SUBGROUPS OF PROFINITE SURFACE GROUPS

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ABSTRACT. We study the subgroup structure of the étale fundamental group Π of a projective curve over an algebraically closed field of characteristic 0. We obtain an analog of the diamond theorem for Π. As a consequence we show that most normal subgroups of infinite index are semi-free. In particular every proper open subgroup of a normal subgroup of infinite index is semi-free.

1. Introduction

Every subgroup of a free group is free, this is the content of the Nielsen-Schreier theorem. The profinite version of the Nielsen-Schreier theorem fails in general and even fails for normal subgroups, for example $\mathbb{Z}_p \leq \widehat{\mathbb{Z}}$. Therefore the question of finding conditions under which a subgroup of a free profinite group is free is natural and of importance. The question was considered by Melnikov, Lubotzky, van der Dries, Jarden, Haran, and others ([7, Chapter 8] and [3, Chapter 25]).

Roughly speaking the most general criteria are Melnikov's characterization of normal (and accessible) subgroups of free profinite groups and Haran's diamond theorem. In this work we consider the étale fundamental group $\Pi = \pi_1(X)$, where X is a curve over an algebraically closed field of characteristic 0 and the genus of its smooth completion is ≥ 2.

If X is affine, then Π is free of finite rank. Therefore Melnikov's characterization is known to hold [7, Chapter 8.6] and similarly Haran's diamond theorem [1]. If X is projective, then Π is a profinite surface group, i.e., the profinite completion of a surface group. Melnikov's characterization for normal subgroups of Π is obtained in [8]. The objective of this work is to obtain the diamond theorem for profinite surface groups:

Theorem 1.1. *Let Π be a profinite surface group of genus $g \geq 2$ and let N be a closed subgroup of Π with $[\Pi : N] = \prod_p p^\infty$ as supernatural numbers, where p runs over all primes. Assume there exist closed normal subgroups K_1, K_2 of Π such that $K_1 \cap K_2 \leq N$ but $K_1 \not\leq N$ and $K_2 \not\leq N$. Then N is a free profinite group of countable rank.*

We note that a necessary condition for a profinite group to be free is that it is projective, and a closed subgroup N of a profinite surface group Π is projective if and only if $[\Pi : N] = \prod_p p^\infty$ as supernatural numbers, where p runs over all primes [8, Proposition 1.2].

Recently a notion of "free not necessarily projective" profinite groups evolved from Galois theory [5, 2], the so called semi-free groups. A second countable profinite group

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is semi-free if and only if every finite split embedding problem is properly solvable. Using this notion we can generalize Theorem 1.1 to any closed subgroup of infinite index:

Theorem 1.2. *Let Π be a profinite surface group of genus $g \geq 2$ and let N be a closed subgroup with $[\Pi : N] = \infty$. Assume there exist closed normal subgroups K_1, K_2 of Π such that $K_1 \cap K_2 \leq N$ but $K_1 \nleq N$ and $K_2 \nleq N$. Then N is semi-free of countable rank.*

Since a semi-free projective group is free [2, Theorem 3.6], Theorem 1.1 follows from Theorem 1.2.

A consequence of Theorem 1.2 is that 'most' closed normal subgroups of Π of infinite index are semi-free in the following sense.

Corollary 1.3. *Let Π be a profinite surface group of genus $g \geq 2$ and let N be a closed normal subgroup with $[\Pi : N] = \infty$. Then every proper open subgroup of N is semi-free.*

We give more examples in Section 4.3.

A typical example of a closed normal subgroup which is not semi-free is the kernel M of the epimorphism from Π to its maximal pro-p quotient (because M has no p-quotient). Note however that M is contained in a semi-free closed normal subgroup of Π. Indeed, there exists a continuous epimorphism $\alpha \colon \Pi \to \mathbb{Z}_p^2$, so $\ker \alpha = K_1 \cap K_2$, where K_1, K_2 are closed normal subgroups of Π with $\Pi/K_i \cong \mathbb{Z}_p$. By Theorem 1.2, $\ker \alpha$ is semi-free, and clearly $M \leq \ker \alpha$.

We show in fact that every closed normal subgroup N of Π of infinite index such that Π/N is not hereditarily just infinite is contained in a semi-free closed normal subgroup. (An infinite profinite group is **just infinite** if it has no proper infinite quotient. It is **hereditarily just infinite** if every open normal subgroup of it is just infinite.)

Theorem 1.4. *Let Π be a profinite surface group of genus $g \geq 2$ and let N be a closed normal subgroup with $[\Pi : N] = \infty$ such that Π/N is not hereditarily just infinite. Then there exists a semi-free closed normal subgroup M of Π such that $N \leq M$.*

Through the paper, all subgroups of a profinite group are assumed to be closed, and all homomorphisms of profinite groups are assumed to be continuous. The paper is organized as follows. Section 2 is dedicated to preliminary results on surface groups. In Section 3 we prove Theorems 1.1 and 1.2. In the last section we prove Corollary 1.3 and Theorem 1.4.

2. Surface groups

The fundamental group $\pi_1(X)$ of an oriented Riemann surface X of genus g is given by the presentation

$$\pi_1(X) = \left\langle x_1, \ldots, x_g, y_1, \ldots, y_g \,\Big|\, \prod_{i=1}^{g} [x_i, y_i] \right\rangle.$$

Here $[x, y] = x^{-1}y^{-1}xy$. A group with this presentation is said to be a surface group of genus g. We shall call its profinite completion Π a profinite surface group of genus g. Note that Π is the same as the ètale fundamental group of X.

