ is singular at Σ_1, that is, at the interface between the cloaked and uncloaked regions, as one of its eigenvalues, namely the one corresponding to the radial direction, tends to 0, and the other tends to ∞ as $|x|$ approaching 1^+. We extend σ to B_1 as an arbitrary smooth, nonsingular (bounded from above and below) conductivity there. Let $\Omega = B_3$; the conductivity σ is then a cloaking conductivity on Ω, as it is indistinguishable from σ_0, *vis-a-vis* electrostatic boundary measurements of electrostatic potentials (treated rigorously as bounded, distributional solutions of the degenerate elliptic boundary value problem corresponding to σ [16, 17]).

A similar construction based on a blow up map F was proposed in Pendry, Schurig and Smith [33] to cloak the region B_1 in $\mathbb{R}^3$ from observation by electromagnetic waves at a positive frequency; see also Leonhardt [24] for a related proposition for the Helmholtz equation in $\mathbb{R}^2$ based on the use of several leaves of an Riemann surface.

2.2. Ideal cloaking for the Helmholtz equation with interior sources. We consider the Helmholtz equation, with source term p, of the form

$$\lambda\nabla\cdot\sigma\nabla u + \omega^2 u = p(x) \qquad \text{on } \Omega \tag{7}$$

corresponding to a cloaking medium with the inverse of the anisotropic mass density and the bulk modulus given by

$$\sigma^{jk} = \left\{\begin{array}{cl} \sigma_0^{jk} & \text{for } \ |x| > 2 \\ (F_*\sigma_0)^{jk} & \text{for } \ 1 < |x| \le 2 \\ \sigma_a^{jk} & \text{for } \ |x| \le 1 \end{array}\right. , \quad \lambda = \left\{\begin{array}{cl} \lambda_0 & \text{for } \ |x| > 2 \\ F_*\lambda_0 & \text{for } \ 1 < |x| \le 2 \\ \lambda_a & \text{for } \ |x| \le 1 \end{array}\right. \tag{8}$$

where (σ_0, λ_0) corresponds to homogeneous background space and (σ_a, λ_a) are arbitrary smooth, nondegenerate medium in cloaked region B_1. The push-forward of tensor $F_*\sigma_0$ is defined by (3) and $F_*\lambda_0$ by

$$(F_*\lambda_0)(x) := [\det(DF)\lambda_0] \circ F^{-1}(x).$$

More specifically, if $(\sigma_0, \lambda_0) = (I, 1)$, one has for $1 < |x| < 2$,

$$\sigma(x) = \frac{|x|-1}{|x|}\Pi(x) + \frac{|x|}{|x|-1}(I - \Pi(x)), \quad \lambda(x) = \frac{|x|}{4(|x|-1)}$$

are both singular at the cloaking surface $\Sigma := \Sigma_1 = \partial B_1$.

In the next section, we study the regularized approximate cloaking and obtain the behavior of acoustic waves in the singular medium by taking the limit of waves propagating in the nonsingular medium.

3. Nonsingular approximate cloaking for the Helmholtz equation with interior sources

For the moment, we assume that σ and λ be homogeneous isotropic in side B_1, i.e., $(\sigma,\lambda) = (\sigma_a\delta^{jk},\lambda_a)$ with σ_a and λ_a arbitrary positive constants.

To start, let $1 < R < 2$, $\rho = 2(R-1)$ and introduce the coordinate transformation $F_R : \mathbb{R}^2\backslash B_\rho \to \mathbb{R}^2\backslash B_R$,

$$x := F_R(y) = \begin{cases} y, & \text{for } |y| > 2,\\ \left(1+\frac{|y|}{2}\right)\frac{y}{|y|}, & \text{for } \rho < |y| \le 2. \end{cases}$$

We define the corresponding approximate mass tensor σ_R and bulk modulus λ_R as

$$\sigma_R^{jk}(x) = \begin{cases} \sigma^{jk}(x) & \text{for } |x| > R,\\ \sigma_a\delta^{jk} & \text{for } |x| \le R. \end{cases} \qquad \lambda_R(x) = \begin{cases} \lambda(x) & \text{for } |x| > R,\\ \lambda_a & \text{for } |x| \le R, \end{cases} \tag{9}$$

where σ^{jk} and λ are as in (8). Note that then $\sigma^{jk}(x) = \left((F_R)_*\,\sigma_0\right)^{jk}(x)$ and $\lambda(x) = \left((F_R)_*\,\lambda_0\right)(x)$ for $|x| > R$. Observe that, for each $R > 1$, the medium is nonsingular, i.e., is bounded from above and below with, however, the lower bound going to 0, and the upper bound going to ∞ as $R \to 1^+$. Consider the solutions of

$$\begin{aligned} (\lambda_R\nabla\cdot\sigma_R\nabla + \omega^2)u_R =& p \quad \text{in } \Omega\\ u_R|_{\partial\Omega} =& f, \end{aligned} \tag{10}$$

As σ_R and λ_R are now non-singular everywhere on Ω, we have the standard transmission conditions on $\Sigma_R := \{x : \ |x| = R\}$,

$$\begin{aligned} u_R|_{\Sigma_R+} &= u_R|_{\Sigma_R-},\\ e_r\cdot\sigma_R\nabla u_R|_{\Sigma_R+} &= e_r\cdot\sigma_R\nabla u_R|_{\Sigma_R-}, \end{aligned} \tag{11}$$

where e_r is the radial unit vector and $\pm$ indicates when the trace on Σ_R is computed as the limit $r \to R^{\pm}$.

