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# A DENSE G-DELTA SET OF RIEMANNIAN METRICS WITHOUT THE FINITE BLOCKING PROPERTY

Marlies Gerber and Wah-Kwan Ku

Abstract. A pair of points $(x, y)$ in a Riemannian manifold $(M, g)$ is said to have the finite blocking property if there is a finite set $P \subset M \setminus \{x, y\}$ such that every geodesic segment from $x$ to $y$ passes through a point of $P$. We show that for every closed $C^\infty$ manifold $M$ of dimension at least two and every pair $(x, y) \in M \times M$, there exists a dense $G_\delta$ set, $\mathcal{G}$, of $C^\infty$ Riemannian metrics on $M$ such that $(x, y)$ fails to have the finite blocking property for every $g \in \mathcal{G}$. Moreover, there exists a dense $G_\delta$ set, $\mathcal{G}_1$, of $C^\infty$ Riemannian metrics on $M$ such that for every $g \in \mathcal{G}_1$, there is a dense $G_\delta$ subset $\mathcal{R} = \mathcal{R}(g)$ of $M \times M$ such that every $(x, y) \in \mathcal{R}$ fails to have the finite blocking property for $g$.

## 1. Introduction

Let $M$ be a closed $C^\infty$ manifold, and let $g$ be a $C^\infty$ Riemannian metric on $M$. We consider a geodesic as a mapping $\gamma : I \to M$, where $I$ is an interval of positive length, and $\gamma$ is parametrized by arc length. Two geodesics $\gamma_i : I_i \to M$, $i = 1, 2$ will be considered to be the same if and only if $\gamma_1 = \gamma_2 \circ \varphi$, where $\varphi$ is a translation that maps $I_1$ onto $I_2$. Let $x$ and $y$ be points in $M$, possibly with $x = y$. When we say that a geodesic $\gamma : [c, d] \to M$ is from $x$ to $y$, we mean $\gamma(c) = x$ and $\gamma(d) = y$.

Given a Riemannian metric $g$ on $M$, a *blocking set* for $(x, y)$ is defined to be a subset $P$ of $M \setminus \{x, y\}$ such that every geodesic from $x$ to $y$ passes through a point in $P$. The pair $(x, y) \in M \times M$ is said to have the *finite blocking property* for $g$ if there exists a finite blocking set for $(x, y)$. If every $(x, y) \in M \times M$ has the finite blocking property, then $(M, g)$ is called *secure*. (See [8] and [5] for an explanation of this terminology.) A Riemannian manifold $(M, g)$ is called *insecure* if it is not secure, and it is called *totally insecure* if no pair $(x, y)$ has the finite blocking property. Furthermore, it is called *uniformly secure* if there exists a positive integer $n$ such that any pair of points $(x, y)$ has a blocking set with at most $n$ elements.

A point $p \in M$ is a *self-intersection point* of a geodesic $\gamma : I \to M$ if there exist $s, t \in I$, $s \neq t$, such that $\gamma(s) = p = \gamma(t)$. If there is no such point $p$ for a geodesic $\gamma$, we say that $\gamma$ is *non-self-intersecting.* We call a pair $(x, y) \in M \times M$ *strongly insecure* for $g$ if for each positive integer $n$, there exist $n$ geodesics $\gamma_i : [c_i, d_i] \to M$, $i = 1, \ldots, n$, from $x$ to $y$ satisfying the following three conditions: (i) the sets $\gamma_i((c_i, d_i))$, $i = 1, \ldots, n$, are pairwise disjoint; (ii) if $x \neq y$, then $\gamma_1, \ldots, \gamma_n$ are non-self-intersecting; and (iii) if $x = y$, then $x \notin \gamma_1((c_1, d_1)) \cup \cdots \cup \gamma_n((c_n, d_n))$, and $\gamma_1, \ldots, \gamma_n$ have no self-intersection points except $x$. It follows already from condition (i) that if $(x, y)$ is strongly insecure, then $(x, y)$ fails to have the finite blocking property.

