SPECTRAL ANALYSIS OF RANDOM WALK OPERATORS ON EUCLIDEAN SPACE

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ABSTRACT. We study the operator associated to a random walk on $\mathbb{R}^d$ endowed with a probability measure. We give a precise description of the spectrum of the operator near 1 and use it to estimate the total variation distance between the iterated kernel and its stationary measure. Our study contains the case of Gaussian densities on $\mathbb{R}^d$.

1. Introduction

Let $\rho \in C^1(\mathbb{R}^d)$ be a strictly positive bounded function such that $d\mu = \rho(x)dx$ is a probability measure. Let $h > 0$ be a small parameter and $B_h(x)$ be the ball of radius h and center x. We consider the natural random walk associated to the density ρ with step h: if the walk is in x at time n, then the position y at time $n+1$ is determined by choosing $y \in \mathbb{R}^d$ uniformly with respect to the measure

$$(1.1) \qquad t_h(x, dy) = \frac{\rho(y)}{\mu(B_h(x))}\, \mathbb{1}_{|x-y|<h}\, dy$$

The associated random-walk operator is defined by

$$(1.2) \qquad T_h f(x) = \frac{1}{\mu(B_h(x))} \int_{B_h(x)} f(x')d\mu(x')$$

for any continuous function f, and the kernel of T_h is $t_h(x, dy)$. This is clearly a Markov kernel. Introduce the measure

$$d\nu_h = \frac{\mu(B_h(x))\rho(x)}{Z_h}dx$$

where Z_h is chosen so that $d\nu_h$ is a probability on $\mathbb{R}^d$. Then, the operator T_h is self-adjoint on $L^2(M, d\nu_h)$ and the measure $d\nu_h$ is stationary for the kernel $t_h(x, dy)$ (this means that $T_h^t(d\nu_h) = d\nu_h$, where T_h^t is the transpose operator of T_h acting on Borel measures).

The aim of this article is to describe the spectrum ot T_h and to address the problem of convergence of the iterated operator to the stationary measure. Such problems have been investigated in compact cases in [2], [7] and [3], and the link between the spectrum of T_h and the Laplacian (with Neumann boundary condition in [2] and [3]) was etablished. In this paper we investigate the case of such operators on the whole Euclidean space. The main difference with the previous works comes from the lack of compactness due to the fact that $\mathbb{R}^d$ is unbounded. We will consider densities satisfyng the following assumptions:

Case 1: tempered density. A density $\rho \in C^1(\mathbb{R}^d)$ is said *tempered* if there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^d$

$$(1.3) \qquad\qquad\qquad |d\rho(x)| \leq C\rho(x)$$

We shall say that it is *smooth tempered of exponential type* if ρ is smooth and if there are some positive numbers $(C_\alpha)_{\alpha \in \mathbb{N}^d}$, $R > 0, \kappa_0 > 0$, such that

$$(1.4) \qquad\qquad\qquad \forall |x| \geq R, \ |\partial_x^\alpha \rho(x)| \leq C_\alpha \rho(x)$$

and, if $\Delta := -\sum_{i=1}^d \partial_{x_i}^2$ is the positive Laplacian,

$$(1.5) \qquad\qquad\qquad \forall |x| \geq R, \ -\Delta\rho(x) \geq \kappa_0 \rho(x).$$

Densities verifying these assumptions can be easily constructed. For instance, if ρ is a smooth non vanishing function such that there exists $\alpha, \beta > 0$ such that for any $|x| > R$ we have $\rho(x) = \beta e^{-\alpha|x|}$, then the above assumptions are satisfied with $\kappa_0 = \alpha^2$. For densities satisfying (1.4), (1.5), we will define

$$(1.6) \qquad\qquad\qquad \kappa = \lim_{R \to \infty} \inf_{|x| \geq R} \frac{-\Delta\rho(x)}{\rho(x)}.$$

The second type of densities we shall consider is the following

Case 2: Gaussian density. We assume that $\rho(x) = (\frac{\alpha}{\pi})^{\frac{d}{2}} e^{-\alpha|x|^2}$ for some $\alpha > 0$.

It can be shown that that if ρ satisfies (1.3) or is Gaussian, there exists a constant $C > 0$ and $h_0 > 0$ such that

$$(1.7) \qquad\qquad \forall x \in \mathbb{R}^d, \ \forall h \in]0, h_0], \ \mu(B_h(x)) \geq Ch^d\rho(x).$$

Let us set $m_h(x) = \mu(B_h(x))$ and define the functions

$$(1.8) \qquad a_h(x) := (\alpha_d h^d \rho(x)/m_h(x))^{1/2}, \quad G_d(\xi) = \frac{1}{\alpha_d} \int_{|z| \leq 1} e^{iz\xi} dz$$

where $\alpha_d := \text{Vol}(B_{\mathbb{R}^d}(0,1))$. Notice that G_d is a real valued function bounded above by 1 and below by some $M > -1$, then define

$$(1.9) \qquad\qquad A_h := \lim_{R \to \infty} \sup_{|x| \geq R} a_h^2(x), \quad M := \min_{\xi \in \mathbb{R}^d} G_d(\xi) > -1.$$

We will show that $A_h = 1 - \frac{\kappa}{2(d+2)}h^2 + O(h^4)$ with κ defined in (1.6).

In order to describe the eigenvalues of T_h, let us also introduce the operator

$$(1.10) \qquad\qquad\qquad L_\rho = \Delta + V(x)$$

with $V(x) := \frac{-\Delta\rho(x)}{\rho(x)}$. Observe that the essential spectrum of this operator is included in $[\kappa, +\infty[$ in the tempered case, and empty in the Gaussian case. Moreover, we have the following factorisation:

$$(1.11) \qquad\qquad\qquad L_\rho = \sum_{j=1}^d \ell_j^* \ell_j$$

where $\ell_j = -\partial_{x_j} + \frac{\partial_{x_j}\rho}{\rho}$. This shows that L_ρ is non-negative on $L^2(\mathbb{R}^d)$. Moreover, since $\ell_j u = 0$ iff u is proportional to ρ, then 0 is a simple eigenvalue associated to the eigenfunction $\rho \in L^1 \cap L^\infty \subset L^2$.

We first prove the following result in the tempered case

Theorem 1.1. *Suppose that ρ is tempered in the sense of (1.3), then:*
(i) the essential spectrum of T_h on $L^2(\mathbb{R}^d, d\nu_h)$ is contained in $[A_h M, A_h]$ where M and A_h are defined in (1.9). If in addition $A_h = \lim_{|x|\to\infty} a_h^2(x)$, then $\sigma_{\mathrm{ess}}(T_h) = [A_h M, A_h]$.
(ii) If (1.4) and (1.5) hold, then $A_h = 1 - \frac{\kappa}{2(d+2)}h^2 + O(h^4)$ with κ defined in (1.6), and for all $\alpha \in]0,1[$ there exist $C > 0, h_0 > 0$ such that, if $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k$ denote the $L^2(\mathbb{R}^d, dx)$ eigenvalues of L_ρ in $[0, \alpha\kappa]$ counted with multiplicities, and if $1 = \lambda_0(h) > \lambda_1(h) \geq \ldots \geq \lambda_k(h)$ denote the k largest eigenvalues of T_h on $L^2(\mathbb{R}^d, d\nu_h)$ counted with multiplicities, then for all $h \in]0, h_0]$ and any $j = 1, \ldots, k$,

$$\left| 1 - \frac{1}{2(d+2)}\mu_k h^2 - \lambda_k(h) \right| \leq Ch^4.$$

Observe that if ρ is only tempered, the statement (i) shows that the essential spectrum can be the whole interval $[M, 1]$: for instance, take a density ρ such that $\rho(x) = |x|^{-m}$ in $\{|x| > R\}$ for some $R > 0$ and $m > d$, then it is easy to check that $m_h(x)/\rho(x) \to 1$ as $|x| \to \infty$ and therefore $A_h = 1$ in this case.

Notice also that there are examples of smooth densities of exponential type ρ such that the discrete spectrum of L_ρ below its continuous spectrum is non-empty. Indeed, take for instance $\rho = e^{-\tau\alpha(x)}$ where $\tau > 0$ and $\alpha(x)$ is smooth, equal to $|x|$ for $|x| > 1$ and $\nabla\alpha(0) = 0$, then

$$P_\tau := \tau^{-2}L_\rho = \tau^{-2}\Delta + |\nabla\alpha|^2 + \tau^{-1}\Delta\alpha$$

is a τ^{-1} semi-classical elliptic differential operator with semi-classical principal symbol $p(x,\xi) = |\xi|^2 + |\nabla\alpha|^2$ (see [8, 5, 4] for the theory of semi-classical pseudodifferential operators). Since $|\nabla\alpha| = 1$ in $|x| > 1$ and $\Delta\alpha = 0$ in $|x| > 1$, the essential spectrum of P_τ is $[1, \infty)$, then we can apply Theorem 9.6 of [4] and the fact that $\mathrm{Vol}\{(x,\xi) \in \mathbb{R}^{2d}; p(x,\xi) \in [0, \frac{1}{2}]\} > 0$ (since $\nabla\alpha(0) = 0$) to conclude that, if $\tau > 0$ is large enough, there exist $C\tau^d$ eigenvalues of P_τ in $[0, \frac{1}{2}]$ for some $C > 0$.

We also emphasize that the result in Theorem 1.1 is used in a fundamental way in the recent paper [1] to analyze random walks on surface with hyperbolic cusps.