Let $\Omega = B_3$. Then u_R defines two functions $v_R^{\pm}$ such that

$$u_R(x) = \begin{cases} v_R^+(F_R^{-1}(x)), & \text{for } R < |x| < 3,\\ v_R^-(x), & \text{for } |x| \le R, \end{cases}$$

and $v_R^{\pm}$ satisfy

$$\begin{aligned} (\nabla^2+\omega^2)v_R^+(y) = p(F_R(y)) \quad \text{in } \rho < |y| < 3,\\ v_R^+|_{\partial B_3} = f, \end{aligned} \tag{12}$$

and

$$(\nabla^2+\kappa^2\omega^2)v_R^-(x) \quad = \quad \kappa^2 p(x), \quad \text{in } |x| < R. \tag{13}$$

where $\kappa^2 = (\sigma_a\lambda_a)^{-1}$ is a constant. Moreover, if we assume ω^2 is not an eigenvalue of the transmission problem, then by the transformation law we have

$$e_r \cdot \sigma_R \nabla u_R\big|_{\partial\Omega} = e_r \cdot \nabla v_R^+\big|_{\partial\Omega}.$$

This implies that the DN-map $\Lambda_{\sigma_R,\lambda_R}$ at $\partial\Omega$ for the approximate cloaking medium (9) is the same as the DN-map at $\partial\Omega$, denoted by Λ_ρ, of a nearly vacuum domain with a small inclusion present in B_ρ.

Next, using polar coordinates (r,θ), $r = |y|$, and $(\widetilde{r},\theta)$, $\widetilde{r} = |x|$, the transmission conditions (11) on the surface Σ_R yield

$$\begin{aligned} v_R^+(\rho,\theta) &= v_R^-(R,\theta), \\ \rho\, \partial_r v_R^+(\rho,\theta) &= \kappa R\, \partial_{\widetilde{r}} v_R^-(R,\theta). \end{aligned} \tag{14}$$

We consider the source term $\kappa^2 p(x)$ where $p(x) \in C^\infty(\mathbb{R}^2)$ with $\operatorname{supp} p \subset B_{R_0}$ $(0 < R_0 < 1)$. It generates a radiating wave $w(x) \in C^\infty(\mathbb{R}^2)$, namely the solution of

$$(\nabla^2 + \kappa^2\omega^2)w = \kappa^2 p \qquad \text{in } \mathbb{R}^2 \tag{15}$$

satisfying the Sommerfeld radiation condition. Moreover, one can write w as

$$w(\widetilde{r},\theta) = \sum_{n=-\infty}^{\infty} p_n H_{|n|}^{(1)}(\kappa\omega\widetilde{r})e^{in\theta}, \qquad \text{for } \widetilde{r} > R_0 \tag{16}$$

where $H_{|n|}^{(1)}(z)$ and $J_{|n|}(z)$ denote the Hankel and Bessel functions, see [8].

In $B_R \backslash \overline{B_{R_0}}$ the function $v_R^-(x)$ differs from w by an entire solution to the homogeneous equation of (13), and thus for $\widetilde{r} \in (R_0, R)$

$$v_R^-(\widetilde{r},\theta) \;=\; \sum_{n=-\infty}^{\infty} (a_n J_{|n|}(\kappa\omega\widetilde{r}) + p_n H_{|n|}^{(1)}(\kappa\omega\widetilde{r}))e^{in\theta},$$

with yet undefined $a_n = a_n(\kappa,\omega;R)$. Similarly, for $\rho < r < 3$,

$$v_R^+(r,\theta) \;=\; \sum_{n=-\infty}^{\infty} \big(c_n H_{|n|}^{(1)}(\omega r) + b_n J_{|n|}(\omega r)\big)e^{in\theta}, \tag{17}$$

with as yet unspecified $b_n = b_n(\kappa,\omega;R)$ and $c_n = c_n(\kappa,\omega;R)$.