Fact 2.1. *Let Π be a profinite surface group of genus g and let U be an open subgroup of index n. Then U is a profinite surface group of genus $n(g-1)+1$.*

Proof. This is well known for surface groups. The map $\pi_1(X) \to \Pi$ induces an index-preserving bijection between finite index subgroups of $\pi_1(X)$ and open subgroups of Π sending a subgroup to its completion [7, Chapter 3]. $\qquad\square$

Let Π be a profinite group. A finite split embedding problem (**FSEP**) for Π consists of finite groups A, B and epimorphisms $f: \Pi \to B$ and $\alpha: A \to B$, such that α splits. We denote it by (f, α). A **weak solution** is a homomorphism $\psi: \Pi \to A$ such that $\alpha \circ \psi = f$. If ψ is surjective we say it is a **proper solution**.

We shall need the following technical lemma.

Lemma 2.2. *Let $(f: \Pi \to B, \alpha: A \to B)$ be a finite split embedding problem for Π of genus $g \geq 2|A|^3$. Then (f, α) is properly solvable.*

Remark 2.3. The bound $g \geq 2|A|^3$ is not the best possible. In fact, if s is the minimal number of generators of $\ker \alpha$ as a normal subgroup of A, then $g \geq s|B|^2(|A| + 1)$ suffices. We will not use this sharper bound here, and hence will not prove it.

Proof. Let $n = |A|$, and $\beta: B \to A$ a section of α. Note that $\ker \alpha$ is generated by $\frac{|A|}{|B|}$ elements. Let $\varphi = \beta \circ f: \Pi \to A$. Then φ is a weak solution.

By [6, Lemma 6.1], it suffices to replace the generators of Π with a different set of generators having the same unique relation such that the first $\frac{|A|^2+|A|}{|B|} \leq \frac{2|A|^2}{|B|}$ new x_i's (resp., y_i's) have the same image under φ. Let $r = \frac{2|A|^2}{|B|}$.

Each of the g pairs (x_i, y_i) has $|B|^2$ possibilities for $(\varphi(x_i), \varphi(y_i))$, hence, since $g \geq 2|A|^3 \geq |B|^2 r$, Dirichlet's box principle gives indexes $j_1 < \cdots < j_r$ for which

$$(1) \qquad \varphi(x_{j_1}) = \cdots = \varphi(x_{j_r}) \qquad \text{and} \qquad \varphi(y_{j_1}) = \cdots = \varphi(y_{j_r})$$

The following argument explains how to replace j_1 with 1, j_2 with 2, and so forth. Let $x^y = y^{-1}xy$. Suppose $j_1 \neq 1$. Then

$$\prod_{i=1}^{g}[x_i, y_i] = [x_{j_1}, y_{j_1}]([x_1, y_1] \cdots [x_{j_1-1}, y_{j_1-1}])^{[x_{j_1}, y_{j_1}]}[x_{j_1+1}, y_{j_1+1}] \cdots [x_g, y_g].$$

For each $i = 1, \ldots, j_1 - 1$, replace the pair of generators x_i, y_i with $x_i^{[x_{j_1}, y_{j_1}]}, y_i^{[x_{j_1}, y_{j_1}]}$. Thus we may assume that $j_1 = 1$. Continuing similarly, we get a new presentation of Π of the same kind for which (1) holds, and hence by [6, Lemma 6.1] (f, α) is solvable. $\qquad\square$

3. Diamond $\diamond$

In this section we prove Theorems 1.2 and 1.1.

3.1. Haran-Shapiro Induction. Let $N \leq \Pi$ be a subgroup of Π. Consider a FSEP

$$\mathcal{E} = (\mu_1: N \to G_1, \alpha_1: A \rtimes G_1 \to G_1)$$

for N. We describe a method to construct an embedding problem $\mathcal{E}_{ind}$ for Π such that a weak solution of $\mathcal{E}_{ind}$ induces a weak solution of $\mathcal{E}$, and under certain conditions, a proper solution of $\mathcal{E}_{ind}$ induces a proper solution of $\mathcal{E}$.

We start by setting up the notation. Let $L \triangleleft \Pi$ be an open normal subgroup of Π. Assume

(2) $$L \cap N \leq \ker \mu_1.$$

Let $\mu \colon \Pi \to G := \Pi/L$ be the natural epimorphism, $G_0 = NL/L \cong N/N \cap L$, and $\mu_0 = \mu|_N \colon N \to G_0$. Then μ_1 factors as $\mu_1 = \nu \circ \mu_0$, where $\nu \colon G_0 \to G_1$ is induced by (2). The group G_0 acts on A via ν, i.e., $a^g := a^{\nu(g)}$, for all $a \in A, g \in G_0$.

$$
\begin{array}{ccc}
 & & N \\
 & & \Big\downarrow{\mu_0} \\
A \rtimes G_0 & \xrightarrow{\alpha_0} & G_0 \; \Big)\mu_1 \\
\Big\downarrow{\rho} & & \nu\Big\downarrow \\
A \rtimes G_1 & \xrightarrow{\alpha_1} & G_1
\end{array}
$$

Here $\rho(a, g) = (a, \nu(g))$ and α_0 is the projection map. The group G acts on

$$\mathrm{Ind}_{G_0}^G(A) = \{f \colon G \to A \mid f(\sigma\tau) = f(\sigma)^\tau, \ \forall \sigma \in G, \tau \in G_0\} \cong A^{(G:G_0)}$$

by $(f^\sigma)(\sigma') = f(\sigma\sigma')$, for all $\sigma, \sigma' \in G$, $f \in \mathrm{Ind}_{G_0}^G(A)$. This gives rise to the so called **twisted wreath product**

$$A \wr_{G_0} G = \mathrm{Ind}_{G_0}^G(A) \rtimes G.$$

Let $\alpha \colon A \wr_{G_0} G \to G$ be the projection map. Then we have the following FSEP for Π induced from $\mathcal{E}$ (w.r.t. L satisfying (2)):