If instead ρ is Gaussian, then $L_\rho = \Delta + 4\alpha^2|x|^2 - 2d\alpha$ and its spectrum is discrete $\sigma(L_\rho) = 4\alpha\mathbb{N}$ and the eigenfunctions associated to $4\alpha k$ have the form $H_k(x)e^{-2\alpha|x|^2}$ for some explicit polynomial H_k. We then have

Theorem 1.2. *Suppose that ρ is Gaussian, then the operator T_h is compact and if $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_K \ldots$ denote the $L^2(\mathbb{R}^d, dx)$ eigenvalues of L_ρ and $1 = \lambda_0(h) > \lambda_1(h) \geq \ldots \lambda_K(h) \geq \ldots$ those of T_h, then for $K \geq 0$ fixed, there exists $C > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$ and any $k = 1, \ldots, K$,*

$$(1.12) \qquad \left| 1 - \frac{1}{2(d+2)}\mu_k h^2 - \lambda_k(h) \right| \leq Ch^4.$$

Moreover, there exists $\delta_0 > 0$ such that for any $\lambda \in [0, \delta_0]$, the number $N(\lambda, h)$ of eigenvalues of T_h in $[1 - \lambda, 1]$ satisfies

$$(1.13) \qquad N(\lambda, h) \leq C(1 + \lambda h^{-2})^d.$$

In the last section of this paper, we also give some consequences on the convergence of the kernel of T_h^n to the stationary measure $d\nu_h$ as $n \to \infty$. In particular we show that, contrary to the compact setting [7], the convergence in L^∞ norm fails, essentially due to the non-compactness of the space.

These theorems, will be proved by using microlocal analysis. We refer to the books, [4], [5] and [8] for standard results in this theory. The organisation of the paper is the following. In the next section we study the essential spectrum of T_h on $L^2(\mathbb{R}^d, d\nu_h)$. In section 3, we collect some a priori estimates (regularity and decay) on the eigenfunctions of T_h. Following the strategy of [7], we use these estimates in section 4 to prove the above theorems. In last section, we adress the problem of total variation estimates: we show that the convergence to stationarity can not be uniform with respect to the starting point. Considering the case where the starting point x belongs to a ball of radius τ we prove total variation bounds in term of the spectral gap and τ.

2. Essential spectrum

We start by studying the essential spectrum of T_h in the tempered and Gaussian cases. From the definition of $d\nu_h$, it is easy to see that there exists some constant $c_1, c_2 > 0$ such that $c_1 h^d \leq Z_h \leq c_2 h^d$. Let us define the operator $\Omega : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, d\nu_h)$ by

$$(2.1) \qquad \Omega f(x) = \sqrt{\frac{Z_h}{m_h(x)\rho(x)}} f(x).$$

which is unitary, and let $\tilde{T}_h$ defined by $\tilde{T}_h = \Omega^* T_h \Omega$ so that

$$\tilde{T}_h f(x) = a_h(x)\overline{T}_h(a_h f)$$

with a_h defined in (1.8) and (with $\alpha_d = \text{Vol}(B_{\mathbb{R}^d}(0,1))$)

$$(2.2) \qquad \overline{T}_h g(x) := \frac{1}{\alpha_d h^d} \int_{|x-y|<h} f(y)dy.$$

Using the semiclassical Fourier transform it is easy to see that

$$\overline{T}_h = G_d(hD_x), \text{ with } G_d(\xi) = \frac{1}{\alpha_d} \int_{|z| \leq 1} e^{iz\xi} dz.$$

This function depends only on $|\xi|$, it is clearly real valued and $-1 < M \leq G_d(\xi) \leq 1$ for all ξ if M is defined in (1.9). Moreover, G_d tends to zero at infinity and $G_d(\xi) = 1$ if and if only $\xi = 0$.

Let us first prove (1.7) assuming (1.3): we have by assumption on ρ that for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq h$

$$-Ch \sup_{z \in B_h(x)} \rho(z) \leq \rho(x) - \rho(y) \leq Ch \sup_{z \in B_h(x)} \rho(z)$$

and therefore if $Ch < 1$

$$(1 - Ch) \sup_{z \in B_h(x)} \rho(z) \le \rho(x) \le \sup_{z \in B_h(x)} \rho(z)$$

which implies

$$\rho(x)\left(1 - \frac{Ch}{1 - Ch}\right) \le \rho(y) \le \rho(x)\left(1 + \frac{Ch}{1 - Ch}\right).$$

and thus (1.7).

The function a_h is then bounded and A_h of (1.9) is well defined. We first prove

Proposition 2.1. *Suppose that ρ is tempered in the sense of (1.3), then $\sigma_{\mathrm{ess}}(T_h) \subset [MA_h, A_h]$. If moreover $A_h = \lim_{|x| \to \infty} a_h^2(x)$, then the inclusion above is an equality.*

Proof. Let $R > 0$, then the operator $\tilde{T}_h$ can be written under the form

$$\tilde{T}_h = \mathbb{1}_{|x|>R}\,\tilde{T}_h\,\mathbb{1}_{|x|>R} + \mathbb{1}_{R<|x|<R+h}\,\tilde{T}_h\,\mathbb{1}_{|x|<R} + \mathbb{1}_{|x|<R}\,\tilde{T}_h\,\mathbb{1}_{|x|<R+h}$$

since $\tilde{T}_h$ increases support by a set of diameter at most h. The kernels of the last two operators in the right hand side is in $L^2(\mathbb{R}^d \times \mathbb{R}^d, dx \otimes dx)$, and thus these operators are compact. We thus deduce that the essential spectrum of $\tilde{T}_h$ is given by that of $S_h^R = \mathbb{1}_{|x|>R}\,\tilde{T}_h\,\mathbb{1}_{|x|>R}$. Since $S_h^R = b_h^R \overline{T}_h b_h^R$ with $b_h^R = \mathbb{1}_{|x|>R}\,a_h(x)$ and since $\overline{T}_h$ is a bounded self-adjoint operator satisfying

$$M||f||_{L^2}^2 \le \langle \overline{T}_h f, f \rangle_{L^2}, \quad ||\overline{T}_h f||_{L^2} \le ||f||_{L^2}$$

and $a_h(x) > 0$ we deduce easily that $\sigma_{\mathrm{ess}}(S_h^R) \subset [-MA_h^R, A_h^R]$ where we set $A_h^R := \sup_{|x| \ge R} a_h(x)^2$. It then suffices to take the limit as $R \to \infty$. Now if in addition $a_h^2(x)$ has a limit A_h when $|x| \to \infty$, we can write

$$(2.3) \qquad \tilde{T}_h = A_h \overline{T}_h + \epsilon_h(x)\overline{T}_h a_h(x) + A_h^{\frac{1}{2}} \overline{T}_h \epsilon_h(x)$$

with $\epsilon_h(x) := a_h(x) - A_h^{\frac{1}{2}}$ converging to 0 as $|x| \to \infty$. In particular, using that $|G_d(\xi)| \to 0$ when $|\xi| \to \infty$, we deduce that the last two operators in (2.3) are compact on L^2. Since, $\overline{T}_h$ is a function of the Euclidean Laplacian (or radial Fourier multiplier) the spectrum of $\overline{T}_h$ on $L^2(\mathbb{R}^d, dx)$ is absolutely continuous and consists of $[M, 1]$, which is the range of $G_d(\xi)$. This achieves the proof since the essential spectrum of $\tilde{T}_h$ is that of $A_h \overline{T}_h$ by (2.3). $\qquad \square$

We also describe the asymptotic behaviour of A_h:

Lemma 2.2. *If ρ satisfies (1.4) and (1.5), then the following asymptotic holds as $h \to 0$*

$$A_h = 1 - \frac{\kappa}{2(d+2)}h^2 + O(h^4)$$

where $\kappa = \liminf_{|x| \to \infty} \frac{-\Delta\rho(x)}{\rho(x)}$.

Proof. If ρ is tempered, we expand $m_h(x) = \mu(B_h(x))$ with respect to h and use assumption (1.4):

$$m_h(x) = h^d \int_{|z|<1} \rho(x+hz)dz$$

$$= \alpha_d h^d \rho(x) + \frac{1}{2} h^{d+2} \sum_{i,j} \partial_{x_i} \partial_{x_j} \rho(x) \int_{|z|\leq 1} z_i z_j dz + O(h^{d+4}\rho_4)$$

$$= \alpha_d h^d \rho(x) - \frac{\beta_d}{2d} h^{d+2} \Delta\rho(x) + O(h^{d+4}\rho_4)$$

with $|\rho_4(x)| \leq \rho(x)$ and $\beta_d := \int_{|z|<1} |z|^2 dz$. Using the definition of a_h, it follows from Lemma 3.2 below that

$$(2.4) \qquad\qquad a_h^2(x) = 1 + h^2 \gamma_d \frac{\Delta\rho(x)}{\rho(x)} + O(h^4)$$

with $\gamma_d = \frac{\beta_d}{2d\alpha_d} = \frac{1}{2(d+2)}$ and the $O(h^4)$ is uniform in $x \in \mathbb{R}^d$. Hence, it follows from (1.6) that

$$A = \limsup_{|x|\to\infty} a_h^2(x) = 1 + \gamma_d h^2 \liminf_{|x|\to\infty} \frac{\Delta\rho(x)}{\rho(x)} + O(h^4)$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.3. *In the tempered case, the operator* $\gamma_d L_\rho = \frac{1}{2(d+2)}(\Delta + \frac{-\Delta\rho}{\rho})$ *has essential spectrum contained in* $[\frac{\kappa}{2(d+2)}, \infty)$. *If in addition* $\kappa = \lim_{|x|\to\infty} \frac{-\Delta\rho}{\rho}$, *then the essential spectrum is exactly* $\sigma_{\mathrm{ess}}(L_\rho) = [\frac{\kappa}{2(d+2)}, \infty)$ *by Theorem 13.9 of* [6].