Rewriting the boundary value f on $\partial\Omega$ as

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta},$$

we obtain, together with transmission conditions (14), the following equations for a_n, b_n and c_n:

$$f_n = b_n J_{|n|}(3\omega) + c_n H_{|n|}^{(1)}(3\omega), \tag{18}$$

$$a_n J_{|n|}(\kappa\omega R) + p_n H_{|n|}^{(1)}(\kappa\omega R) = b_n J_{|n|}(\omega\rho) + c_n H_{|n|}^{(1)}(\omega\rho), \tag{19}$$

$$\begin{aligned} &\kappa R(\kappa\omega a_n (J_{|n|})'(\kappa\omega R) + \kappa\omega p_n (H_{|n|}^{(1)})'(\kappa\omega R)) \\ &\quad = \rho(b_n\omega(J_{|n|})'(\omega\rho) + \omega c_n (H_{|n|}^{(1)})'(\omega\rho)). \end{aligned} \tag{20}$$

Solve for a_n and c_n from (19)-(20) in terms of p_n and b_n, and use the solutions obtained and the equation (18) to solve for b_n in terms of f_n and p_n. This yields

$$\begin{aligned} b_n &= \frac{1}{J_{|n|}(3\omega)+s_nH^{(1)}_{|n|}(3\omega)}(f_n+\widetilde{s}_nH^{(1)}_{|n|}(3\omega)p_n), \\ c_n &= s_nb_n-\widetilde{s}_np_n, \\ a_n &= t_nb_n-\widetilde{t}_np_n \end{aligned} \tag{21}$$

where

$$\begin{aligned} s_n &= \frac{1}{D_n}\left\{\rho J_{|n|}(\kappa\omega R)J'_{|n|}(\omega\rho)-\kappa^2RJ'_{|n|}(\kappa\omega R)J_{|n|}(\omega\rho)\right\}, \\ t_n &= \frac{1}{D_n}\left\{\rho H^{(1)}_{|n|}(\omega\rho)J'_{|n|}(\omega\rho)-\rho(H^{(1)}_{|n|})'(\omega\rho)J_{|n|}(\omega\rho)\right\}, \\ \widetilde{s}_n &= \frac{1}{D_n}\left\{\kappa^2R(H^{(1)}_{|n|})'(\kappa\omega R)J_{|n|}(\kappa\omega R)-\kappa^2RJ'_{|n|}(\kappa\omega R)H^{(1)}_{|n|}(\kappa\omega R)\right\}, \\ \widetilde{t}_n &= \frac{1}{D_n}\left\{\kappa^2RH^{(1)}_{|n|}(\omega\rho)(H^{(1)}_{|n|})'(\kappa\omega R)-\rho(H^{(1)}_{|n|})'(\omega\rho)H^{(1)}_{|n|}(\kappa\omega R)\right\} \end{aligned}$$

with D_n the common denominator given by

$$D_n=\kappa^2RJ'_{|n|}(\kappa\omega R)H^{(1)}_{|n|}(\omega\rho)-\rho J_{|n|}(\kappa\omega R)(H^{(1)}_{|n|})'(\omega\rho). \tag{22}$$

3.1. Resonances inside the cloak. Suppose that the boundary data vanishes, i.e., $f\equiv 0$. Then by (21), we have

$$b_n=\frac{\widetilde{s}_nH^{(1)}_{|n|}(3\omega)}{J_{|n|}(3\omega)+s_nH^{(1)}_{|n|}(3\omega)}p_n.$$

Therefore, one can show

$$\begin{aligned} a_n &= \frac{(t_n\widetilde{s}_n-\widetilde{t}_ns_n)H^{(1)}_{|n|}(3\omega)-\widetilde{t}_nJ_{|n|}(3\omega)}{J_{|n|}(3\omega)+s_nH^{(1)}_{|n|}(3\omega)}p_n \\ &= \frac{\kappa^2R(H^{(1)}_{|n|})'(\kappa\omega R)l_1-\rho H^{(1)}_{|n|}(\kappa\omega R)l_2}{\rho J_{|n|}(\kappa\omega R)l_2-\kappa^2RJ'_{|n|}(\kappa\omega R)l_1}p_n:=\frac{A_n}{B_n}p_n \end{aligned} \tag{23}$$

where

$$\begin{aligned} l_1 &= J_{|n|}(\omega\rho)H^{(1)}_{|n|}(3\omega)-H^{(1)}_{|n|}(\omega\rho)J_{|n|}(3\omega), \\ l_2 &= J'_{|n|}(\omega\rho)H^{(1)}_{|n|}(3\omega)-(H^{(1)}_{|n|})'(\omega\rho)J_{|n|}(3\omega). \end{aligned} \tag{24}$$

For small arguments $0<x\ll 1$,

(25)

$$J_{|n|}(x)\sim\begin{cases}1 & n=0,\\ \frac{1}{n!}\left(\frac{x}{2}\right)^n & n\geq 1,\end{cases}\qquad J'_{|n|}(x)\sim\begin{cases}-\frac{x}{2} & n=0,\\ \frac{1}{2(n-1)!}\left(\frac{x}{2}\right)^{n-1} & n\geq 1,\end{cases}$$

$$H^{(1)}_{|n|}(x)\sim\begin{cases}\frac{2i}{\pi}\ln\left(\frac{x}{2}\right) & n=0,\\ -\frac{i(n-1)!}{\pi}\left(\frac{2}{x}\right)^n & n\geq 1,\end{cases}\qquad (H^{(1)}_{|n|})'(x)\sim\begin{cases}2ix^{-1}/\pi & n=0,\\ in!2^nx^{-n-1}/\pi & n\geq 1,\end{cases}$$

when $n \geq 1$, where we denote $f \sim g$ if $f - g = o(g)$ as $x \to 0$. Thus, for $n \geq 1$, we have