Given a manifold $M$, it is natural to ask the following:

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**Question.** *Which pairs of points $(x, y) \in M \times M$ and which Riemannian metrics $g$ on $M$ are such that $(x, y)$ has the finite blocking property for $g$?*

Our contribution in this direction is Theorem 1.1 below, which implies that any given pair of points $(x, y)$ fails to have the finite blocking property for a dense $G_\delta$ set of metrics. We will give the proof in Section 3.

We let $\mathbb{G}$ denote the set of $C^\infty$ Riemannian metrics on $M$. For $k = 1, 2, \ldots, \infty$, there exists a complete metric on $\mathbb{G}$ whose topology coincides with the $C^k$ topology on $\mathbb{G}$. In particular, the Baire category theorem applies to $\mathbb{G}$ with the $C^k$ topology. When we refer to the $C^k$ topology on $M \times \mathbb{G}$ or $M \times M \times \mathbb{G}$, we mean the product topology, where we take the manifold topology on $M$ and the $C^k$ topology on $\mathbb{G}$.

**Theorem 1.1.** *Let $M$ be a closed $C^\infty$ manifold of dimension at least two, and let $\mathbb{G}$ be the space of $C^\infty$ Riemannian metrics on $M$. The following three statements hold.*

(1) *Let $x$ and $y$ be two points in $M$, possibly with $x = y$. Let $\mathcal{G} := \{g \in \mathbb{G} : (x, y)$ fails to have the finite blocking property for $g\}$. Then $\mathcal{G}$ contains the intersection of a countable collection of sets that are $C^1$-open and $C^\infty$-dense in $\mathbb{G}$.*

(2) *Let $\tilde{\mathcal{G}} := \{(x, y, g) \in M \times M \times \mathbb{G} : (x, y)$ fails to have the finite blocking property for $g\}$. Then $\tilde{\mathcal{G}}$ contains the intersection of a countable collection of sets that are $C^1$-open and $C^\infty$-dense in $M \times M \times \mathbb{G}$.*

(3) *Let $\hat{\mathcal{G}} := \{(x, g) \in M \times \mathbb{G} : (x, x)$ fails to have the finite blocking property for $g\}$. Then $\hat{\mathcal{G}}$ contains the intersection of a countable collection of sets that are $C^1$-open and $C^\infty$-dense in $M \times \mathbb{G}$.*

*If $M$ has dimension at least three, then "fails to have the finite blocking property" can be sharpened to "is strongly insecure" in all three statements.*

If $k \in \{1, 2, \ldots, \infty\}$, then a $C^1$ open subset of $\mathbb{G}$ is $C^k$ open, and a $C^\infty$ dense subset of $\mathbb{G}$ is $C^k$ dense. Thus we obtain the following corollary of Theorem 1.1.

**Corollary 1.2.** *If $M$ is a closed $C^\infty$ manifold of dimension at least two and $\mathbb{G}$ is the space of $C^\infty$ Riemannian metrics on $M$, then the sets $\mathcal{G}$, $\tilde{\mathcal{G}}$, and $\hat{\mathcal{G}}$ in Theorem 1.1 contain dense $G_\delta$ sets in the $C^k$ topology for $k = 1, 2, \ldots, \infty$.*

Corollary 1.2 (for $\tilde{\mathcal{G}}$ and $\hat{\mathcal{G}}$) implies the corollary below.

**Corollary 1.3.** *Let $M$ be a closed $C^\infty$ manifold of dimension at least two and suppose $k \in \{1, 2, \ldots, \infty\}$. The following two statements hold.*

(1) *There exists a dense $G_\delta$ set $\mathcal{G}_1$ in $\mathbb{G}$ with the $C^k$ topology, so that for each $g \in \mathcal{G}_1$, there is a dense $G_\delta$ subset $\mathcal{R}_1 = \mathcal{R}_1(g)$ of $M \times M$ such that each $(x, y) \in \mathcal{R}_1$ fails to have the finite blocking property for $g$.*

(2) *There exists a dense $G_\delta$ set $\mathcal{G}_2$ in $\mathbb{G}$ with the $C^k$ topology, so that for each $g \in \mathcal{G}_2$, there is a dense $G_\delta$ subset $\mathcal{R}_2 = \mathcal{R}_2(g) \subseteq M$ such that for each $x \in \mathcal{R}_2$, $(x, x)$ fails to have the finite blocking property for $g$.*

*Again, if $M$ has dimension at least three, then "fails to have the finite blocking property" can be replaced by "is strongly insecure" in both statements.*

V. Bangert and E. Gutkin obtained stronger results for the case when the dimension of $M$ is two and the genus is positive [2]. They proved that if $M$ has genus greater than one, then every Riemannian metric is totally insecure. Moreover, if $M$ has genus one, they showed that non-flat metrics are insecure and a $C^2$-open, $C^\infty$-dense set of metrics are totally insecure. These results provide evidence that (c) follows from (a) in the following conjecture, which originally appeared in [5] and [11]. A proof that (c) implies (b) is given in [9].