Now for the Gaussian case

Proposition 2.4. *If ρ is Gaussian, then T_h is a compact operator.*

Proof. The symbol $G_d(\xi)$ of $\overline{T}_h$ is decaying to 0 as $|\xi| \to 0$, a standard argument shows that if $\lim_{|x|\to\infty} a_h(x) \to 0$, then $\overline{T}_h a_h$ is compact on L^2. We write

$$(2.5) \qquad\qquad\qquad \frac{m_h(x)}{h^d \rho(x)} = \int_{|z|\leq 1} e^{-2hx.z - h^2|z|^2} dz$$

and by bounding below this integral by a dz integral on a conic region $-z.x > |z|.|x|/2$, we see that it converges to ∞ when $|x| \to \infty$, which proves the claim. $\qquad\square$

Remark 2.5. *In the Gaussian case, the operator $L_\rho = \Delta + 4\alpha^2|x|^2 - 2d\alpha$ has compact resolvent and discrete spectrum.*

Notational convention: For the following sections, all the tempered densities we shall consider will be smooth tempered densities of exponential type (ie. satisfying (1.4) and (1.5)), and therefore we will abuse notation and just call them tempered.

3. Spectral analysis of T_h

We recall here some notations. Let $a = a(x, \xi; h)$ be an h-dependent family of $C^\infty(\mathbb{R}^{2d})$ function and $m(x, \xi)$ be an order function as in [4]. We say that a belongs to the symbol class $S(m)$ if there exists some $h_0 > 0$ and constants $C_{\alpha,\beta}$ such that for any $\alpha, \beta \in \mathbb{N}^d$, any $0 < h \leq h_0$

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha,\beta} m(x, \xi)$$

For any $a \in S(m)$, we define $\mathrm{Op}_h(a)$ by

$$\mathrm{Op}_h(a) f(x) = \frac{1}{(2\pi h)^d} \int e^{\frac{i(x-y)\cdot\xi}{h}} a(x, \xi; h) f(y) dy d\xi$$

The standard theory of such operators is developped in [4], [5], [8].

3.1. Preliminary estimates. Let us start by some estimates on the symbols of the operator $\tilde{T}_h$, which will be useful to study its eigenfunctions.

Lemma 3.1. *The function $G_d(\xi)$ belongs to $S(\langle \xi \rangle^{-\max(1, \frac{d-1}{2})})$.*

Proof. Suppose first that $d \geq 2$. It is clear that the function G_d is smooth. When $|\xi| \geq 1$, one has

$$\partial_\xi^\beta G_d(\xi) = \frac{1}{\alpha_d} \int_{|z| \leq 1} (iz)^\beta e^{iz\xi} dz.$$

Let $\chi \in C_0^\infty(B_{\mathbb{R}^d}(0,1))$ be a radial cut-off equal to 1 on $B_{\mathbb{R}^d}(0, \frac{1}{2})$. Then the non-stationary phase theorem shows that

$$\int_{|z| \leq 1} \chi(z) z^\beta e^{iz\xi} dz = O(|\xi|^{-\infty}).$$

On the other hand,

$$I_\chi := \int_{|z| \leq 1} (1 - \chi(z)) z^\beta e^{iz\xi} dz = \int_{\frac{1}{2}}^1 (1-\chi)(r) r^{d-1+|\beta|} \left(\int_{S^{d-1}} e^{ir\omega\xi} \omega^\beta d\omega \right) dr.$$

For any $r \geq \frac{1}{2}$ the phase $\omega \mapsto \omega\xi$ has only two stationary points: these points are non-degenerate so that the stationary phase theorem implies $I_\chi = O(|\xi|^{-\frac{d-1}{2}})$. In the case $d = 1$, the computation is simpler since $G_d(\xi) = \frac{\sin(\xi)}{\xi}$. We leave it to the reader. $\square$

We will also need the following result on the function a_h.

Lemma 3.2. *The function a_h is smooth and the following estimates hold true:*

- *if ρ is tempered, then*

$$(3.1) \qquad \forall \alpha \in \mathbb{N}^d \setminus \{0\}, \exists C_\alpha > 0, \forall h \in]0, 1], |\partial_x^\alpha a_h(x)| \leq C_\alpha h^2$$

and there exists $C > 0$ such that

$$(3.2) \qquad \forall x \in \mathbb{R}^d, \left| \frac{1}{a_h^2(x)} - 1 - \frac{h^2}{2(d+2)} \frac{-\Delta\rho}{\rho} \right| \leq C h^4$$

- *if ρ is Gaussian, then*

(3.3) $$\forall \alpha \in \mathbb{N}^d, \ \exists C_\alpha > 0, \ \forall h \in]0,1], \ |\partial_x^\alpha a_h(x)| \leq C_\alpha h^{|\alpha|}.$$

and

(3.4)
$$\forall M > 0, \exists C_M > 0, \forall |x| < Mh^{-1}, \left| \frac{1}{a_h^2(x)} - 1 - \frac{(4\alpha^2|x|^2 - 2d\alpha)}{2(d+2)}h^2 \right| \leq C_M |x|^4 h^4,$$

(3.5) $$\exists C, R > 0, \forall |x| \geq R, \ \frac{1}{a_h^2(x)} \geq \max(1 + Ch^2|x|^2, Ce^{h|x|})$$

Proof. It follows from (1.8) that $a_h(x) = F \circ g_h(x)$ with $F(z) = z^{-1/2}$ and $g_h(x) = \frac{m_h(x)}{\alpha_d h^d \rho(x)}$. Following the arguments of the proof of Lemma 2.2, we have when ρ is tempered (using $\int_{|z|<1} z_i dz = \int_{|z|<1} z_i z_j z_k dz = 0$)

(3.6)
$$g_h(x) = 1 - \frac{h^2}{2(d+2)} \frac{\Delta\rho}{\rho} + \frac{h^4}{\alpha_d \rho(x)} \int_{|z|<1} \rho_4(x,z)dz$$
$$= 1 - \frac{h^2}{2(d+2)} \frac{\Delta\rho}{\rho} + h^4 r_4(x)$$

where $\rho_4(x,z)$ is a function which satisfies for all $\alpha \in \mathbb{N}^d$:

$$|\partial_x^\alpha \rho_4(x,z)| \leq C_\alpha \rho(x)$$

uniformly with respect to $x \in \mathbb{R}^d, |z| \leq 1$ and $r_4(x)$ has all its derivatives uniformly bounded on $\mathbb{R}^d$. In particular, for any $\alpha \in \mathbb{N}^d \setminus \{0\}$, $\partial_x^\alpha g_h(x) = O(h^2)$. Hence, for $h > 0$ small enough, Faà di Bruno formula combined with (3.6) shows that a_h is a smooth bounded function such that

(3.7) $$\forall \alpha \in \mathbb{N}^d \setminus \{0\}, \ \partial_x^\alpha a_h(x) = O(h^2).$$

This shows that a_h enjoys estimate (3.1) while (3.2) is a direct consequence of (3.6).

Suppose now that $\rho(x)$ is Gaussian. To enlight the notations, we consider the case $\alpha = 1$, i.e. $\rho(x) = \pi^{-\frac{d}{2}} e^{-|x|^2}$. It follows from (2.5) that

$$g_h(x) = \frac{1}{\alpha_d} \int_{|z|\leq 1} e^{-2hx.z - h^2|z|^2} dz.$$

Hence, there exists $c_0 > 0$ such that for all $x \in \mathbb{R}^d$, $h \in]0,1]$, $g_h(x) \geq c_0$. Moreover, for all $\alpha \in \mathbb{N}^d$ we have

$$\partial_x^\alpha g_h(x) = \frac{1}{\alpha_d} \int_{|z|<1} (-2hz)^\alpha e^{-2hz.x - h^2|z|^2} dz$$

so that there exists $C_\alpha > 0$ such that

(3.8) $$\forall h \in]0,1], \ \forall x \in \mathbb{R}^d, \ |\partial_x^\alpha g_h(x)| \leq C_\alpha h^{|\alpha|} |g_h(x)|.$$

Using again Faà di Bruno formula, we easily get that a_h is a smooth function such that for any $\alpha \in \mathbb{N}^d$,

(3.9) $$\partial_x^\alpha a_h(x) = \sum_{\pi \in \Pi_{|\alpha|}} C_{|\pi|} g_h(x)^{-\frac{1}{2}(2|\pi|+1)} \Pi_{B \in \pi} \frac{\partial^{|B|} g_h(x)}{\Pi_{j \in B} \partial x_j}$$

where $\Pi_{|\alpha|}$ denotes the set of all partitions of $\{1, \ldots, |\alpha|\}$, $|\pi|$ denotes the number of blocks in the partition π and $|B|$ denotes the cardinal of B, and $C_{|\pi|}$ is an explicit constant depending on $|\pi|$. Combining this formula with estimate (3.8), we get

$$(3.10) \qquad |\partial_x^\alpha a_h(x)| \leq \sum_{\pi \in \Pi_{|\alpha|}} C_{|\pi|}|g_h(x)|^{-\frac{1}{2}(2|\pi|+1)} \Pi_{B \in \pi}|hg_h(x)|^{|B|} \leq C|a_h(x)|h^{|\alpha|}$$

which proves (3.3).

Let us now prove the estimates on $a_h^{-2} = g_h$. The same computation as in the tempered case remains valid if we assume that $|hx.z|$ is bounded, which holds true if $h|x|$ is bounded. This shows (3.4). In order to prove (3.5), we observe that there exist constants $c, C > 0$ such that for all $0 < h < 1$

$$a_h(x)^{-2} = \alpha_d^{-1} \int_{S^{d-1}} \int_{0 < r \leq 1} e^{-2hrx.\theta - h^2 r^2} r^{d-1} dr d\theta$$

$$= \alpha_d^{-1} \int_{S^{d-1}} \int_{0 < r \leq 1} \left(1 + 4r^2 h^2 (x.\theta)^2 \int_0^1 (1-t) e^{-2thrx.\theta} dt \right) e^{-h^2 r^2} r^{d-1} dr d\theta$$

$$\geq 1 + 4h^2 \alpha_d^{-1} \int_{S^{d-1}} \int_{0 < r \leq 1} r^2 (x.\theta)^2 \left(\int_0^1 (1-t) e^{-2thrx.\theta} dt \right) e^{-h^2 r^2} r^{d-1} dr d\theta$$

$$\qquad - ch^2$$

$$a_h(x)^{-2} \geq 1 + Ch^2 |x|^2 - ch^2$$

for $|x| > R$ with $R > 0$ large, the last inequality being proved by the same argument as for Proposition 2.4. Enlarging R and modifying $C > 0$ if necessary, this shows the quadratic bound in (3.5). The exponential bound in (3.5) follows easily from the inequality above, by bounding below the integral by an integral on a region $\{\theta.x/|x| < -(1 - \epsilon), r \geq 1 - \epsilon\}$ for some small $\epsilon > 0$. $\qquad \square$

3.2. Regularity and decay of eigenfunctions. We are now in position to prove the first estimates on the eigenfunctions of $\tilde{T}_h$.