(3) $$\mathcal{E}_{ind}(L) = (\mu \colon \Pi \to G, \alpha \colon A \wr_{G_0} G \to G).$$

Let $\mathrm{Sh} \colon \mathrm{Ind}_{G_0}^G(A) \rtimes G_0 \to A \rtimes G_0$ be defined by $\mathrm{Sh}((f, \sigma)) = f(1)\sigma$. Clearly Sh is surjective, it is also a homomorphism, since

$$\mathrm{Sh}(f^\sigma) = f^\sigma(1) = f(\sigma) = f(1)^\sigma = \mathrm{Sh}(f)^\sigma.$$

Now, a weak solution $\psi \colon \Pi \to A \wr_{G_0} G$ of $\mathcal{E}_{ind}$ induces the weak solution $\psi^{ind} = \rho \circ \mathrm{Sh} \circ \psi|_N$ of $\mathcal{E}$:

$$
N \xrightarrow{\psi|_N} \mathrm{Ind}_{G_0}^G(A) \rtimes G_0 \xrightarrow{\mathrm{Sh}} A \rtimes G_0 \xrightarrow{\rho} A \rtimes G_1
$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\psi^{ind}}$$

(Note $\psi(N) \leq \mathrm{Ind}_{G_0}^G(A) \rtimes G_0$ since $\mu(N) = \mu_0(N) = G_0$, hence $\mathrm{Sh} \circ \psi|_N$ is well defined.)

Assume ψ is surjective. In general this does not imply surjectivity of ψ^{ind}. The following result gives a working sufficient condition on L for ψ^{ind} to be surjective.

Proposition 3.1 ([2, Proposition 4.5]). *Let $N \leq \Pi$ be profinite groups and let*

$$\mathcal{E} = (\mu_1 \colon N \to G_1, \alpha_1 \colon A \rtimes G_1 \to G_1)$$

be a FSEP for N. Let D, Π_0, L be subgroups of Π such that

(4a) *D is an open normal subgroup of Π with $N \cap D \leq \ker \mu_1$,*

(4b) *Π_0 is an open subgroup of Π with $N \leq \Pi_0 \leq ND$,*

(4c) *L is an open normal subgroup of Π with $L \leq \Pi_0 \cap D$.*

In particular $L \cap N \leq D \cap N \leq \ker \mu_1$, so (2) holds. Let $\mathcal{E}_{ind}$ be as in (3).

Assume that there is a closed normal subgroup $\mathcal{N}$ of Π with $\mathcal{N} \leq N \cap L$ such that there is NO nontrivial quotient $\bar{A}$ of A through which the action of G_0 on A descends and for which the FSEP

(5) $$\bar{\mathcal{E}}_{ind,\mathcal{N}}(L) = (\bar{\mu}\colon \Pi/\mathcal{N} \to G, \bar{\alpha}\colon \bar{A} \wr_{G_0} G \to G),$$

where $\bar{\mu}$ is the quotient map, $G = \Pi/L$, and $G_0 = \Pi_0/L$, is properly solvable. Then a proper solution ψ of $\mathcal{E}_{ind}$ induces a proper solution ψ^{ind} of $\mathcal{E}$.

3.2. Condition ($\diamond$). Let $N \leq \Pi$ be profinite groups with $[\Pi : N] = \infty$ and assume there exist normal subgroups N_1, N_2 of Π such that $N_1 \cap N_2 \leq N$ but $N_1 \not\leq N$ and $N_2 \not\leq N$. Without loss of generality we can assume that $[N_1 N : N] = \infty$. Indeed, if $[N_1 N : N] < \infty$, then Π has an open subgroup N_2' such that $N_2' \cap (N_1 N) \leq N$. Then $N_1 \cap N_2' \leq N$ and $[N_2' N : N] = \infty$. Replace N_1 with N_2' and N_2 with N_1 to get the assumption.

The following result will be used in the sequel.

Lemma 3.2. *Let $N \leq \Pi$ be profinite groups with $[\Pi : N] = \infty$ and assume there exist normal subgroups N_1, N_2 of Π such that $N_1 \cap N_2 \leq N$, $[N_1 : N_1 \cap N] = \infty$, and $N_2 \not\leq N$. Let*

$$\mathcal{E} = (\mu_1\colon N \to G_1, \alpha_1\colon A \rtimes G_1 \to G_1)$$

be a FSEP for N. Let L be an open normal subgroup of Π satisfying

  (i) $L \cap N \leq \ker \mu_1$,
  (ii) $[N_1 N L : N L] \geq 3$,
  (iii) $[N_2 N L : N L] \geq 2$, *and*
  (iv) $[\Pi : N L] \geq 3$.

Let $G = \Pi/L$, $G_0 = NL/L \cong N/N \cap L$ and let

$$\mathcal{E}_{ind} = (\mu\colon \Pi \to G, \alpha\colon A \wr_{G_0} G \to G)$$

be as defined the induced embedding problem of Equation (3). Then a proper solution ψ of $\mathcal{E}_{ind}$ induces a proper solution ψ^{ind} of $\mathcal{E}$.

Proof. To prove the assertion we use Proposition 3.1. Choose an open normal subgroup D of Π with $N \cap D \leq \ker \mu_1$, let $\Pi_0 = ND$. We can choose an open normal subgroup L of Π such that

(6') $N_1 L, N_2 L \not\leq NL$ (use $N_1, N_2 \not\leq N$),

(7') $[\Pi : NL] > 2$ (use $[\Pi : N] > 2$),

(8') $(N_1 NL : NL) > 2$ (use $[N_1 N : N] > 2$).