**Conjecture 1.4.** *Let $(M, g)$ be a closed $C^\infty$ Riemannian manifold. The following statements are equivalent.*

(a) *$(M, g)$ is secure.*
(b) *$(M, g)$ is uniformly secure.*
(c) *$g$ is a flat metric.*

While Conjecture 1.4 concerns the finite blocking property for *all pairs of points*, Theorem 1.1 shows that the finite blocking property can be destroyed for *any given pair of points*, under some small perturbation of metric.

In the next section, we will present some results which will be used to prove Theorem 1.1. We refer the reader to [7] for background information about geodesics and conjugate points.

## 2. Some preliminary results

We begin with the following classical result by J. P. Serre [14], [12], [3], [13].

**Theorem 2.1.** *Let $(M, g)$ be a closed $C^\infty$-Riemannian manifold, and let $x, y \in M$. Then there exist infinitely many geodesics from $x$ to $y$. That is, $\exp_x^{-1}\{y\}$ is an infinite subset of $T_xM$.*

For $a, b > 0$, we let $I_a$ denote the open interval $(-a, a) \subset \mathbb{R}$, and we let $B_b$ denote the open ball $\{\mathbf{w} \in \mathbb{R}^{n-1} : |\mathbf{w}| < b\}$, where $n$ is the dimension of the manifold $M$ under consideration.

**Definition 2.2.** If $(M, g)$ is a closed Riemannian manifold of dimension $n \geq 2$, and $a, b > 0$, let $\mathcal{F}(a, b, g) = \{f \in C^\infty(I_a \times B_b, M) \mid f \text{ satisfies (i),(ii),(iii) below}\}$.

(i) The map $f$ is a $C^\infty$-diffeomorphism onto its image.
(ii) For all $\mathbf{p} \in B_b$ , the map $t \mapsto f(t, \mathbf{p})$ , for $t \in I_a$, is a geodesic (for the metric $g$).
(iii) For all $t \in I_a$, the $(n-1)$-dimensional submanifold $\{f(t, \mathbf{p}) : \mathbf{p} \in B_b\}$ is perpendicular (in the metric $g$) to all the geodesics in (ii).

That is, each element of $\mathcal{F}(a, b, g)$ is a $C^\infty$ coordinate chart that is foliated by geodesics and is also foliated by codimension one submanifolds perpendicular to the geodesics. We endow $\mathcal{F}(a, b, g)$ with the relative topology induced from the $C^\infty$ compact-open topology on $C^\infty(I_a \times B_b, M)$.

The following lemma allows us to "merge" two foliations by geodesics for a Riemannian metric $g$ into a new foliation by geodesics for a small perturbation of $g$, provided the two original foliations are $C^\infty$-close.

**Lemma 2.3.** *Let $(M, g)$ be a closed $C^\infty$ Riemannian manifold of dimension $n \geq 2$, and let $\mathbb{G}$ be the set of $C^\infty$ Riemannian metrics on $M$. Suppose $\mathcal{N}$ is an open neighborhood of $g$ in $\mathbb{G}$ with the $C^\infty$ topology. Let $a, b > 0$, and let $\mathcal{F}(a, b, g)$ be as in Definition 2.2. Suppose $f_0 \in \mathcal{F}(a, b, g)$.*

*Then there exists an open neighborhood* $\mathcal{F}_0 \subseteq \mathcal{F}(a,b,g)$ *of* $f_0$ *such that for all* $f_1, f_2 \in \mathcal{F}_0$*, there exists* $\tilde{g} \in \mathcal{N}$ *such that the following conditions are satisfied.*

(1) $\tilde{g}$ *agrees with* $g$ *on the complement of* $f_1(I_{a/2} \times B_{b/2}) \cap f_2(I_{a/2} \times B_{b/2})$.