Observe that for any $1/2 > \delta > 0$ small, there exists $s_\delta > 0$ such that $|G_d(\xi)| \leq 1 - 2\delta$ when $|\xi|^2 \geq s_\delta$.

Lemma 3.3. *Let $C > 0$ and $\lambda_h \in [1 - Ch^2, 1]$ be an eigenvalue of T_h (which can belong to the essential spectrum) in the tempered case, and $\lambda_h \in [1 - \delta, 1]$, $\delta > 0$ in the Gaussian case. Let $e_h \in L^2(\mathbb{R}^d, dx)$ satisfy $\tilde{T}_h e_h = \lambda_h e_h$, $\| e_h \|_{L^2(\mathbb{R}^d)} = 1$. Then e_h belongs to all Sobolev spaces and for all $s \in \mathbb{R}$*

$$(3.11) \qquad \| e_h \|_{H^s(\mathbb{R}^d)} = O\left(\left(1 + \frac{1 - \lambda_h}{h^2}\right)^{\frac{s}{2}}\right).$$

Moreover,

$$(3.12) \qquad \| (1 - \chi)(h^2 \Delta) e_h \|_{H^s(\mathbb{R}^d)} = O(h^\infty)$$

where $\chi \in C_0^\infty(\mathbb{R})$ is equal to 1 near 0 in the tempered case and $\chi = 1$ on $[-s_\delta, s_\delta]$ in the Gaussian case.

Proof. We use arguments similar to those of [7], the difference is that now we are working in $\mathbb{R}^d$ instead of a compact manifold: let us write $\lambda_h = 1 - h^2 z_h$ with $0 < z_h < \kappa \gamma_d$ in the tempered case and $0 < z_h < \delta h^{-2}$ in the Gaussian case; and start

from $(\tilde{T}_h - \lambda_h)e_h = 0$. Since $\tilde{T}_h = a_h \overline{T}_h a_h$, it follows from Lemmas 3.1 and 3.2 that $\tilde{T}_h$ is a semiclassical pseudodifferential operator on $\mathbb{R}^d$ of order $m \leq -1$. In particular, it maps $L^2(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$ and $\| \tilde{T}_h \|_{L^2 \to H^1} = O(h^{-1})$. Since $e_h = \frac{1}{\lambda_h}\tilde{T}_h e_h$ and λ_h is bounded from below, we deduce that $\| e_h \|_{H^1} = O(h^{-1})$. Iterating this argument, we finally get

$$(3.13) \qquad \qquad \| e_h \|_{H^s} = O(h^{-s})$$

for any $s > 0$ (using interpolation for non integral s). Let us denote $p_h(x,\xi)$ the symbol of $\tilde{T}_h$. It follows from usual symbolic calculus and Lemma 3.2 that

$$(3.14) \qquad \qquad p_h(x,\xi) = a_h^2(x)G_d(\xi) + h^m r_h(x,\xi)$$

for some symbol $r_h \in S(\langle \xi \rangle^{-\max(1, \frac{d-1}{2})})$ with $m = 3$ if ρ is tempered and $m = 2$ if ρ is Gaussian.

Suppose that ρ is smooth tempered and let $\chi \in C_0^\infty(\mathbb{R})$ be equal to 1 near 0. Since $|G_d(\xi)| \leq 1$ with $G_d(\xi) \to 0$ as $|\xi| \to \infty$ and $G_d(\xi) = 1 \iff \xi = 0$, we deduce that for any cut-off function $\tilde{\chi}$ equal to 1 near 0, we have

$$(1 - \tilde{\chi}(\xi))G_d(\xi) \leq (1 - \epsilon)(1 - \tilde{\chi}(\xi))$$

for some $\epsilon > 0$ depending on $\tilde{\chi}$. Since $\lambda_h = 1 + O(h^2)$ and $a_h = 1 + O(h^2)$, the symbol

$$q_h(x,\xi) = (1 - \tilde{\chi}(\xi))(\lambda_h - p_h(x,\xi))$$

is bounded from below by $\frac{\epsilon}{2}(1 - \tilde{\chi}(\xi))$ for $h > 0$ small enough. Moreover it is, up to a lower order symbol, equal to the symbol of $(1 - \tilde{\chi}(h^2\Delta))(\lambda_h - \tilde{T}_h)$ and thus by taking $(1 - \tilde{\chi}) = 1$ on the support of $(1 - \chi)$, we can construct a parametrix L_h with symbol $\ell_h(x,\xi) \in S(1)$ such that

$$L_h(1 - \tilde{\chi}(h^2\Delta))(\lambda_h - \tilde{T}_h) = (1 - \chi(h^2\Delta)) + h^\infty \mathrm{Op}_h(w_h)$$

for some symbol $w_h \in S(1)$. This clearly shows that

$$(3.15) \qquad \qquad \| (1 - \chi(h^2\Delta))e_h \|_{L^2} = O(h^\infty)$$

and by interpolation with (3.13) we get

$$(3.16) \qquad \qquad \| (1 - \chi(h^2\Delta))e_h \|_{H^s} = O(h^\infty).$$

It remains to show that $\chi(h^2\Delta)e_h$ is bounded in H^s. We have

$$(\mathrm{Op}_h(p_h) - 1 + h^2 z_h)e_h = 0.$$

Let $b_h(x,\xi) = p_h(x,\xi) - 1 + h^2 z_h$, then since z_h is bounded, we know from (3.14) that

$$(3.17) \qquad \qquad b_h(x,\xi) = a_h^2(x)G_d(\xi) - 1 + h^2 r_h(x,\xi)$$

for some $r_h \in S^0(1)$. By Taylor expansion of $G_d(\xi)$ at $\xi = 0$, we see that there exists a smooth function F on $\mathbb{R}^+$, strictly positive and such that $1 - G_d(\xi) = |\xi|^2 F(|\xi|^2)$. Since $a_h^2(x) = 1 + O(h^2)$, we get

$$b_h(x,\xi) = -|\xi|^2 F(|\xi|^2) + h^2 \tilde{r}_h(x,\xi)$$

with $\tilde{r}_h \in S^0(1)$. Combined with (3.15), this shows that for any $\chi \in C_0^\infty(\mathbb{R}^d)$

$$h^2 \Delta F(h^2\Delta)\chi(h^2\Delta)e_h = O_{L^2}(h^2).$$

Since F is strictly positive on the support of χ, we can construct a parametrix like above and obtain that

$$\| \chi(h^2\Delta)e_h \|_{H^2} = O(1).$$

Iterating this process, it follows that the above bounds hold in all Sobolev spaces.

Consider now the case of a Gaussian density and let us prove (3.12). For $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 on $[-s_\delta, s_\delta]$ (and $0 \leq \chi \leq 1$) we get

$$(1 - \chi)(|\xi|^2)G_d(\xi) \leq (1 - 2\delta)(1 - \chi)(|\xi|^2).$$

Since we have $a_h \leq 1 + O(h^2)$ and $\lambda_h \geq 1 - \delta$ for small $h > 0$, this shows that $(1 - \chi(|\xi|^2))(\lambda_h - p_h(x, \xi)) \geq \frac{\delta}{2}(1-\chi)(|\xi|^2)$ for h small and (3.15), (3.16) are still valid. Let us prove (3.11). By definition, we have $\text{Op}_h(b_h)e_h = 0$ with $b_h(x, \xi) = a_h^2(x)G_d(\xi) - \lambda_h + h^2 r_h(x, \xi)$ for some $r_h \in S(1)$. Thanks to (3.10), we have $|\partial_x^\alpha r_h(x, \xi)| \leq C_\alpha |a_h^2(x)|$ for any α and $|\partial_x^\alpha (a_h^{-2} r_h(x, \xi))| \leq C_\alpha$. Using again the structure of G_d and dividing by a_h^2, it follows that

$$(3.18) \qquad h^2\Delta F(h^2\Delta)e_h = (1 - \lambda_h a_h^{-2}(x) + h^2\text{Op}_h(\tilde{r}_h))e_h$$

for some symbol $\tilde{r} \in S(1)$. Taking the scalar product with $\chi(h^2\Delta)^2 e_h$ and using the fact that $\text{Op}_h(\tilde{r}_h)$ is bounded on L^2, we get