(26)
$$A_n = \frac{i2^n\omega^{-n-1}(n-1)!}{\pi} J_n(3\omega)\big[\omega\kappa^2 R(H_n^{(1)})'(\kappa\omega R) + nH_n^{(1)}(\kappa\omega R)\big]\rho^{-n} + O(\rho^{-n+1}),$$
$$B_n = \frac{-i2^n\omega^{-n-1}(n-1)!}{\pi} J_n(3\omega)\big[\omega\kappa^2 R J_n'(\kappa\omega R) + nJ_n(\kappa\omega R)\big]\rho^{-n} + O(\rho^{-n+1}),$$

as $\rho \to 0$ (or equivalently, as $R \to 1^+$). Now we observe that $|a_n| \to \infty$ as $R \to 1^+$ if

$$\big[\omega\kappa^2 R(J_{|n|})'(\kappa\omega R) + |n|J_{|n|}(\kappa\omega R)\big]\Big|_{R=1} = 0. \tag{27}$$

Note that then

$$\big[\omega\kappa^2 R(H_{|n|}^{(1)})'(\kappa\omega R) + |n|H_{|n|}^{(1)}(\kappa\omega R)\big]\Big|_{R=1} \neq 0. \tag{28}$$

This implies that if κ is outside a discrete set and if ω is such that (27) and (28) are satisfied by functions $J_{|n|}$ and $H_{|n|}^{(1)}$ for some n, then there are sources p for which the H^1-norm of the solution u_R goes to infinity in the cloaked region (i.e., when resonance happens) as $R \to 1^+$ (i.e., $\rho \to 0$). We remark that condition (27) implies condition (28) automatically and is equivalent to that the function

$$V_{\pm n}(\widetilde{r}, \theta) := J_{|n|}(\kappa\omega\widetilde{r})e^{\pm in\theta} \tag{29}$$

satisfies the boundary value problem

$$\begin{aligned} (\Delta + \kappa^2\omega^2)V &= 0 \qquad \text{in } \ B_1, \\ \Big[\kappa\widetilde{r}\partial_{\widetilde{r}}V + (-\partial_\theta^2)^{1/2}V\Big]\Big|_{\widetilde{r}=1^+} &= 0. \end{aligned} \tag{30}$$

Next we consider the frequencies ω for which

$$\begin{cases} \Big[\omega\kappa^2 R J_{|n|}'(\kappa\omega R) + |n|J_{|n|}(\kappa\omega R)\Big]\Big|_{R=1} \neq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad J_{|n|}(3\omega) \neq 0, \end{cases} \qquad \text{for any } n \in \mathbb{Z}. \tag{31}$$

3.2. Non-local boundary condition with non-resonant frequencies. In the following, we show that when we have in $B_2\backslash\overline{B_R}$, $1 < R < 2$ the approximative cloaking material parameters, then for the non-resonant frequencies ω, the boundary condition in (30) holds for all solutions when $R \to 1^+$.

Lemma 3.1. *Assume that in $\Omega = B_3$ we have the material parameters (σ_R, λ_R). Moreover, suppose ω is such that (31) holds. When $R > 1$ is sufficiently close to 1, then for any source $p \in L^2(\Omega)$ supported compactly in B_1 and for $f \in H^{1/2}(\partial\Omega)$ the Helmholtz equation (10) has a unique solution u_R. Moreover, as $R \to 1^+$, Fourier coefficients of the solution $v_R^- := u_R|_{B_R}$ in the cloaked region,*

$$v_{R,n}^-(\widetilde{r}) = \int_0^{2\pi} e^{-in\theta} v_R^-(\widetilde{r}, \theta)d\theta$$

the limits $\lim_{R\to 1+} v^-_{R,n}(\widetilde{r})$ *exists and we have for all* $n \in \mathbb{Z}$

$$\lim_{R\to 1^+} (\kappa\widetilde{r}\partial_{\widetilde{r}} v^-_{R,n}(\widetilde{r}) + |n| v^-_{R,n}(\widetilde{r}))\big|_{\widetilde{r}=R} = 0, \tag{32}$$

$$\lim_{R\to 1^+} (\kappa\widetilde{r}\partial_{\widetilde{r}} v^-_{R,n}(\widetilde{r}) + |n| v^-_{R,n}(\widetilde{r}))\big|_{\widetilde{r}=1} = 0. \tag{33}$$

Proof. We suppose that ω is not an eigenvalue of (30). Then

$$\begin{aligned} v^-_R(\widetilde{r},\theta) &= \sum_{n=-\infty}^{\infty} (a_n J_{|n|}(\kappa\omega\widetilde{r}) + p_n H^{(1)}_{|n|}(\kappa\omega\widetilde{r}))e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{A_n}{B_n} J_{|n|}(\kappa\omega\widetilde{r}) + H^{(1)}_{|n|}(\kappa\omega\widetilde{r})\right] p_n e^{in\theta} \end{aligned}$$

where as in (23)

$$\begin{aligned} A_n(R) &= \kappa^2 R (H^{(1)}_{|n|})'(\kappa\omega R) l_1 - \rho H^{(1)}_{|n|}(\kappa\omega R) l_2 \\ B_n(R) &= \rho J_{|n|}(\kappa\omega R) l_2 - \kappa^2 R J'_{|n|}(\kappa\omega R) l_1. \end{aligned}$$

In following, we sometimes denote shortly $A_n(R) = A_n$ and $B_n(R) = B_n$.