(2) *There is a family of* $\tilde{g}$*-geodesics* $\gamma_{\mathbf{p}} : I_a \to f_1(I_a \times B_b) \cup f_2(I_a \times B_b)$*, for* $\mathbf{p} \in B_{b/4}$*, such that*

$$\gamma_{\mathbf{p}}(t) = \begin{cases} f_1(t,\mathbf{p}), & \text{if } t \in (-a,-a/4); \\ f_2(t,\mathbf{p}), & \text{if } t \in (a/4,a). \end{cases}$$

(3) *If* $f_1(t,\mathbf{0}) = f_2(t,\mathbf{0})$ *for all* $t \in I_a$*, then* $\gamma_{\mathbf{0}}(t) = f_1(t,\mathbf{0})$*. This implies that the map* $t \mapsto f_1(t,\mathbf{0})$ *for* $t \in I_a$*, is a geodesic for* $\tilde{g}$ *as well.*

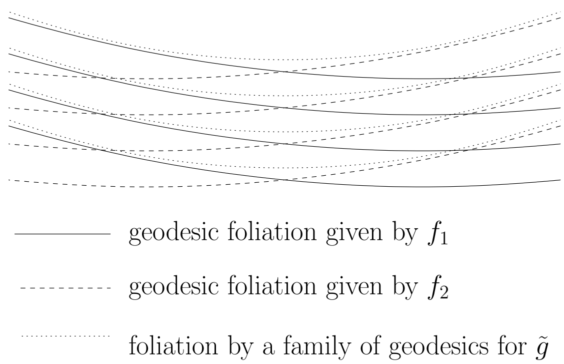


**Figure 1.** Merging of the geodesic foliations given by $f_1$ and $f_2$ into a new foliation by geodesics for $\tilde{g}$, as in Lemma 2.3.

*Proof.* Let $(a_i)_{0 \le i \le 5}$ and $(b_j)_{0 \le j \le 5}$ be strictly decreasing sequences of positive numbers, where $a_0 = a, a_3 = a/2, a_5 = a/4, b_0 = b, b_1 = b/2$, and $b_5 = b/4$. Let $R_{i,j} = I_{a_i} \times B_{b_j}$, for $0 \le i,j \le 5$.

Let $h : \mathbb{R} \to [0,1]$ be a $C^\infty$ function such that

$$h(t) = \begin{cases} 0, & \text{if } t \le -a_5; \\ 1, & \text{if } t \ge a_5, \end{cases}$$

and let $H : M \to [0,1]$ be a $C^\infty$ function such that

$$H(x) = \begin{cases} 0, & \text{if } x \in M \setminus f_0(R_{3,3}); \\ 1, & \text{if } x \in f_0(\overline{R_{4,4}}). \end{cases}$$

Given $f_0 \in \mathcal{F}(a,b,g)$, the required open neighborhood $\mathcal{F}_0$ will be chosen so that functions $f_1, f_2 \in \mathcal{F}_0$ satisfy the properties given below. We begin by requiring $f_1, f_2$ to be sufficiently close to $f_0$ in the $C^0$ topology so that

$$f_2(\overline{R_{i+1,j+1}}) \subseteq f_1(R_{i,j}) \text{ and } f_1(\overline{R_{i+1,j+1}}) \subseteq f_2(R_{i,j}), \text{ for } 0 \le i,j \le 4. \tag{2.1}$$

We define $\phi : \overline{R_{1,1}} \to R_{0,0}$ by

$$\phi(t,\mathbf{p}) = (1-h(t))(t,\mathbf{p}) + h(t)(f_1^{-1} \circ f_2(t,\mathbf{p}))$$

for $(t, \mathbf{p}) \in \overline{R_{1,1}}$, where '+' denotes the usual vector addition in $\mathbb{R}^n$. We have $\phi(t, \mathbf{p}) \in R_{0,0}$ , because $f_1^{-1} \circ f_2(\overline{R_{1,1}}) \subseteq R_{0,0}$ (by (2.1)), and $\phi(t, \mathbf{p})$ is a convex combination of $f_1^{-1} \circ f_2(t, \mathbf{p})$ and $(t, \mathbf{p})$.