Let $G = \Pi/L$, $\mu\colon \Pi \to G$ be the natural epimorphism, $G_0 = \mu(N) = NL/L \cong N/N \cap L$, and $G_i = \mu(N_i) = N_i L/L \cong N_i/N_i \cap L$. Then taking the above conditions modulo L gives the following conditions.

(6) $G_1, G_2 \not\leq G_0$.

(7) $[G : G_0] > 2$.

(8) $[G_1 G_0 : G_0] > 2$.
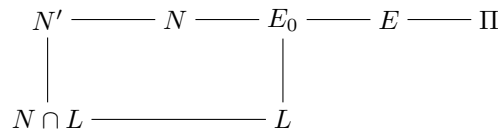
Let $\mathcal{N} = N_1 \cap N_2 \cap L$.

Let $\bar{A}$ be a non-trivial quotient of A through which the action of G_0 descends. By Proposition 3.1 it suffices to show that $\bar{\mathcal{E}}_{ind,\mathcal{N}}$ appearing in (5) is not properly solvable.

Assume $\psi\colon \Pi \to \bar{A} \wr_{G_0} G$ is an epimorphism with $\alpha \circ \psi = \mu$ that factors through $\Pi/\mathcal{N}$. Then $\psi(\mathcal{N}) = 1$. For $i = 1, 2$ put $H_i = \psi(N_i)$. Then $H_i \lhd \bar{A} \wr_{G_0} G$ and $\alpha(H_i) = \mu(N_i) = G_i$. By (6) there is an $h_2 \in H_2$ with $\alpha(h_2) \notin G_0$. Recalling (8), [3, Lemma 13.7.4(a)] gives an $h_1 \in H_1$ for which $\alpha(h_1) = 1$ and $[h_1, h_2] \neq 1$.

For $i = 1, 2$, lift h_i to $y_i \in N_i$ (i.e., $\psi(y_i) = h_i$). Then $\mu(y_1) = \alpha(h_1) = 1$. So, $y_1 \in L$. Then $[y_1, y_2] \in [L, N_2] \cap [N_1, N_2] \le L \cap (N_1 \cap N_2) = \mathcal{N}$. So, $[h_1, h_2] = [\psi(y_1), \psi(y_2)] \in \psi(\mathcal{N}) = 1$. This contradiction proves that ψ as above does not exist. $\qquad\square$

We write $f \uparrow \infty$ for an increasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{x\to\infty} f(x) = \infty$.

We say that a subgroup N of Π with $[\Pi : N] = \infty$ satisfies **Condition ($\diamond$)** in Π if there exist normal subgroups N_1, N_2 of Π such that $N_1 \cap N_2 \le N$, $[N_1 : N_1 \cap N] \ge 3$, $[N_2 : N_2 \cap N] \ge 2$, and for every $f \uparrow \infty$, $r \in \mathbb{N}$, and open subgroup N' of N there exists a diagram of subgroups

$$
\begin{array}{ccccccccc}
N' & \text{---} & N & \text{---} & E_0 & \text{---} & E & \text{---} & \Pi \\
| & & & & | & & & & \\
N \cap L & \text{------} & & & L & & & &
\end{array}
$$

such that

(9a) $L \le E_0 \le E$ are open in Π;
(9b) L is normal in E;
(9c) $[N_1 \cap E : N_1 \cap E_0] \ge 3$;
(9d) $[N_2 \cap E : N_2 \cap E_0] \ge 2$;
(9e) $f([\Pi : E]) \ge r \cdot [E : L]$.

In the sequel we use the notion of sparse and abundant subgroups ([1, Defintion 2.1]) and some of their basic properties.

Definition 3.3. A closed subgroup M of a profinite group Π of infinite index is called **sparse** if for every $n \in \mathbb{N}$ there exists an open subgroup K of Π containing M such that for every proper open subgroup L of K containing M we have $[K : L] \ge n$.

It follows that one can take K with arbitrarily large index in Π. See [2, Definition 5.1].

A subgroup of Π is called **abundant** if it is not open and not sparse

Proposition 3.4. *Let Π, N, N_1, N_2 be profinite groups such that N, N_1, N_2 are subgroups of Π, N_1, N_2 are normal in Π, $[\Pi : N] = \infty$, $N_1 \cap N_2 \le N$, $[N_1 N : N] = \infty$, and $N_2 \not\le N$. Each of the following implies that N satisfies Condition ($\diamond$) in an open subgroup of Π.*

(10a) $[\Pi : N N_1 N_2] = \infty$.
(10b) $[\Pi : N N_1 N_2] < \infty$ *and* $N N_1$ *is abundant in* Π.
(10c) $[\Pi : N N_1 N_2] < \infty$ *and* $N N_2$ *is abundant in* Π.
(10d) $[\Pi : (N N_1) \cap (N N_2)] < \infty$ *and* N *is abundant in* Π.

We need two lemmas for the proof.

Lemma 3.5. *Let Π be a profinite group and N a subgroup of Π of infinite index. Let N_1, N_2 be normal subgroups of Π such that $N_1 \cap N_2 \leq N$, $[N_1 : N_1 \cap N] \geq 3$ and $[N_2 : N_2 \cap N] \geq 2$. Assume that for every $f \uparrow \infty$, $s \in \mathbb{N}$, Π has open subgroups $E_1 \leq E$ containing N such that $f([\Pi : E]) \geq s \cdot [E : E_1]!$ and for each $i \in \{1, 2\}$ either*

(11a) $N_i \leq E$ *or*
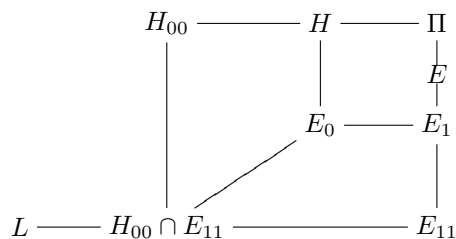(11b) $N_i E_1 = \Pi$ *and* $[E : E_1] \geq 3$.
Then N satisfies Condition $(\diamond)$.