Denote

$$\Phi_n(\widetilde{r}) := \frac{A_n}{B_n} J_{|n|}(\kappa\omega\widetilde{r}) + H^{(1)}_{|n|}(\kappa\omega\widetilde{r}).$$

Apparently $\widetilde{\Phi}(\widetilde{r},\theta) := \Phi_n(\widetilde{r})e^{in\theta}$ satisfies the Helmholtz equation

$$(\Delta + \kappa^2\omega^2)\widetilde{\Phi} = 0 \qquad \text{in } B_R.$$

To prove the existence of the limits $\lim_{R\to 1+} v^-_{R,n}(\widetilde{r})$ and (32), it is sufficient to show

$$\widetilde{r}\kappa\partial_{\widetilde{r}}\Phi_n(\widetilde{r}) + n\Phi_n(\widetilde{r})\big|_{\widetilde{r}=R} \to 0 \quad \text{as } R \to 1^+. \tag{34}$$

Indeed, by

$$\partial_{\widetilde{r}}\Phi_n(\widetilde{r}) = \frac{A_n}{B_n}\kappa\omega J'_{|n|}(\kappa\omega\widetilde{r}) + \kappa\omega (H^{(1)}_{|n|})'(\kappa\omega\widetilde{r}),$$

and $\lim_{R\to 1} B_n(R) \neq 0$ (corresponding to the non-resonant case), we have

$$\begin{aligned} &\widetilde{r}\kappa\partial_{\widetilde{r}}\Phi_n(\widetilde{r}) + n\Phi_n(\widetilde{r})\big|_{\widetilde{r}=R} \\ &= \frac{\kappa^2 R}{B_n}(nl_1 + \omega\rho l_2)\Big((H^{(1)}_{|n|})'(\kappa\omega R) J_{|n|}(\kappa\omega R) - J'_{|n|}(\kappa\omega R) H^{(1)}_{|n|}(\kappa\omega R)\Big) \end{aligned} \tag{35}$$

where l_1 and l_2 are given by (24).

For $n \geq 1$, as $\rho \to 0^+$,

$$\begin{aligned} nl_1 + \omega\rho l_2 =& \omega\rho\Big(J_{|n|-1}(\omega\rho) H^{(1)}_{|n|}(3\omega) - H^{(1)}_{|n|-1}(\omega\rho) J_{|n|}(3\omega)\Big) \\ =& \mathcal{O}(\rho^{-n+2}). \end{aligned} \tag{36}$$

Combining (35), (36) and (26), one has

$$\widetilde{r}\kappa\partial_{\widetilde{r}}\Phi_n(\widetilde{r}) + |n|\Phi_n(\widetilde{r})\big|_{\widetilde{r}=R} = \mathcal{O}(\rho^2) \qquad \text{as } R \to 1^+ \text{ (i.e. } \rho \to 0^+), \tag{37}$$

which proves (34).

For $n = 0$, from (25), one has

$$\begin{aligned} A_0 =& -\frac{2i\kappa^2 R}{\pi}(H_0^{(1)})'(\kappa\omega R)J_0(3\omega)\ln\left(\frac{\omega\rho}{2}\right) + \kappa^2 R(H_0^{(1)})'(\kappa\omega R)H_0^{(1)}(3\omega) \\ &+ \frac{2i}{\pi\omega}H_0^{(1)}(\kappa\omega R)J_0(3\omega) + \mathcal{O}(\rho), \\ B_0 =& \frac{2i\kappa^2 R}{\pi}J_0'(\kappa\omega R)J_0(3\omega)\ln\left(\frac{\omega\rho}{2}\right) - \kappa^2 R J_0'(\kappa\omega R)H_0^{(1)}(3\omega) \\ &- \frac{2i}{\pi\omega}J_0(\kappa\omega R)J_0(3\omega) + \mathcal{O}(\rho). \end{aligned}$$

Therefore,

$$\partial_{\widetilde{r}}\Phi_0(R) = \frac{A_0}{B_0}\kappa\omega J_0'(\kappa\omega R) + \kappa\omega(H_0^{(1)})'(\kappa\omega R) \tag{38}$$

has denominator B_0 and numerator

$$\kappa\omega[A_0 J_0'(\kappa\omega R) + B_0(H_0^{(1)})'(\kappa\omega R)] = \frac{2\kappa i}{\pi}W_n(\kappa\omega R)J_0(3\omega) + \mathcal{O}(\rho),$$

where $W_n(x) = H_0^{(1)}(x)J_0'(x) - (H_0^{(1)})'(x)J_0(x)$. This implies

$$\partial_{\widetilde{r}}\Phi_0(R) \sim \frac{W_n(\kappa\omega R)}{\kappa R J_0'(\kappa\omega R)\ln\left(\frac{\omega\rho}{2}\right)} \to 0 \quad \text{as } \rho \to 0^+,$$

i.e., the boundary condition (34) is satisfied for $n = 0$ and moreover,

$$\widetilde{r}\kappa\partial_{\widetilde{r}}\Phi_0(\widetilde{r})\big|_{\widetilde{r}=R} = \mathcal{O}\left(\frac{1}{\ln\left(\frac{\omega\rho}{2}\right)}\right) \qquad \text{as } R \to 1^+. \tag{39}$$