Next we consider $\hat{f} := f_1 \circ \phi : \overline{R_{1,1}} \to f_1(R_{0,0})$. If $f_1$ and $f_2$ are close to $f_0$ in $C^\infty(R_{0,0}, M)$, then $\phi$ is close to the inclusion map $\overline{R_{1,1}} \hookrightarrow R_{0,0}$ in $C^\infty(\overline{R_{1,1}}, R_{0,0})$, and $\hat{f}$ is close to $f_0$ in $C^\infty(\overline{R_{1,1}}, M)$. We require $f_1$ and $f_2$ to be sufficiently close to $f_0$ in $C^\infty(R_{0,0}, M)$ so that the following four conditions are satisfied:

$$\hat{f} : \overline{R_{1,1}} \to M \text{ is a diffeomorphism onto its image,} \tag{2.2}$$

$$\hat{f}(\overline{R_{i+1,j+1}}) \subseteq f_0(R_{i,j}) \cap f_1(R_{i,j}) \cap f_2(R_{i,j}), \text{ for } 0 \le i, j \le 4, \tag{2.3}$$

$$f_0(\overline{R_{3,3}}) \subseteq \hat{f}(R_{2,2}), \text{ and} \tag{2.4}$$

$$(f_1((-a_1, -a_2] \times B_{b_2}) \cup f_2([a_2, a_1) \times B_{b_2})) \cap \hat{f}(\overline{R_{5,2}}) = \emptyset. \tag{2.5}$$

For $(t, \mathbf{p}) = (t, p_1, \ldots, p_{n-1}) \in R_{2,2}$, we define a Riemannian metric $\hat{g}$ at $\hat{f}(t, \mathbf{p}) \in \hat{f}(R_{2,2})$ by

$$\hat{g}\left(\frac{\partial \hat{f}}{\partial t}, \frac{\partial \hat{f}}{\partial t}\right) = 1, \tag{2.6}$$

$$\hat{g}\left(\frac{\partial \hat{f}}{\partial t}, \frac{\partial \hat{f}}{\partial p_k}\right) = 0, \text{ and} \tag{2.7}$$

$$\hat{g}\left(\frac{\partial \hat{f}}{\partial p_k}, \frac{\partial \hat{f}}{\partial p_l}\right) = [1 - h(t)]g\left(\frac{\partial f_1}{\partial p_k}, \frac{\partial f_1}{\partial p_l}\right) + h(t)g\left(\frac{\partial f_2}{\partial p_k}, \frac{\partial f_2}{\partial p_l}\right),$$
$$\text{for } 1 \le k, l \le n-1.$$

We know that, for $i = 0, 1, 2$, the original metric $g$ satisfies

$$g\left(\frac{\partial f_i}{\partial t}, \frac{\partial f_i}{\partial t}\right) = 1, \text{ and}$$
$$g\left(\frac{\partial f_i}{\partial t}, \frac{\partial f_i}{\partial p_k}\right) = 0, \text{ for } k = 1, \ldots, n-1,$$

in the region $f_i(R_{0,0})$.

We define the required Riemannian metric as

$$\tilde{g} = H\hat{g} + (1 - H)g,$$

where we interpret $H\hat{g}$ to be 0 when $H = 0$.

If $(t, \mathbf{p}) \in [-a_1, -a_5] \times B_{b_1}$, then $h(t) = 0$ and $\phi(t, \mathbf{p}) = (t, \mathbf{p})$; if $(t, \mathbf{p}) \in [a_5, a_1] \times B_{b_1}$, then $h(t) = 1$ and $\phi(t, \mathbf{p}) = f_1^{-1} \circ f_2(t, \mathbf{p})$. Thus

$$\hat{f}(t, \mathbf{p}) = \begin{cases} f_1(t, \mathbf{p}), & \text{if } (t, \mathbf{p}) \in [-a_1, -a_5] \times B_{b_1}; \\ f_2(t, \mathbf{p}), & \text{if } (t, \mathbf{p}) \in [a_5, a_1] \times B_{b_1}. \end{cases} \tag{2.8}$$