Proof. Let $f \uparrow \infty$, $r \in \mathbb{N}$ and N' an open subgroup of N. Then there exists an open normal subgroup D of Π such that $D \cap N \leq N'$. Since $[N_1 : N_1 \cap N] \geq 3$, and $[N_2 : N_2 \cap N] \geq 2$, Π has an open normal subgroup H containing N such that

(12) $\qquad\qquad [N_1 : N_1 \cap H] \geq 3 \qquad$ and $\qquad [N_2 : N_2 \cap H] \geq 2$.

Put $s = r \cdot [\Pi : H]![\Pi : D]$.

Our condition gives open subgroups $E_1 \leq E$ containing N such that $f([\Pi : E]) \geq s \cdot [E : E_1]!$ and for each $i \in \{1, 2\}$ either (11a) or (11b) holds. Set $E_0 = H \cap E_1$. Let $E_{11} = \bigcap_{\sigma \in E} E_1^\sigma$ (resp., $H_{00} = \bigcap_{\sigma \in \Pi} H^\sigma$) be the normal core of E_1 (resp., H) in E (resp., Π). Finally let $L = H_{00} \cap E_{11} \cap D$. Then $L \leq H_{00} \cap E_{11} \leq H \cap E_1 = E_0$.

We have

$$
\begin{aligned}
[E : L] &= [E : H_{00} \cap E_{11} \cap D] \\
&= [E : E_{11}][E_{11} : H_{00} \cap E_{11}][H_{00} \cap E_{11} : H_{00} \cap E_{11} \cap D] \\
&\leq [E : E_{11}][\Pi : H_{00}][\Pi : D] \\
&\leq [E : E_1]![\Pi : H]![\Pi : D] \leq \frac{1}{s} f([\Pi : E])[\Pi : H]![\Pi : D] \\
&= \frac{1}{r} f([\Pi : E]).
\end{aligned}
$$

It remains to show that $[N_1 \cap E : N_1 \cap E_0] \geq 3$ and $[N_2 \cap E : N_2 \cap E_0] \geq 2$. First assume (11a), i.e., $N_i \leq E$. Then, since $E_0 \leq H$,

$$[N_i \cap E : N_i \cap E_0] \geq [N_i : N_i \cap H],$$

and we are done by (12).

Next assume (11b), i.e., $N_i E_1 = \Pi$ and $[E : E_1] \geq 3$. Then $(N_i \cap E)E_1 = E$, so

$$[N_i \cap E : N_i \cap E_0] \geq [N_i \cap E : N_i \cap E_1] = [E : E_1],$$

as needed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.6. *Let N be an abundant subgroup of a profinite group Π. Then for every $f \uparrow \infty$ and $s \in \mathbb{N}$ there exist open subgroups $N \leq E_1 \leq E \leq \Pi$ such that $f([\Pi : E]) \geq s \cdot [E : E_1]!$ and $[E : E_1] \geq 3$.*

Proof. Since N is abundant in Π, there exist $m, n \in \mathbb{N}$ such that for every open subgroup Π_0 of Π containing N with $[\Pi : \Pi_0] \geq m$ there exists an open subgroup Π_1 of Π_0 containing N such that $1 < [\Pi_0 : \Pi_1] \leq n$.

Let $f \uparrow \infty$ and $s \in \mathbb{N}$. By definition, $[\Pi : N] = \infty$. Thus there exists an open subgroup Π_0 of Π containing N with $f([\Pi : \Pi_0]) \geq \max\{s \cdot n!, s \cdot 4!, f(m)\}$. In particular $f([\Pi : \Pi_0]) \geq f(m)$, thus $[\Pi : \Pi_0] \geq m$. By assumption, Π_0 has an open subgroup Π_1 containing N such that $1 < [\Pi_0 : \Pi_1] \leq n$.

If $[\Pi_0 : \Pi_1] \geq 3$, then the subgroups $E = \Pi_0$ and $E_1 = \Pi_1$ satisfy the conclusion of the lemma because $f([\Pi : \Pi_0]) \geq s \cdot n!$. Otherwise, $[\Pi_0 : \Pi_1] = 2$. By assumption Π_1 has an open subgroup Π_2 containing N such that $1 < [\Pi_1 : \Pi_2] \leq n$. If $[\Pi_1 : \Pi_2] \geq 3$, then, similarly to the previous case, the subgroups $E = \Pi_1$ and $E_1 = \Pi_2$ satisfy the conclusion of the lemma. Otherwise, $[\Pi_1 : \Pi_2] = 2$, thus $[\Pi_0 : \Pi_2] = 4$, so $E = \Pi_0$, $E_1 = \Pi_2$ satisfy the conclusion of the lemma because $f([\Pi : \Pi_0]) \geq s \cdot 4!$. . $\square$

Proof of Proposition 3.4. Let $f \uparrow \infty$ and $s \in \mathbb{N}$. By Lemma 3.5 it suffices to find open subgroups $E_1 \leq E$ of Π containing N such that $f([\Pi : E]) \geq s \cdot [E : E_1]!$ and for each $i \in \{1, 2\}$ either

  (i) $N_i \leq E$ or
  (ii) $N_i E_1 = \Pi$ and $[E : E_1] \geq 3$.