This proves (32). The equation (33) follows similarly by evaluating (35) and (38) at $\widetilde{r} = 1$ instead of $\widetilde{r} = R$. $\square$

Now we are ready to prove our main result of the limit of the waves u_R of the physical approximate cloaking medium as $R \to 1^+$. We recall that $\Sigma = \partial B_1$.

Theorem 3.2. *Let ω be such that (31) is satisfied. Assume that u_R is the solution of (10) with $f = 0$ and $p \in C_0^\infty(B_{R_0})$ with $R_0 < 1$. Then as $R \to 1^+$, u_R converges uniformly in compact subsets of $B_3 \setminus \Sigma$ to the limit u_1 satisfying*

$$(\nabla^2 + \kappa^2\omega^2)u_1 = \kappa^2 p \qquad \text{in } B_1, \tag{40}$$

$$\kappa\partial_{\widetilde{r}}u_1 + (-\partial_\theta^2)^{1/2}u_1\big|_{\partial B_1} = 0, \tag{41}$$

and

$$u_1\big|_{B_2\setminus\overline{B_1}} = 0. \tag{42}$$

We note that solutions of (40)-(42) with $f \neq 0$ and $p = 0$ have been analyzed in [11].

Proof. Let $R_0 < R_1 < R$. Recall that solution $w \in C^\infty(\mathbb{R}^2)$ of (15) is the radiating solution produced by source $\kappa^2 p$ in $\mathbb{R}^2$ and it has the expansion (16) for $\widetilde{r} > R_1$. Consider the Fourier coefficients

$$w_n(\widetilde{r}) = \int_0^{2\pi} e^{-in\theta}w(\widetilde{r},\theta)d\theta$$

and denote $P_n(|x|) := -(\nabla^2 + \kappa^2\omega^2 - n^2/|x|^2)w_n(|x|)$. As $w \in C^\infty(\mathbb{R}^2)$, we see using integration by parts that

$$\|P_n(|x|)\|_{L^2(B_3)} \le C_M(1+|n|)^{-M} \qquad \text{for arbitrary } M>0. \tag{43}$$

We consider the Fourier coefficients

$$v_{R,n}^-(\widetilde{r}) = \int_0^{2\pi} e^{-in\theta} v_R^-(\widetilde{r},\theta)d\theta, \quad v_{R,n}^+(r) = \int_0^{2\pi} e^{-in\theta} v_R^+(r,\theta)d\theta.$$

They satisfy the following problem

$$\left(-\nabla^2 - \kappa^2\omega^2 + \frac{n^2}{|x|^2}\right) v_{R,n}^-(|x|) = P_n(|x|) \qquad \text{for } 0 \le |x| \le R, \tag{44}$$

$$\left(-\nabla^2 - \omega^2 + \frac{n^2}{|y|^2}\right) v_{R,n}^+(|y|) = 0 \qquad \text{for } \rho \le |y| \le 2, \tag{45}$$

$$v_{R,n}^+|_{\partial B_2} = 0, \tag{46}$$

$$v_{R,n}^-|_{\partial B_R^-} = v_{R,n}^+|_{\partial B_\rho^+}, \quad \kappa R(v_{R,n}^-)'(|x|)|_{\partial B_R^-} = \rho(v_{R,n}^+)'(|y|)|_{\partial B_\rho^+}. \tag{47}$$

By the transmission conditions (47), we see for $V_{R,n}^\pm(x) = v_{R,n}^\pm(|x|)$

$$\int_{\partial B_R} \partial_{\widetilde{r}} V_{R,n}^- \overline{V_{R,n}^-} dS_x = \frac{R}{\rho}\int_{\partial B_\rho} \frac{\rho}{\kappa R}\partial_r V_{R,n}^+ \overline{V_{R,n}^+} dS_y = \int_{\partial B_\rho} \frac{1}{\kappa}\partial_r V_{R,n}^+ \overline{V_{R,n}^+} dS_y.$$