(3.19)
$$\langle h^2\Delta F(h^2\Delta)\chi(h^2\Delta)e_h, \chi(h^2\Delta)e_h \rangle = \langle (1 - \lambda_h a_h^{-2}(x))\chi(h^2\Delta)e_h, \chi(h^2\Delta)e_h \rangle + O(h^2)$$
$$= I_R(h) + J_R(h) + O(h^2)$$

where

$$I_R(h) := \langle \psi_R(x)(1 - \lambda_h a_h^{-2}(x))\chi(h^2\Delta)e_h, \chi(h^2\Delta)e_h \rangle$$

$$J_R(h) := \langle (1 - \psi_R(x))(1 - \lambda_h a_h^{-2}(x))\chi(h^2\Delta)e_h, \chi(h^2\Delta)e_h \rangle$$

with $\psi_R(x) := \mathbb{1}_{|x| \leq R}$. Hence, it follows from (3.4) that $I_R(h) = O(h^2 R^2 + 1 - \lambda_h)$. On the other hand, setting $R^2 = (1 - \lambda_h)/(h^2\epsilon)$ with $\epsilon > 0$ small enough but independent of h, (3.5) gives that $1 - \lambda_h a_h^{-2}(x) \leq -\lambda_h C h^2 |x|^2 + (1 - \lambda_h) < 0$ if $|x| \geq R$, and hence $J_R(h) \leq 0$. Combined with the estimate on I_R, this shows that

$$\langle h^2\Delta F(h^2\Delta)\chi(h^2\Delta)e_h, \chi(h^2\Delta)e_h \rangle = O(1 - \lambda_h).$$

Dividing by h^2 and using again the fact that $F > 0$ we obtain $\|\Delta\chi(h^2\Delta)e_h\|_{L^2} = O(1 + \frac{1-\lambda_h}{h^2})$. Iterating this argument and using interpolation, we obtain the desired estimates for any H^s. $\qquad\square$

In order to control the multiplicity of the eigenvalues as in [7], we need some compactness of the family $(e_h)_h$. Since $\mathbb{R}^d$ is not bounded, the regularity of the eigenfunctions is not sufficient and we need some decay property of the eigenfunctions near infinity. For $R > 0$, let χ_R be a smooth function equal to 1 for $|x| \geq R + 1$ and zero for $|x| \leq R$.

Lemma 3.4. *Let us assume that ρ is tempered and let $\alpha \in]0,1[$. Suppose that $\lambda_h \in [1 - \alpha h^2 \frac{\kappa}{2(d+2)}, 1]$ and that $e_h \in L^2(\mathbb{R}^d, dx)$ satisfies $\tilde{T}_h e_h = \lambda_h e_h$ and $\|e_h\|_{L^2(\mathbb{R}^d)} = 1$. Let $\phi \in C_0^\infty(\mathbb{R})$, then there exists $R > 0$ such that*

$$\| \chi_R(x)\phi(h^2\Delta)e_h \|_{L^2(\mathbb{R}^d)} = O(h).$$

As a by-product, for any $s \in \mathbb{R}$, $\chi_R e_h$ goes to 0 in $H^s(\mathbb{R}^d)$ when h goes to 0.

Proof. From the preceding Lemma, we know that

$$(\Delta F(h^2 \Delta) + \mathrm{Op}_h(\widetilde{r}_h))\phi(h^2 \Delta)e_h = O(h^\infty).$$

for some $\widetilde{r}_h \in S(1)$. On the other hand, this term can be made more precise : it follows from Lemma 3.2 and equation (3.14) that

$$O(h) = \left(-\Delta F(h^2 \Delta) - \frac{1}{2}\gamma_d(V(x)\mathrm{Op}_h(G_d(\xi)) + \mathrm{Op}_h(G_d(\xi))V(x)) + z_h \right)\phi(h^2 \Delta)e_h$$

$$= -(\Delta F(h^2 \Delta) + \gamma_d V(x) - z_h)e_h + O(h^2 \|e_h\|_{H^2})$$

with $\lambda_h = 1 - h^2 z_h$ and using (3.11), we obtain

$$(\Delta \widetilde{F}(h^2 \Delta) + V(x) - \tilde{z}_h)\tilde{f}_h = O(h)$$

with $\tilde{f}_h := \phi(h^2 \Delta)e_h$, $\tilde{z}_h := z_h/\gamma_d$ and $\widetilde{F} = F/\gamma_d$. Let $q_h(x, \xi) := |\xi|^2 \widetilde{F}(|\xi|^2) + V(x) - \tilde{z}_h$. Since $F \geq 0$, it follows from assumption (1.5) that there exists $R > 0$ such that for any $\xi \in \mathbb{R}^d$ and any $|x| \geq R$, we have $q_h(x, \xi) \geq (1-\alpha)\kappa/2$ if $1 - \lambda_h \leq \alpha\kappa h^2/2(d+2)$. Hence we can build a parametrix for q_h on the support of χ_R and this shows that $\| \chi_R \tilde{f}_h \|_{L^2} = O(h)$. Using interpolation and the fact that $(e_h)_h$ is bounded in H^s, we obtain directly the same bounds in H^s. □

Lemma 3.5. *Suppose that ρ is Gaussian. Let $\delta > 0$ and $\chi \in C_0^\infty(\mathbb{R})$ be equal to 1 on $[-s_\delta, s_\delta]$, then there exists h_0 such that, for any $k, s \in \mathbb{N}$ there exists $C_{k,s} > 0$ such that for all $h \leq h_0$ and any eigenfunction $e_h \in L^2(\mathbb{R}^d)$ of $\widetilde{T}_h$ with eigenvalue $\lambda_h \in [1 - h^2\delta, 1]$, we have*

$$(3.20) \qquad \|\langle x \rangle^k \chi(h^2 \Delta)e_h\|_{H^s(\mathbb{R}^d)} \leq C_{k,s}\|\chi(h^2 \Delta)e_h\|_{H^{s+k}(\mathbb{R}^d)}$$

Proof. It follows from (3.18) and (3.12) that

$$(3.21) \qquad (1 - \lambda_h a_h^{-2}(x))\chi(h^2 \Delta)e_h = h^2 \mathrm{Op}_h(r_h)\chi(h^2 \Delta)\Delta e_h$$

for some $r_h \in S(1)$. Let $R > 0$ be sufficiently large so that $a_h^{-2}(x) \geq 1 + Ch^2|x|^2$ for $|x| \geq R$. Then, if $\lambda_h = 1 - h^2 z_h$, one has for $|x| > R$

$$(3.22) \qquad -1 + \lambda_h a_h^{-2}(x) \geq h^2(C|x|^2 - z_h) \geq C'h^2(1 + |x|^2)$$

for some $C' > 0$ independent of h. We take $\psi_R \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 for $|x| \geq R+1$ and 0 for $|x| \leq R$, then by (3.22) and (3.3), we deduce easily that $\langle x \rangle^2 h^2(-1 + \lambda_h a_h^{-2})^{-1}\psi_R \in S(1)$ and therefore

$$\langle x \rangle^2 \psi_R(x)\chi(h^2 \Delta)e_h = \mathrm{Op}_h(\widetilde{r}_h)\chi(h^2 \Delta)\Delta e_h$$

for some $\tilde{r}_h \in S(1)$. Therefore, for any $s \geq 0$, we have

$$\|\langle x \rangle^2 \chi(h^2 \Delta)e_h\|_{H^s(\mathbb{R}^d)} \leq C\|\chi(h^2 \Delta)e_h\|_{H^{s+2}(\mathbb{R}^d)}.$$

Iterating this argument $k/2$ times and using (3.11), we get (3.20). □

4. Proof of Theorem 1.1 and 1.2

4.1. Spectrum localisation. We work as in [7] and we only give a sketch of the proof since it is rather similar. The main difference with the situation in [7] is that we work on unbounded domains, so that Sobolev embedding do not provide directly compactness. In both tempered and Gaussian case, we will use the following observation: suppose that φ is a smooth function, then it follows from Lemma 3.2 and the expansion $G_d(\xi) = 1 - \gamma_d |\xi|^2 + O(|\xi|^4)$ as $|\xi| \to 0$ that

$$(4.1) \qquad \frac{1 - \tilde{T}_h}{h^2} \varphi = \gamma_d L_\rho \varphi + h^2 \psi$$

where $\|\psi\|_{L^2(\mathbb{R}^d)} = O(\|\varphi\|_{H^4(\mathbb{R}^d)})$ in the tempered case and $\|\psi\|_{L^2(B(0,Mh^{-1}))} = O(\||x|^4 \varphi\|_{H^4(B(0,Mh^{-1}+1))})$ for any h-independent $M > 0$ in the Gaussian case.

We start with the case of a tempered density and follow the strategy of [7]. Since T_h and $\tilde{T}_h$ are unitarily conjugated by $\Omega : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, d\nu_h)$, the eigenvalues of T_h on $L^2(\mathbb{R}^d, d\nu_h)$ (and their multiplicities) are exactly those of $\tilde{T}_h$ on $L^2(\mathbb{R}^d, dx)$. First, assume that $(L_\rho - \mu)e = 0$ for some $\mu \in [0, \kappa)$ and $e \in H^2(\mathbb{R}^d)$, $\|e\|_{L^2(dx)} = 1$. Then, e is in fact in C^∞ and using (4.1) with $\varphi = e$, we get easily

$$\frac{1 - \tilde{T}_h}{h^2} e = \gamma_d \mu e + O_{L^2}(h^2).$$

Since $\tilde{T}_h$ is self-adjoint, this shows that $\mathrm{dist}(\gamma_d \mu, \sigma(\Delta_h)) = O(h^2)$ with

$$\Delta_h := \frac{1 - \tilde{T}_h}{h^2},$$

and that there exist $C_0 > 0, C_1 > 0, h_0 > 0$ such that for all $0 < h \le h_0$ and $\mu \in \sigma(L_\rho) \cap [0, \kappa - C_1 h^2)$, the number of eigenvalues of Δ_h in $[\gamma_d \mu - C_0 h^2, \gamma_d \mu + C_0 h^2]$ is bounded from below by the multiplicity of μ.