We distinguish between the four cases:

In the first case we have $[\Pi : N N_1 N_2] = \infty$. Then there exists an open subgroup E of Π containing $N N_1 N_2$ such that $f([\Pi : E]) \geq s$. Put $E_1 = E$. Then $N_1, N_2 \leq E$ and $f([\Pi : E]) \geq s \cdot [E : E_1]!$.

In the second case, we assume that $[\Pi : N N_1 N_2] < \infty$ and $N N_1$ is abundant in Π. By [1, Corollary 2.3], $N N_1$ is abundant in every open subgroup that contains it, so $N N_1$ is abundant in $N N_1 N_2$. Thus we can replace Π by $N N_1 N_2$ in order to assume that $\Pi = N N_1 N_2$; it suffices to prove (i) and (ii) for this Π. Lemma 3.6 gives open subgroups $E_1 < E$ of Π that contain $N N_1$ for which $f([\Pi : E]) \geq s \cdot [E : E_1]!$ and $[E : E_1] \geq 3$. Then, $N_1 \leq E$ and $E_1 N_2 = \Pi$.
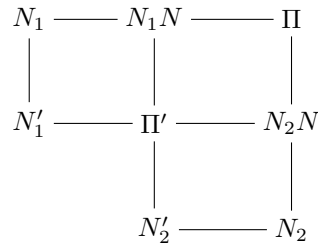
The third case is the same as the second case, after exchanging the indices 1 and 2.

In the last case we assume that $[\Pi : (N N_1) \cap (N N_2)] < \infty$ and N is abundant in Π. In particular $N N_i$ is open in Π, so

(13) $[N_i : N_i \cap N] = [N N_i : N] = \infty, \qquad i = 1, 2.$

Let $\Pi' = (N N_1) \cap (N N_2)$. Then since Π' is open in Π, it follows that N is abundant in Π'.

Put $N_1' = N_1 \cap \Pi'$ and $N_2' = N_2 \cap \Pi'$. Then $NN_1' = NN_2' = \Pi'$. Since $[N_i : N_i'] < \infty$, by (13), it follows that $[N_i' : N_i' \cap N] = \infty$.

$$
\begin{array}{ccccc}
N_1 & \longrightarrow & N_1 N & \longrightarrow & \Pi \\
\vert & & \vert & & \vert \\
N_1' & \longrightarrow & \Pi' & \longrightarrow & N_2 N \\
& & \vert & & \vert \\
& & N_2' & \longrightarrow & N_2
\end{array}
$$

Replace Π by Π', N_1 by N_1', and N_2 by N_2', if necessary, to assume that $NN_1 = \Pi$ and $NN_2 = \Pi$; it suffices to prove (i) and (ii) for this Π. Lemma 3.6 gives open subgroups $E_1 \le E$ of Π containing N with $f([\Pi : E]) \ge s \cdot [E : E_1]!$ and $[E : E_1] \ge 3$. Meanwhile, for $i = 1, 2$,

$$\Pi = NN_i \le E_1 N_i \le \Pi,$$

hence these subgroups satisfy (ii). $\qquad\square$

3.3. Proof of Theorem 1.2. Let Π be a profinite surface group of genus $g \ge 2$. We start with two lemmas.

Lemma 3.7. *A sparse subgroup of Π is semi-free of countable rank.*

Proof. Assume $N \le \Pi$ is sparse. Since Π is finitely generated, the rank of N is at most $\aleph_0$. Thus it suffices to solve any finite split embedding problem $(\mu \colon N \to B, \alpha \colon A \to B)$ for N [2, Lemma 3.4].

Choose an open normal subgroup $D \triangleleft \Pi$ with $D \cap N \le \ker \mu$ and set $H = ND$. Then H is open in Π and μ extends to an epimorphism $\mu' \colon H \to B$ by setting $\mu'(nd) = \mu(n)$ for all $n \in N$, $d \in D$.

Since N is sparse in Π, by [1, Lemma 2.2], there is an open subgroup H_0 of H that contains N such that $[\Pi : H_0] \ge 2|A|^3$ and every proper open subgroup $N \le H_1 \lneq H_0$ satisfies $[H_0 : H_1] > |A|$. Note that $\mu_0 = \mu'|_{H_0}$ is surjective, since $\mu'(H_0) \ge \mu'(N) = B$.

By Fact 2.1, we get that H_0 is a profinite surface group of genus

$$g_0 = [\Pi : H_0](g - 1) + 1 > [\Pi : H_0] \ge 2|A|^3.$$

By Lemma 2.2, the split embedding problem

$$(\mu_0 \colon H_0 \to B, \alpha \colon A \to B)$$

is properly solvable; let $\gamma \colon H_0 \to A$ be a solution. It suffices to show that $\gamma(N) = A$, or equivalently $N \ker \gamma = H_0$, since then $\gamma|_N$ is a solution of (μ, α). Indeed, as $[H_0 : N \ker \gamma] \le [H_0 : \ker \gamma] \le |A|$, we have $N \ker \gamma = H_0$. $\qquad\square$

Lemma 3.8. *Assume N satisfies Condition $(\diamond)$ in Π. Then N is semi-free.*

Proof. Since Π is finitely generated we get that N is countably generated. Hence it suffices to show that every finite split embedding problem

$$\mathcal{E} = (\mu_1 \colon N \to G_1, \alpha_1 \colon A \rtimes G_1 \to G_1)$$

with $A \neq 1$ is properly solvable.