Thus, using integration by parts, we obtain

$$\begin{aligned}
I_1 :=& \int_{B_R} P_n \overline{V_{R,n}^-} dy \\
=& \int_{B_R} \left(-\nabla^2 - \kappa^2\omega^2 + \frac{n^2}{|x|^2}\right) V_{R,n}^- \overline{V_{R,n}^-} dx \\
&+ \frac{1}{\kappa}\int_{B_2\setminus\overline{B_\rho}} \left(-\nabla^2 - \omega^2 + \frac{n^2}{|y|^2}\right) V_{R,n}^+ \overline{V_{R,n}^+} dy \\
=& -\int_{\partial B_R} \partial_{\widetilde{r}} V_{R,n}^- \overline{V_{R,n}^-} dS_x - \left(\int_{\partial B_2} - \int_{\partial B_\rho}\right)\frac{1}{\kappa}\partial_r V_{R,n}^+ \overline{V_{R,n}^+} dS_y \\
&+ \int_{B_R}\left(|\nabla V_{R,n}^-|^2 + \left(-\kappa^2\omega^2 + \frac{n^2}{|x|^2}\right)|V_{R,n}^-|^2\right)dx \\
&+ \frac{1}{\kappa}\int_{B_2\setminus\overline{B_\rho}}\left(|\nabla V_{R,n}^+|^2 + \left(-\omega^2 + \frac{n^2}{|y|^2}\right)|V_{R,n}^+|^2\right)dy,
\end{aligned}$$

and then

$$\begin{aligned}
I_1 \ge& \int_{B_R}\left(|\nabla V_{R,n}^-|^2 + \left(-\kappa^2\omega^2 + \frac{n^2}{R^2}\right)|V_{R,n}^-|^2\right)dx \\
&+ \frac{1}{\kappa}\int_{B_2\setminus\overline{B_\rho}}\left(|\nabla V_{R,n}^+|^2 + \left(-\omega^2 + \frac{n^2}{2^2}\right)|V_{R,n}^+|^2\right)dy \\
\ge& \int_{B_R}\left(-\kappa^2\omega^2 + \frac{n^2}{R^2}\right)|V_{R,n}^-|^2 dx + \frac{1}{\kappa}\int_{B_2\setminus\overline{B_\rho}}\left(-\omega^2 + \frac{n^2}{2^2}\right)|V_{R,n}^+|^2 dy.
\end{aligned}$$

For $|n| \geq N_0$ with $\frac{N_0^2}{2^2} \geq \max\{\kappa^2\omega^2, \omega^2\}$, as $R < 2$, the above and

$$I_1 \leq \|V_{R,n}^-\|_{L^2} \|P_n\|_{L^2}$$

imply first that

$$\left(\|V_{R,n}^-\|_{L^2(B_R)} + \|V_{R,n}^+\|_{L^2(B_2 \setminus \overline{B_\rho})} \right) \leq C_{N_0} \|P_n\|_{L^2(B_3)} \tag{48}$$

and second that

$$\left(\|V_{R,n}^-\|_{H^1(B_R)} + \|V_{R,n}^+\|_{H^1(B_2 \setminus \overline{B_\rho})} \right) \leq C'_{N_0} \|P_n\|_{L^2(B_3)} \tag{49}$$

where C_{N_0} and C'_{N_0} are independent of R and n.

By Lemma 3.1, for each $n \in \mathbb{Z}$ and $\widetilde{r} \in [0, 1)$ and $r \in (0, 2)$ there exists limits

$$v_n^-(\widetilde{r}) = \lim_{R \to 1^+} v_{R,n}^-(\widetilde{r}), \qquad v_n^+(r) = \lim_{R \to 1^+} v_{R,n}^+(r) \tag{50}$$

and we denote $V_n^\pm(x) = v_n^\pm(|x|)$. Let now $0 < r_1 < R_1 < 1$. Then by (49) the restrictions $v_{R,n}^-|_{(r_1,R_1)}$, $R > R_1$ are uniformly bounded in $H^1(r_1, R_1)$. By Sobolev embedding theorem, the set $\{v_{R,n}^-|_{[r_1,R_1]};\ R_0 < 1 < R\}$ is relatively compact in $H^s(r_1, R_1) \subset C([r_1, R_1])$, $1/2 < s < 1$. Thus any sequence $(v_{R_j,n}^-|_{[r_1,R_1]})_{j=1}^\infty$ with $R_j \to 1$ has a subsequence converging in $C([r_1, R_1])$ which limit has to coincide with v_n^- by (50). Thus $v_{R,n}^-$ have to converge to v_n^- in $C([r_1, R_1])$ and hence $V_{R,n}^-$ have to converge to V_n^- in $C(\overline{B}_{R_1} \setminus B_{r_1})$ as $R \to 1$. Similarly, for all $\rho_1 > 0$ we see using (49) that $V_{R,n}^+$ have to converge to V_n^+ in $C(\overline{B}_2 \setminus B_{\rho_1})$ as $R \to 1$. Now, as the Sobolev norm $u \mapsto \|u\|_{H^1(B_{R_1} \setminus \overline{B}_{r_1})}$ is a lower semi-continuous function in $L^2(B_{R_1} \setminus \overline{B}_{r_1})$ and (49) holds, we see that

$$\|V_n^-\|_{H^1(B_{R_1} \setminus \overline{B}_{r_1})} \leq C'_{N_0} \|P_n\|_{L^2(B_3)}$$

for all $0 < r_1 < R_1 < 1$. Hence, by (50) we see using e.g. monotone convergence theorem and [18] that

$$\begin{aligned} \|V_n^-\|_{H^1(B_1)} = \|V_n^-\|_{H^1(B_1 \setminus 0)} &= \lim_{r_1 \to 0, R_1 \to 1} \|V_n^-\|_{H^1(B_{R_1} \setminus \overline{B}_{r_1})} \\ &\leq C'_{N_0} \|P_n\|_{L^2(B_3)}. \end{aligned} \tag{51}$$