Conversely, consider an eigenfunction e_h of Δ_h corresponding to an eigenvalue $z_h \in [0, \gamma_d \kappa)$, then using Lemma (3.3), we get

$$z_h e_h = \Delta_h e_h = \gamma_d L_\rho e_h + O_{L^2}(h^2).$$

This shows that all the eigenvalues of Δ_h are at distance at most Ch^2 of the spectrum of $\gamma_d L_\rho$. Let us now consider an orthonormal set of eigenfunctions e_h^j of Δ_h associated to the eigenvalues z_h^j contained in $[\gamma_d \mu - C_0 h^2, \gamma_d \mu + C_0 h^2]$ for some $\mu \in \sigma(L_\rho) \cap [0, \alpha\kappa]$, where with C_0, C_1 are the constants given above. Let $R > 0$ be fixed as in Lemma 3.4. From Lemmas 3.4 and 3.3, each eigenfunction can be decomposed as

$$e_h^j = u_h^j + v_h^j$$

with u_h^j bounded in any H^s and supported in $B(0, R)$ and v_h^j converging to 0 in H^s when h goes to 0. Since $H^s(B(0,R))$ is compactly embedded in H^2 for s larger than 2, we can assume (extracting a subsequence if necessary) that the e_h^j converge to some f^j in $H^2(\mathbb{R}^d, dx)$ and z_h^j converges to $\gamma_d \mu$. Hence, the $(f^j)_j$ provide an orthonormal family of eigenfunctions of L_ρ associated to the eigenvalue μ. This shows that the number of eigenvalues of Δ_h in $[\gamma_d \mu - C_0 h^2, \gamma_d \mu + C_0 h^2]$ is exactly the multiplicity of μ as an eigenvalue of L_ρ, and achieves the proof of Theorem 1.1.

Notice in particular that our proof does not rule out the possibility of an infinite sequence of eigenvalues z_h^j for Δ_h converging to the bottom of the essential spectrum

κ.

Assume now that ρ is Gaussian and start with $(L_\rho - \mu)e = 0$ with $\|e\|_{L^2} = 1$. It follows from (4.1) that

$$\Delta_h e = \mathbb{1}_{|x| < h^{-1}} \gamma_d L_\rho e + \mathbb{1}_{|x| \geq h^{-1}} \Delta_h e + h^2 \psi$$

with ψ supported in $B(0, Mh^{-1})$ and $\|\psi\|_{L^2} = O(\|\langle x \rangle^4 e\|_{H^4(B(0,Mh^{-1}+1))})$. Since $e = p(x)e^{-\alpha|x|^2}$ for some polynomial p, then $\|\psi\|_{L^2}$ is bounded uniformly with respect to h. The same argument and $\mathbb{1}_{|x| \geq h^{-1}} \Delta_h = \mathbb{1}_{|x| \geq h^{-1}} \Delta_h \mathbb{1}_{|x| \geq h^{-1}-h}$ shows that $\| \mathbb{1}_{|x| \geq h^{-1}} \Delta_h e\|_{L^2} = O(h^{-2}e^{-ch^{-2}})$. This implies that

$$\Delta_h e = \gamma_d \mu e + O_{L^2}(h^2).$$

Like in the tempered case, it follows that $\mathrm{dist}(\gamma_d \mu, \sigma(\Delta_h)) = O(h^2)$ and that for any given $L > 0$ there exists $C_0 > 0, h_0 > 0$ such that for all $0 < h \leq h_0$ and all $\mu \in \sigma(L_\rho)$ with $\mu \leq L$, the number of eigenvalues of Δ_h in $[\gamma_d\mu - C_0 h^2, \gamma_d\mu + C_0 h^2]$ is bounded from below by the multiplicity of μ.

Conversely, suppose now that $\tilde{T}_h e_h = (1 - h^2 z_h)e_h$ for some $e_h \in L^2(\mathbb{R}^d)$ such that $\|e_h\|_{L^2} = 1$ and $z_h \in [0, L]$, $L > 0$ being fixed. From Lemmas 3.3 and 3.5, we know that

$$z_h e_h = \Delta_h e_h = \gamma_d L_\rho e_h + O_{L^2}(h^2),$$

this shows that the distance of the eigenvalues of Δ_h (less than L) to $\sigma(\gamma_d L_\rho)$ is of order $O(h^2)$.

To get the equality between the multiplicities, we work as in the tempered case and consider an orthonormal family of eigenfunctions e_h^j of Δ_h associated to the eigenvalues z_h^j contained in $[\gamma_d\mu - C_0 h^2, \gamma_d\mu + C_0 h^2]$. It follows from Lemmas 3.4 and 3.5 that

$$e_h^j = u_h^j + O(h^\infty)$$

with $u_h^j := \chi(h^2\Delta)e_h^j$ bounded uniformly with respect to h in $\langle x \rangle^{-k} H^s(\mathbb{R}^d)$ for any $k, s \geq 0$. Then the family $(u_h^j)_{h>0}$ is compact in $H^2(\mathbb{R}^d)$ and extracting a subsequence if necessary, we can then assume that both u_h^j and e_h^j converge to some f^j in H^2 and z_h converges to $\gamma_d\mu$ when $h \to 0$. We split u_h^j into $\psi_h(x)u_h^j + (1 - \psi_h(x))u_h^j$ where ψ_h is smooth, supported in $|x| \leq 1/h$ and equal to 1 in $|x| \leq 1/2h$. In particular we have that $\|(1 - \psi_h)u_h^j\|_{H^4} = O(h^\infty)$. On the other hand, it follows from (4.1) that

$$\begin{aligned}
z_h^j e_h^j = \Delta_h e_h^j &= \Delta_h(\psi_h u_h^j) + O(h^\infty) \\
&= \gamma_d L_\rho(\psi_h u_h^j) + O(h^2 \|\langle x \rangle^4 \psi_h u_h^j\|_{H^4}) + O(h^\infty) \\
&= \gamma_d L_\rho(e_h^j) + O(h^2 \|\langle x \rangle^4 e_h^j\|_{L^2}) + O(h^\infty) \\
z_h^j e_h^j &= \gamma_d L_\rho(e_h^j) + O(h^2 \|e_h^j\|_{L^2}) + O(h^\infty)
\end{aligned}$$

where we used Lemma 3.5 in the last line. Making $h \to 0$, we show that $(f^j)_j$ is an orthonormal family of eigenfunctions of L_ρ associated to the eigenvalue μ. This achieves the proof of (1.12).

4.2. The Weyl estimate. It remains to prove the Weyl estimate on the number of eigenvalues in the Gaussian density case. Fix $\delta > 0$ small, then for $\tau > 0$, let us define the operator on $\mathbb{R}^d$

$$P_\tau = \tau(\chi^2(\sqrt{\Delta/\tau}) + \chi^2(\sqrt{|x|^2/\tau}))$$

where $\chi \in C^\infty((0,\infty))$ is a positive increasing function which satisfies $\chi(x) = x$ for $x < 1 - \delta$ and $\chi(x) = 1$ for $x > 1$. Clearly P_τ is a self-adjoint bounded operator on $L^2(\mathbb{R}^d)$ with norm less or equal to 2τ and since for any function $f \in L^2$ such that f is supported in $|x| > \tau$ or $\hat{f}$ is supported in $|\xi| > \tau$, one has $\langle P_\tau f, f \rangle \geq \tau \|f\|_{L^2}^2$, the essential spectrum is contained in the interval $[\tau, 2\tau]$. Let $\Pi_{\tau/2} = \mathbb{1}_{[0,\tau/2]}(P_\tau)$ be the orthogonal spectral projector, it is then finite rank by what we just said. We shall prove that there is $\epsilon > 0$, $C > 0$ such that for all f in the range of $1 - \Pi_{\tau/2}$ and all $h > 0$ small enough and $\tau \leq \epsilon h^{-2}$, we have

(4.2) $$\langle T_h f, f \rangle \leq (1 - C\tau h^2)\|f\|_{L^2}^2.$$

Notice that if $(1 - \Pi_{\frac{\tau}{2}})f = f$, we have $\langle P_\tau f, f \rangle \geq \frac{1}{2}\tau\|f\|_{L^2}^2$ and thus

(4.3) $$\|\chi(\sqrt{\Delta/\tau})f\|^2 + \|\chi(\sqrt{|x|^2/\tau})f\|^2 \geq \frac{1}{2}\|f\|^2.$$

We first assume that $\|\chi(\sqrt{|x|^2/\tau})f\|^2 \geq \frac{1}{4}\|f\|^2$, then using that $\bar{T}_h$ has $L^2 \to L^2$ norm bounded by 1 we deduce

$$\langle a_h \bar{T}_h a_h f, f \rangle = \langle \bar{T}_h a_h f, a_h f \rangle \leq \|a_h f\|_{L^2}^2.$$

But from (3.4) and (3.5), we also have that there is $\epsilon > 0$, $C > 0$ independent of τ, h such that if $\tau \leq \epsilon h^{-2}$,

$$a_h^2(x) \leq 1 - Ch^2\tau\chi(\sqrt{|x|^2/\tau})^2.$$

Thus we obtain by combining with (4.3)

(4.4) $$\langle a_h \bar{T}_h a_h f, f \rangle \leq (1 - Ch^2\tau/4)\|f\|_{L^2}^2.$$

Assume now that $\|\chi(\sqrt{|x|^2/\tau})f\|^2 < \frac{1}{4}\|f\|^2$, then since $(1 - \Pi_{\tau/2})f = f$ this implies

(4.5) $$\|\chi(\sqrt{\Delta/\tau})f\|^2 \geq \frac{1}{4}\|f\|^2$$

and we shall prove that (4.4) holds as well in that case. Using $a_h^2 \leq 1 + Ch^2$ for some $C > 0$, let us write for $f \in L^2$

(4.6)
$$\langle a_h \bar{T}_h a_h f, f \rangle = \langle a_h^2 \bar{T}_h f, f \rangle + \langle a_h[\bar{T}_h, a_h]f, f \rangle$$
$$\leq (1 + Ch^2)\|\bar{T}_h f\|_{L^2}\|f\|_{L^2} + \langle a_h[\bar{T}_h, a_h]f, f \rangle.$$