Therefore $\hat{g}$ agrees with $g$ on $\hat{f}(R_{2,2}\backslash R_{5,2})$. If $f_1$ and $f_2$ are close to $f_0$ in $C^\infty(R_{0,0}, M)$, then $\hat{g}$ is $C^\infty$-close to $g$ on $\hat{f}(R_{2,2}) \supseteq f_0(\overline{R_{3,3}})$. Since $\tilde{g} = g$ on $M \setminus f_0(\overline{R_{3,3}})$, we may choose $\mathcal{F}_0$ sufficiently small so that $\tilde{g} \in \mathcal{N}$ for $f_1, f_2 \in \mathcal{F}_0$.

To summarize, we have chosen $\mathcal{F}_0$ sufficiently small so that if $f_1, f_2 \in \mathcal{F}_0$, then (2.1),(2.2),(2.3),(2.4), and (2.5) hold, and $\tilde{g} \in \mathcal{N}$.

Now we verify that (1), (2), and (3) hold.

The region where $\hat{g}$ is defined and not equal to $g$ is contained in $\hat{f}(R_{5,2})$, which is a subset of $f_1(R_{3,1}) \cap f_2(R_{3,1})$, by (2.3). Therefore $\tilde{g} = g$ on the complement of $f_1(R_{3,1}) \cap f_2(R_{3,1})$, which is conclusion (1).

Since $H = 1$ on $f_0(\overline{R_{4,4}}) \supseteq \hat{f}(R_{5,5})$, we have $\tilde{g} = \hat{g}$ on $\hat{f}(R_{5,5})$. For each $\mathbf{p} \in B_{b_5}$, we define a curve $\gamma_{\mathbf{p}} : I_a \to M$ as

$$\gamma_{\mathbf{p}}(t) = \begin{cases} f_1(t, \mathbf{p}), & \text{if } t \in (-a, -a_2]; \\ \hat{f}(t, \mathbf{p}), & \text{if } t \in (-a_2, a_2); \\ f_2(t, \mathbf{p}), & \text{if } t \in [a_2, a). \end{cases}$$

It follows from (2.8) that these curves are smooth. Moreover, these curves are $\tilde{g}$-geodesics, because $\tilde{g} = g$ on $f_1((-a, -a_2] \times B_{b_5}) \cup f_2([a_2, a) \times B_{b_5})$ (by (2.5)), $\hat{g} = g = \tilde{g}$ on $\hat{f}((\overline{I_{a_2}} \setminus I_{a_5}) \times B_{b_5}) = f_1([-a_2, -a_5] \times B_{b_5}) \cup f_2([a_5, a_2] \times B_{b_5})$, $\tilde{g} = \hat{g}$ on $\hat{f}(R_{5,5})$, and the curves $t \mapsto \hat{f}(t, \mathbf{p})$ are $\hat{g}$-geodesics for all $\mathbf{p} \in B_{b_2}$ (by (2.6) and (2.7)). This proves conclusion (2). If $f_1(t, \mathbf{0}) = f_2(t, \mathbf{0})$ for $t \in I_a$, then $\phi(t, \mathbf{0}) = (t, \mathbf{0})$ and $\hat{f}(t, \mathbf{0}) = f_1(t, \mathbf{0})$ for $t \in \overline{I_{a_1}}$. Therefore the $\tilde{g}$-geodesic $\gamma_{\mathbf{0}}$ is the same as $t \mapsto f_1(t, \mathbf{0})$, which establishes (3).

□

We now define a notion of *merging* for two geodesics. This will be used in Lemma 2.6 below.