By (43) we see that $u_1(\widetilde{r}, \theta) = \sum_{n \in \mathbb{Z}} v_n^-(\widetilde{r}) e^{in\theta}$ is a well defined function in $H^1(B_1)$ satisfying (40). By (49),

$$\begin{aligned} \|V_{R,n}^-\|_{C(\overline{B}_{R_1} \setminus B_{r_1})} = \|v_{R,n}^-\|_{C([r_1,R_1])} &\leq C_{N_0,r_1,R_1} \|v_{R,n}^-\|_{H^1(r_1,R_1)} \\ &\leq C'_{N_0,r_1,R_1} \|P_n\|_{L^2(B_3)}, \end{aligned}$$

where C'_{N_0,r_1,R_1} does not depend on R or $n > N_0$. Thus, using (43) and the convergence of $V_{R,n}^-$ to V_n^- in $C(\overline{B}_{R_1} \setminus B_{r_1})$ we see that u_R converge to u_1 uniformly in compact subsets of $B_1 \setminus \{0\}$. Using equation (44), we see the uniform convergence in a neighborhood of zero, too. Similarly, we see the uniform convergence in compact subsets of $B_3 \setminus \overline{B}_1$. By (43) and (51), we see that

$$\begin{aligned}\|u_1\|^2_{H^1(B_1)} &\leq \sum_{n=-\infty}^{\infty} (1+n^2)\|V_n^-\|^2_{H^1(B_1)} \\ &\leq \sum_{|n|<-N_0} (1+n^2)\|V_n^-\|^2_{H^1(B_1)} + \sum_{|n|\geq N_0} (1+n^2)(C'_{N_0}\|P_n\|_{L^2(B_3)})^2 < \infty.\end{aligned}$$

Hence, $u_1 \in H^1(B_1)$ and we have $u_1|_{\partial B_1} \in H^{1/2}(\partial B_1)$. Moreover, by (40) we have $\nabla^2 u_1 = -\kappa^2\omega^2 u_1 + \kappa^2 p \in L^2(B_1)$. For $\psi \in H^{1/2}(\partial B_1)$ there is an extension $E_\psi \in H^1(B_1)$ such that $E_\psi|_{\partial B_1} = \psi$ and $\|E_\psi\|_{H^1} \leq C'\|\psi\|_{H^{1/2}}$ with some $C' > 0$. Thus, analogously to [27], we see that the Neumann boundary value $\partial_r u_1|_{\partial B_1} \in H^{-1/2}(\partial B_1)$ is well defined by the identity

$$\langle \partial_r u_1|_{\partial B_1}, \psi\rangle_{H^{-1/2}(\partial B_1)\times H^{1/2}(\partial B_1)} = \int_{B_1} ((-\kappa^2\omega^2 u_1 + \kappa^2 p)E_\psi - \nabla u_1 \cdot \nabla E_\psi)dx,$$

for any $\psi \in H^{1/2}(\partial B_1)$ and

$$\|\partial_r u_1|_{\partial B_1}\|_{H^{-1/2}(\partial B_1)} \leq C'(\| -\kappa^2\omega^2 u_1 + \kappa^2 p\|_{L^2(B_1)} + \|u_1\|_{H^1(B_1)}).$$

Using this for (44) we obtain

$$\|\partial_r V_n^-\|_{H^{-1/2}(\partial B_1)} \leq C_2(1+n^2)\|V_n^-\|_{H^1(B_1\setminus \overline{B}_{R_0})}$$

where $C_2 > 0$ is independent of n. Thus using (33), (43), and (51) we see that the boundary value $\kappa\partial_r u_1 + (-\partial_\theta^2)^{1/2}u_1$, vanishes. Hence, the boundary condition (41) is satisfied.

Equation (42) for $u_1|_{B_3\setminus\overline{B_1}}$ follows using (43), (49) and the fact that by (21), (22) for any fixed $n \in \mathbb{Z}$ we have in (17)

$$\begin{aligned}&c_n(\rho) = \mathcal{O}(\rho^{|n|}), \quad b_n(\rho) = \mathcal{O}(\rho^{|n|}) \qquad \text{as } \rho \to 0^+, \quad |n| \geq 1,\\ &c_0(\rho) = \mathcal{O}((\log\rho)^{-1}), \quad b_0(\rho) = \mathcal{O}((\log\rho)^{-1}) \qquad \text{as } \rho \to 0^+.\end{aligned}$$

□

We conclude our discussion by remarking that our analysis also explains the limit of approximate electromagnetic cloaks in the cylindrical case with TE/TM polarized incoming waves, as the solutions of Maxwell's equations in this case satisfy the Helmholtz equation.

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