Let $f(x) = \log x$ and take $N' = \ker \mu_1$. Choose r such that

$$(14) \qquad e^{ry} \geq 2|A|^{3y}y^3, \ \forall y \geq 2.$$

By Condition ($\diamond$) applied to f, r and N', there exist normal subgroups N_1, N_2 of Π such that $N_1 \cap N_2 \leq N$, $[N_1 : N_1 \cap N] \geq 3$, $[N_2 : N_2 \cap N] \geq 2$ and a diagram of subgroups

$$
\begin{array}{ccccccccc}
N' & \!\!\!\!\text{------} & N & \text{------} & E_0 & \text{------} & E & \text{------} & \Pi \\
| & & & & | & & & & \\
N \cap L & & \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\text{------} & & L & & & &
\end{array}
$$

satisfying the conditions (9). In particular, (9e) is equivalent to $[\Pi : E] \geq e^{r \cdot [E:L]}$. Then E is a surface group of genus

$$(15) \qquad g_0 = [\Pi : E](g - 1) + 1 \geq e^{r \cdot [E:L]} \geq 2|A|^{3[E:L]}[E : L]^3.$$

Let $N_i' = N_i \cap E$. We apply Lemma 3.2 with E replacing Π and N_i' replacing N_i. Let $G = E/L$, $G_0 = NL/L \cong N/N \cap L$, and

$$\mathcal{E}_{ind} = (\mu \colon E \to G, \alpha \colon A \wr_{G_0} G \to G)$$

the induced embedding problem. We claim that the conditions (i)-(iv) of the lemma are satisfied. Indeed, $L \cap N \leq N'$ by the diagram, so we have (i). Since $NL \leq E_0 \leq E$ we have

$$[N_i'NL : NL] = [N_i' : N_i' \cap NL] = [N_i \cap E : (N_i \cap E) \cap NL] \geq [N_i \cap E : N_i \cap E_0].$$

So (9c) and (9d) imply (ii) and (iii). Finally, by (9c) we have

$$[E : NL] \geq [E : E_0] \geq [N_1 \cap E : N_1 \cap E_0] \geq 3.$$

So (iv) is satisfied.

Now by (15) we have that $g_0 \geq 2|A \wr_{G_0} G|^3$, hence by Lemma 2.2, $\mathcal{E}_{ind}$ is properly solvable, and thus by Lemma 3.2 so is $\mathcal{E}$. $\square$

Proof of Theorem 1.2. As explained at the beginning of Section 3.2, we can assume that $[N_1N : N] = \infty$.
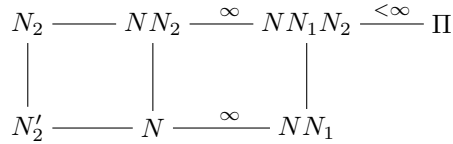
Note that by Lemma 3.8, if one of the conditions of Proposition 3.4 is satisfied, then N is semi-free. Hence we assume that none of them holds.

If $[\Pi : (NN_1) \cap (NN_2)] < \infty$, then the negation of Condition 10d of Proposition 3.4 gives that N is sparse in Π. Hence N is semi-free of countable rank (Lemma 3.7). Thus we may assume that $[\Pi : (NN_1) \cap (NN_2)] = \infty$. W.l.o.g. $[\Pi : NN_1] = \infty$. Then the negation of ((10a) $\vee$ (10b)) gives that NN_1 is sparse in Π. Then Lemma 3.7 gives that NN_1 is semi-free of countable rank.

Put $N_2' = (NN_1) \cap N_2$. Then $N_1, N_2' \triangleleft NN_1$, $N_1 \cap N_2' \leq N$ and $N_1 \not\leq N$. If $N_2' \not\leq N$, then the diamond theorem for semi-free groups ([2, Main Theorem VI]) gives that N is semi-free of countable rank.

We are left with the case $N_2' \leq N$. Then $N = NN_1 \cap NN_2$. By the negation of (10a) of Proposition 3.4, $[\Pi : NN_1N_2] < \infty$, hence

$$[\Pi : NN_2] = [\Pi : NN_1N_2][NN_1 : N] = \infty.$$

$$
\begin{array}{ccccccc}
N_2 & \text{———} & NN_2 & \xrightarrow{\ \infty\ } & NN_1N_2 & \xrightarrow{\ <\infty\ } & \Pi \\
| & & | & & | & & \\
| & & | & & | & & \\
N_2' & \text{———} & N & \xrightarrow{\ \infty\ } & NN_1 & &
\end{array}
$$

The negation of (10c) of Proposition 3.4 gives that NN_2 is sparse in Π and hence in the open subgroup NN_1N_2 of Π ([1, Corollary 2.3]). But since $NN_1/N_2' \cong NN_1N_2/N_2$, this implies that N is sparse in the semi-free group NN_1, and hence N is semi-free of countable rank ([1, Lemma 2.4]). $\qquad\square$

3.4. Proof of Theorem 1.1. Since $[\Pi : N] = \prod_p p^\infty$, N is projective [8, Proposition 1.2]. By Theorem 1.2, N is semi-free. Therefore, by [2, Theorem 3.6] N is free. $\qquad\square$

4. Applications

4.1. Proof of Theorem 1.4. Let Π be a surface group of genus $g \geq 2$ and let $N \lhd \Pi$ be a normal subgroup of infinite index such that Π/N is not hereditary just infinite . We need to prove that N is contained in a semi-free normal subgroup.

If there exists a normal subgroup $N \lneqq M \lhd \Pi$ with $[\Pi : M] = \infty$, then there exists $N \leq U \lhd \Pi$ open in Π such that $M \cap U \neq M$ (recall that N is the intersection of all open subgroups containing it). So $M \cap U$ is semi-free by Theorem 1.2, and we are done.