Using the fact that $\bar{T}_h = G_d(hD_x)$ is a semiclassical pseudo-differential operator with symbol $G_d \in S(1)$ defined in (1.8) and the estimates $|\partial_x^\alpha a_h| = O(h)$ if $|\alpha| > 0$ of Lemma 3.2, we deduce from the composition law of semiclassical pseudo-differential operators that $[\bar{T}_h, a_h] = h^2\text{Op}_h(c_h)$ where $c_h \in S(1)$ is a uniformly bounded symbol in h. Therefore by Calderón-Vaillancourt theorem, $\|a_h[\bar{T}_h, a_h]\|_{L^2 \to L^2} = O(h^2)$ and thus

(4.7) $$\langle a_h[\bar{T}_h, a_h]f, f \rangle \leq Ch^2\|f\|_{L^2}^2$$

for some $C > 0$ uniform in h and independent of τ. Now using Plancherel, $\langle \bar{T}_h^2 f, f \rangle = \int_{\mathbb{R}^d} G_d^2(h\xi)|\hat{f}(\xi)|^2 d\xi$ where G_d is defined in (1.8). Now since $G_d(\xi) \to 0$ as $\xi \to \infty$ and

$G_d^2(\xi) \le 1 - C|\xi|^2$ for some C when ξ is small, we directly obtain that there is $\epsilon > 0$ independent of τ, h such that if $\tau \le \epsilon h^{-2}$, the bound $G_d^2(h\xi) \le 1 - Ch^2\tau\chi(\sqrt{|\xi|^2/\tau})^2$. Combined with (4.5), this implies that $||\bar{T}_h f||_{L^2} \le (1 - Ch^2\tau)||f||_{L^2}$ and thus, by combining this with (4.6) and (4.7), (4.4) holds if $\tau > \tau_0$ for some $\tau_0 > 0$ independent of h and we have proved (4.2).

By the min-max principle, one deduces from (4.2) that the number of eigenvalues of T_h in $[1 - Ch^2\tau, 1]$ counted with multiplicites is bounded by the rank of $\Pi_{\tau/2}$. Now, to prove the Weyl estimate (1.13), it remains to show that $\text{Rank}(\Pi_{\tau/2}) = O(\tau^d)$. This is a rather standard result (see for instance [4, page 115] for a comparable estimate), but we write some details

Let us consider $\hbar := 1/\sqrt{\tau}$ as a semiclassical parameter. The operator $P(\hbar) := \hbar^2 P_{\hbar^{-2}}$ is a $\hbar$ semi-classical operator with a symbol in the class $S(1)$ given by $p_\hbar(x, \xi) = \chi^2(|\xi|) + \chi^2(\hbar|x|)$, more precisely $P(\hbar)$ is the Weyl quantization of the symbol $p_\hbar(x, \xi)$. Let $f \in C_0^\infty(\mathbb{R})$ be such that $f(s) = 1$ for $|s| \le 1$, $f(s) = 0$ for $|s| \ge 2$ and $0 \le f \le 1$. Consider the harmonic oscillator on $\mathbb{R}^d$, $H = \Delta + |x|^2$ and define the operator

$$(4.8) \qquad\qquad \Pi_\hbar^H = f(\hbar^2 H).$$

Then $\Pi_\hbar^H$ is a non-negative self-adjoint operator, its norm is bounded by 1, its rank satisfies $\text{rank}(\Pi_\hbar^H) = O(\hbar^{-2d})$. From the min-max principle, to prove a Weyl estimate for $P(\hbar)$, it suffices to show that for all $u \in L^2$

$$(4.9) \qquad\qquad \langle P(\hbar)u, u \rangle + \langle \Pi_\hbar^H u, u \rangle \ge c$$

for some $c > 0$. First, we claim that the operator $\Pi_\hbar^H$ can be written under the form

$$(4.10) \qquad \Pi_\hbar^H = \text{Op}_\hbar(f(\hbar^2|x|^2 + |\xi|^2)) + R_\hbar, \ \text{ where } ||R_\hbar||_{L^2 \to L^2} = O(\hbar).$$

Let Ω be a fixed compact subset of $\mathbb{C}$ whose intersection with $\mathbb{R}$ contains $\text{supp}(f)$. Then, it is easy to check that for all $s \in \Omega \cap (\mathbb{C} \setminus \mathbb{R})$

$$(\hbar^2 H - s)\text{Op}_\hbar\Big(\frac{1}{\hbar^2|x|^2 + |\xi|^2 - s}\Big) = 1 + \hbar\text{Op}_\hbar(q_\hbar(x, \xi; s))$$

for some symbol $q_\hbar(x, \xi; s) \in S(1)$, satisfying for any α, β

$$|\partial_x^\alpha \partial_\xi^\beta q_\hbar(x, \xi; s)| \le C_{\alpha,\beta}|\text{Im}(s)|^{-3-|\alpha|-|\beta|}$$

for some $C_{\alpha,\beta}$ uniform in h, s. Then this implies

$$(\hbar^2 H - s)^{-1} = \text{Op}_\hbar\Big(\frac{1}{\hbar^2|x|^2 + |\xi|^2 - s}\Big) - \hbar(\hbar^2 H - s)^{-1}\text{Op}_\hbar(q_\hbar(x, \xi; s))$$

but by the Calderon-Vaillancourt theorem and the spectral theorem for H, we deduce that

$$(4.11) \qquad\qquad (\hbar^2 H - s)^{-1} = \text{Op}_\hbar\Big(\frac{1}{\hbar^2|x|^2 + |\xi|^2 - s}\Big) + \hbar W_\hbar$$

for some bounded operator $W_\hbar$ on L^2 with norm $O(\hbar|\text{Im}(s)|^{-N})$ for some N depending only on the dimension d. It remains to apply Helffer-Sjöstrand formula [4, Th 8.1] with $\tilde{f} \in C_0^\infty(\Omega)$ an almost analytic extension of f

$$f(\hbar^2 H) = \frac{1}{2i\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(s)(\hbar^2 H - s)^{-1}ds \wedge d\bar{s}$$

and we deduce directly (4.10) from (4.11). Observe that the symbol of $P(\hbar) + \Pi_{\hbar}^{H}$ satisfies that there exists $C > 0$ such that

$$\chi^2(|\xi|) + \chi^2(\hbar|x|) + f(\hbar^2|x|^2 + |\xi|^2) \geq C$$

for all $0 < \hbar \leq \hbar_0$. Therefore, by Gärding inequality, (4.9) is satisfied for some $c > 0$, and using the min-max principle, this implies easily that the number of eigenvalues of $P(\hbar)$ less or equal to $C/2$ is bounded above by $\text{rank}(\Pi_{\hbar}^{H}) = O(\hbar^{-2d})$, and this concludes the proof of the Weyl estimate for T_h.

5. Convergence to stationarity

In this section, we study the convergence of the iterated kernel $T_h^n(x, dy)$ towards its stationnary measure $d\nu_h$ when n goes to infinity. The measure $d\nu_h$ is associated to the orthogonal projection $\Pi_{0,h}$ onto constant functions in $L^2(d\nu_h)$:

$$(5.1) \qquad \Pi_{0,h}(f) = \int_{\mathbb{R}^d} f(y) d\nu_h(y)$$

On the other hand, let us define the spectral gap $g(h) = \text{dist}(1, \sigma(T_h) \setminus \{1\})$. It follows from Theorems 1.1 and 1.2 that

$$(5.2) \qquad g(h) = h^2 \frac{\mu_1}{2(d+2)} + O(h^4)$$

in the Gaussian case, and

$$(5.3) \qquad g(h) = \frac{h^2}{2(d+2)} \min(\kappa, \mu_1) + O(h^4)$$

in the tempered case, where κ denotes the bottom of the essential spectrum and μ_1 the first non-zero eigenvalue of L_ρ with the convention $\mu_1 = +\infty$ if L_ρ has no eigenvalue. The following proposition gives a convergence result in L^2 norm.

Proposition 5.1. *Let $\alpha > 0$ be fixed. There exists $C > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$ and all $n \in \mathbb{N}$, we have*

$$(5.4) \qquad \| T_h^n - \Pi_{0,h} \|_{L^2(d\nu_h) \to L^2(d\nu_h)} \leq C e^{-ng(h)}.$$

Proof. This is a direct consequence of the spectral theorem and the fact that 1 is a simple eigenvalue proved in Theorems 1.1, 1.2. $\qquad \square$

Let us now introduce the total variation distance, which is much stronger than the L^2 norm. If μ and ν are two probability measures on a set E, their total variation distance is defined by

$$\|\mu - \nu\|_{TV} = \sup_A |\mu(A) - \nu(A)|$$

where the sup is taken over all measurable sets. Then, a standard computation shows that

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{\|f\|_{L^\infty} = 1} |\mu(f) - \nu(f)|$$

The following theorem shows that the convergence in total variation distance can not be uniform with respect to the starting point x. This has to be compared with the results in the case of compact state space [2], [3] and [7] where the convergence is uniform in x.

Theorem 5.2. *There exists $C > 0$ such that for any $n \in \mathbb{N}$, $h \in]0,1]$ and $\tau > 0$, we have*

$$(5.5) \qquad \inf_{|x| \geq \tau + (n+1)h} \|T_h^n(x, dy) - d\nu_h\|_{TV} \geq 1 - Cp(\tau)$$

where $p(\tau) = e^{-2\alpha\tau(\tau - h)}$ if $\rho = (\frac{\alpha}{\pi})^{\frac{d}{2}} e^{-\alpha|x|^2}$ is Gaussian and $p(\tau) = \int_{|y| \geq \tau} \rho(y)^2 dy$ if ρ is tempered.