Therefore we can assume that $J = \Pi/N$ is just infinite. By [4, Theorem 3(b)], there exists an open normal subgroup J_0 of J such that either J_0 is hereditarily just infinite or $J_0 = A \times B$, where A and B are infinite groups. The former is not possible, since if U is an open normal subgroup of J with a proper infinite quotient, then $U \cap J_0$ is an open normal subgroup of J_0 with a proper infinite quotient.

Let Π_0, K_1, K_2 be the respective preimages of J_0, A, B under the map $\Pi \to J$. Then Π_0 is a surface group of genus ≥ 2 and $N = K_1 \cap K_2$. So by Theorem 1.2, N is semi-free. $\qquad\square$

Remark 4.1. Let N be a normal subgroup of Π such that Π/N is hereditarily just infinite. We do not know whether N is necessarily semi-free.

4.2. Proof of Corollary 1.3. Let Π be a surface group of genus at least 2, N a normal subgroup of Π of infinite index, and M a proper open subgroup of N. There exists an open normal subgroup $U \lhd \Pi$ such that $U \cap N \leq M$. Note that U is not contained in M because $[\Pi : M] = \infty$. By Theorem 1.2, M is semi-free. $\qquad\square$

4.3. Some examples. Let Π be a surface group of genus at least 2 and N a closed subgroup of infinite index. The following result provides many interesting examples of semi-free subgroups of a surface group.

Proposition 4.2. *If $N \lhd \Pi$ and if every open subgroup of Π/N can be generated by d elements, for some $d \geq 1$, or if N is sparse in Π, then N is semi-free.*

Proof. Let

$$\mathcal{E}(N) = (\mu_0 \colon N \to A, \alpha_0 \colon C \rtimes A \to A)$$

be a FSEP for N. Since N is countably generated, it suffices to show that $\mathcal{E}(N)$ has a proper solution. Assume first that every open subgroup of Π/N is generated by d elements. Choose $r, n \geq 1$ such that $n > ((|C||A|)!)^d$ and $r \geq 2|A|^3|C|^{3n}$.

Choose $L \lhd \Pi$ to be an open subgroup of Π such that $L \cap N \leq \ker \mu_0$. Choose $\tilde{\Pi}$ be an open subgroup of Π such that $N \leq \tilde{\Pi} \leq LN$ and such that $[\Pi : \tilde{\Pi}] \geq r$. Then we can extend μ_0 to $\mu \colon \tilde{\Pi} \to A$ by $\mu(nl) = \mu_0(n)$, for every $n \in N$, $l \in L$, for which $nl \in \tilde{\Pi}$. By Fact 2.1 the genus of $\tilde{\Pi}$ is at least r. Without loss of generality we can replace Π with $\tilde{\Pi}$ to assume μ_0 can be extended to $\mu \colon \Pi \to A$ and $g \geq r$. (Note that the rank of $\tilde{\Pi}/N$ is bounded by the rank of Π/N.)

Consider the FSEP

$$\mathcal{E}_n(\Pi) = (\mu \colon \Pi \to A, \alpha \colon C^n \rtimes A \to A),$$

where A acts component-wise on C^n. Since $g \geq r \geq 2|A|^3|C|^{3n}$, by Lemma 2.2, there exists a proper solution $\Psi \colon \Pi \to C^n \rtimes A$ of $\mathcal{E}_n(\Pi)$. For each $i = 1, \ldots, n$, let ψ_i be the composition of Ψ with the projection $C^n \rtimes A \to C \rtimes A$ on the ith coordinate. Let $L_i = \ker \psi_i$. Then $L_i L_j = \ker \mu$, for every $i \neq j$.

If $L_i N = \Pi$ for some i, then $\psi_i(N) = C \rtimes A$, so $\psi_i|_N$ is a proper solution of $\mathcal{E}(N)$, and we are done.

Otherwise, assume that $L_i N \neq \Pi$ for every i. But since $(L_i N)(L_j N) = (L_i L_j)N = \ker \mu N = \Pi$, we get that $L_i N$ are distinct subgroups of index $\leq |C||A|$. So Π/N has at least $n > ((|C||A|)!)^d$ open subgroups of index $\leq |C||A|$. This is a contradiction because each such a subgroup induces a distinct homomorphism to the symmetric group $S_{|C||A|}$ defined by the action on the cosets, and the number of these homomorphisms is bounded by $((|C||A|)!)^d$.

Next assume that N is sparse in Π. Replace Π by an open subgroup $\tilde{\Pi}$ of index $[\Pi : \tilde{\Pi}] \geq 2|C|^3|A|^3$ that contains N such that $\tilde{\Pi}$ has no proper subgroups of index $\leq |C||A|$ that contain N. Then arguing as above with $n = 1$, we get that $L_1 N \leq \Pi$ and $[\Pi : L_1 N] \leq |C||A|$, so $L_1 N = \Pi$. So $\psi_1|_N$ is a proper solution of $\mathcal{E}(N)$. $\square$

Examples 4.3. Each of the following conditions implies that N is semi-free.
  (1) $\Pi/N = \mathbb{Z}_p$ (every subgroup is cyclic)
  (2) $\Pi/N = K_1 \times K_2$ (N is the intersection of the preimages of K_1, K_2 in Π, hence by Theorem 1.2, is semi-free).
  (3) Π/N is abelian (Π/N is either $\mathbb{Z}_p$ or direct product).
  (4) Π/N is pro-nilpotent but not pro-p (Π/N is a direct product).
  (5) $[\Pi : N] = \prod_p p^{n(p)}$, where $0 \leq n(p) < \infty$ (this implies that N is sparse in Π).

Notice that (2) gives a new proof that the congruence kernel of an arithmetic lattice in $SL_2(\mathbb{R})$ is a free profinite group of countable rank, see [8] for more details.

Acknowledgements

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