Proof. Let $\tau > 0$ and $n \in \mathbb{N}$. Consider the function

$$(5.6) \qquad \begin{aligned} f_\tau(x) &= 1\!\!1_{[\tau, +\infty[}(|x|) - 1\!\!1_{[0, \tau[}(|x|) \\ &= -1 + 2\,1\!\!1_{[\tau, +\infty[}(|x|). \end{aligned}$$

For $x \in \mathbb{R}^d$ such that $|x| \geq \tau + (n+1)h$, thanks to finite speed propagation we have

$$(5.7) \qquad\qquad\qquad T_h^n f_\tau(x) = 1.$$

On the other hand, we also have

$$(5.8) \qquad \begin{aligned} \Pi_{0,h} f_\tau &= \int_{\mathbb{R}^d} f_\tau(y) d\nu_h(y) = -1 + 2 \int_{|y| \geq \tau} d\nu_h(y) \\ &= -1 + \frac{2}{Z_h} \int_{|y| \geq \tau} m_h(y)\rho(y) dy. \end{aligned}$$

If ρ is tempered, then $m_h(y) \leq Ch^d \rho(y)$ for some constant $C > 0$. Hence, $\Pi_{0,h} f_\tau \leq -1 + Cp(\tau)$ with $p(\tau) = \int_{|y| \geq \tau} \rho(y)^2 dy$. Combined with (5.7), this shows the anounced result in the tempered case.

Suppose now that $\rho(x) = \beta e^{-\alpha|x|^2}$ is Gaussian for some $\alpha, \beta > 0$. Then $m_h(y) \leq Ch^d e^{-\alpha|y|^2 + 2h\alpha|y|}$ for any $h \in]0,1]$. Hence,

$$(5.9) \qquad \Pi_{0,h} f_\tau \leq -1 + C \int_{|y| \geq \tau} e^{-2\alpha(|y|^2 - h|y|)} dy \leq -1 + Cp(\tau)$$

with $p(\tau) = e^{-2\alpha\tau(\tau - h)}$. Using again (5.7), this shows the anounced result in the Gaussian case. $\qquad\square$

In the following theorem, $g(h)$ denotes the spectral gap of T_h, whose asymptotics is given in (5.2) and (5.3).

Theorem 5.3. *There exists $C > 0$ and $h_0 > 0$ such that for any $n \in \mathbb{N}$, $h \in]0, h_0]$, $\tau > 0$,*

$$(5.10) \qquad \sup_{|x| < \tau} \|T_h^n(x, dy) - d\nu_h\|_{TV} \leq Cq(\tau, h) e^{-ng(h)}$$

where $q(\tau, h) = h^{-\frac{d}{2}} \sup_{|x| < \tau} \frac{1}{\rho(x)}$ if ρ is tempered and $q(\tau, h) = e^{\alpha\tau(\tau + 3h)}$ if $\rho = (\frac{\alpha}{\pi})^{\frac{d}{2}} e^{-\alpha|x|^2}$ is Gaussian.

Proof. Assume that $h_0 > 0$ is such that the results of the previous section hold true for $h \in]0, h_0]$. Observe that

$$(5.11) \qquad \begin{aligned} \sup_{|x| \leq \tau} \|T_h^n(x, dy) - d\nu_h\|_{TV} &= \frac{1}{2} \sup_{|x| \leq \tau} \sup_{\|f\|_{L^\infty} = 1} |T_h^n f(x) - \Pi_{0,h} f| \\ &= \frac{1}{2} \|T_h^n - \Pi_{0,h}\|_{L^\infty(\mathbb{R}^d) \to L^\infty(|x| \leq \tau)}. \end{aligned}$$

Suppose first that ρ is tempered and denote B_τ the ball of radius τ centred in 0 and $I_n(\tau, h) = \|T_h^n - \Pi_{0,h}\|_{L^\infty(\mathbb{R}^d) \to L^\infty(B_\tau)}$. Then, denoting $L^2(d\nu_h)$ for $L^2(\mathbb{R}^d, d\nu_h)$,

(5.12)
$$I_n(\tau, h) \leq \|T_h\|_{L^2(d\nu_h) \to L^\infty(B_\tau)} \|T_h^{n-2} - \Pi_{0,h}\|_{L^2(d\nu_h) \to L^2(d\nu_h)} \|T_h\|_{L^\infty(\mathbb{R}^d) \to L^2(d\nu_h)}$$

$$\leq \|T_h\|_{L^2(d\nu_h) \to L^\infty(B_\tau)} e^{-(n-2)g(h)}$$

where we have used Proposition 5.1 and the fact that $\|T_h\|_{L^\infty(\mathbb{R}^d) \to L^2(d\nu_h)} = 1$. To estimate T_h from $L^2(d\nu_h)$ into $L^\infty(B_\tau)$ we consider $f \in L^2(d\nu_h)$ such that $\|f\|_{L^2(d\nu_h)} = 1$. Then,

(5.13)
$$|T_h f(x)| \leq \frac{1}{m_h(x)} \Big(\int_{|x-y|<h} \frac{Z_h^2}{m_h(y)^2} d\nu_h \Big)^{\frac{1}{2}}$$

$$\leq \frac{Z_h^{\frac{1}{2}}}{m_h(x)} \Big(\int_{|x-y|<h} \frac{\rho(y)}{m_h(y)} dy \Big)^{\frac{1}{2}}.$$

Since ρ is tempered we have $m_h(z) \geq Ch^d \rho(z)$ for some $C > 0$ and we deduce from the above estimate that $|T_h f(x)| \leq C/(h^{\frac{d}{2}} \rho(x))$. Taking the supremum over $x \in B_\tau$ we obtain the announced result in the tempered case.

Suppose now that $\rho = (\frac{\alpha}{\pi})^{\frac{d}{2}} e^{-\alpha|x|^2}$ is Gaussian. Since T_h is Markov and $g(h)$ is of order h^2, we can assume $n > h^{-2}$. For $k \in \mathbb{N}$ let $\sigma_k(h) = \frac{1-\lambda_k(h)}{h^2}$, where $1 = \lambda_0(h) > \lambda_1(h) \geq \lambda_2(h) \geq \ldots \geq \lambda_k(h)$ denote the eigenvalues of T_h. Denote also $e_{k,h}$ the eigenvector associated to $\lambda_k(h)$ normalized in $L^2(d\nu_h)$ and $\Pi_{k,h} = \langle ., e_{k,h} \rangle_{L^2(d\nu_h)} e_{k,h}$ the associated projector. We write the eigenvalues under the form $\lambda_k(h) = 1 - h^2 \sigma_k(h)$, then the spectral gap $g(h) = h^2 \sigma_1(h)$. Let $\delta > 0$ and decompose $T_h = T_{h,1} + T_{h,2}$ with

(5.14)
$$T_{h,1} = \sum_{\sigma_1(h) \leq \sigma_k(h) < (1-\delta)h^{-2}} (1 - h^2 \sigma_k(h)) \Pi_{k,h}.$$

From the spectral theorem, we deduce that $\| T_{h,2}^{n-2} \|_{L^2 \to L^2} \leq (1-\delta)^{n-2}$. On the other hand, for ρ Gaussian, we have $m_h(z) \geq Ch^d \rho(z) e^{-2h\alpha|z|}$. Combining this estimate with (5.13), we get

(5.15)
$$\| T_h \|_{L^2(\mathbb{R}^d) \to L^\infty(B_\tau)} \leq Ch^{-\frac{d}{2}} e^{\alpha\tau(\tau+3h)}.$$

Since $T_{h,2}^n = T_h T_{h,2}^{n-2} T_h$, we can combine this with the L^2 estimate, to get

(5.16)
$$\| T_{h,2}^n \|_{L^\infty(\mathbb{R}^d) \to L^\infty(B_\tau)} \leq Ch^{-\frac{d}{2}} e^{\alpha\tau(\tau+3h)} (1-\delta)^{n-2} \leq q(\tau, h) e^{-ng(h)}$$

since $h^{-\frac{d}{2}}(1-\delta)^{n-2} \ll e^{-ng(h)}$. Hence, it remains to study $T_{h,1}^n$.

Since $d\nu_h$ is a probability, then

(5.17)
$$\| \Pi_{k,h} \|_{L^\infty(\mathbb{R}^d) \to L^\infty(B_\tau)} \leq \| e_{k,h} \|_{L^\infty(B_\tau)} \| e_{k,h} \|_{L^1(d\nu_h)} \leq \| e_{k,h} \|_{L^\infty(B_\tau)} .$$

From Lemma 3.3 and Sobolev embedding, we know that $\| \Omega^* e_{k,h} \|_{L^\infty(\mathbb{R}^d)} \leq C\sigma_{k,h}^{\frac{d}{2}}$. Hence,

(5.18)
$$\| \Pi_{k,h} \|_{L^\infty(\mathbb{R}^d) \to L^\infty(B_\tau)} \leq \sup_{B_\tau} \Big(\frac{Z_h}{m_h(x)\rho(x)} \Big)^{\frac{1}{2}} \| \Omega^* e_{k,h} \|_{L^\infty(\mathbb{R}^d)} \leq C\sigma_{k,h}^{\frac{d}{2}} e^{\alpha\tau(\tau+3h)}$$

Using this estimate, we get

$$(5.19) \quad \parallel T_{h,1}^n \parallel_{L^\infty(\mathbb{R}^d) \to L^\infty(B_\tau)} \leq C e^{\alpha\tau(\tau+3h)} \sum_{\sigma_1(h) \leq \sigma_k(h) < (1-\delta)h^{-2}} (1 - h^2\sigma_k(h))^n \sigma_{k,h}^{\frac{d}{2}}.$$

Using the Weyl estimate (1.13) and the same argument as in [7], we get
(5.20)

$$\parallel T_{h,1}^n \parallel_{L^\infty(\mathbb{R}^d) \to L^\infty(B_\tau)} \leq C e^{\alpha\tau(\tau+3h)} \int_{\sigma_{1,h}}^\infty (1+x)^N e^{-nh^2 x} dx \leq C e^{\alpha\tau(\tau+3h)} e^{-nh^2\sigma_{1,h}}$$

for some $N > 0$. This completes the proof in the Gaussian case. $\qquad\square$

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