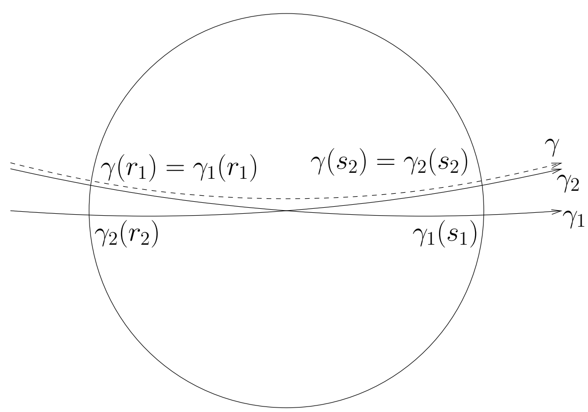


**Figure 2.** Merging of $\gamma_1$ into $\gamma_2$ within $U$, as in Definition 2.4.

**Definition 2.4.** Let $M$ be a $C^\infty$-manifold, and let $g, \tilde{g}$ be Riemannian metrics on $M$. Suppose $U$ is an open set in $M$, $t_0 \in \mathbb{R}$, and $\gamma_i : [\hat{r}_i, \hat{s}_i] \to M$, $i = 1, 2$, are $g$-geodesics such that

(2.9) $$\{t \in [\hat{r}_i, \hat{s}_i] : \gamma_i(t) \in U\} = (r_i, s_i), \text{ where } \hat{r}_i < r_i < t_0 < s_i < \hat{s}_i.$$

We say that a $\tilde{g}$-geodesic $\gamma : [\hat{r}_1, \hat{s}_2] \to M$, *merges* $\gamma_1$ *into* $\gamma_2$ *within* $U$ if there exist $\tilde{r}, \tilde{s}$ such that $r_1 < \tilde{r} < t_0 < \tilde{s} < s_2$, $\gamma((\tilde{r}, \tilde{s})) \subseteq U$, $\gamma(t) = \gamma_1(t)$ for $\hat{r}_1 \leq t \leq \tilde{r}$, and $\gamma(t) = \gamma_2(t)$ for $\tilde{s} \leq t \leq \hat{s}_2$.

In Lemma 2.6, it will be convenient to assume that the set within which the merging occurs is convex (as defined below). Definition 2.5 is stronger than the usual definition of convexity, but it follows from Theorem 3.7 and Proposition 4.2 in Chapter 3 of [7] that every point has an open neighborhood $U$ that satisfies Definition 2.5.

**Definition 2.5.** Let $(M, g)$ be a closed $C^\infty$ Riemannian manifold. We call a subset $U$ of $M$ *convex* if the following holds: for all $x, y \in U$, there is a unique geodesic from $x$ to $y$ whose image is contained in $U$, and this geodesic is length-minimizing in $M$.

If $U$ is a convex (with respect to a given metric $g$) open set in $M$, $t_0 \in \mathbb{R}$, and $\gamma_0 : (-\infty, \infty) \to M$ is a $g$-geodesic with $\gamma_0(t_0) \in U$, then there exist $\hat{r}_0, \hat{s}_0, r_0, s_0$ such that (2.9) is satisfied for $i = 0$. Moreover, $\hat{r}_0$ and $\hat{s}_0$ can be chosen so that $r_0 - \hat{r}_0 \geq \mathcal{R}$ and $\hat{s}_0 - s_0 \geq \mathcal{R}$, where $\mathcal{R}$ is the injectivity radius of $M$. That is, every geodesic that starts in $U$ must leave $U$ (in forward and backward time) and must stay outside $U$ for a time interval of length at least $\mathcal{R}$. Another useful consequence of Definition 2.5 is that if $V$ and $U$ are convex sets and $V \subset U$, then any geodesic that starts inside $V$ and then leaves $V$, cannot return to $V$ before it leaves $U$. This implies that if two geodesics $\gamma_i : [\hat{r}_i, \hat{s}_i] \to M$, $i = 1, 2$, that satisfy (2.9) with $\gamma_i(t_0) \in V$ are merged within $V$ by a $\tilde{g}$-geodesic $\gamma$, where $\tilde{g}$ agrees with $g$ on $M \setminus V$, then they are also merged within $U$.

The following lemma allows us to merge two geodesics according to Definition 2.4. K. Burns and G. Paternain have a similar result in the 2-dimensional case [6]. We also note that Lemma 2 from D. Anosov's proof of the bumpy metric theorem [1], plus the observation in [10] that Anosov's proof does not require his stated assumption that the geodesic in his Lemma 2 is closed, can be used to give an alternate proof of our Lemma 2.6.

**Lemma 2.6.** *Let $(M, g)$ be a closed $C^\infty$ Riemannian manifold of dimension $n \geq 2$, and let $\mathcal{N}$ be an open neighborhood of $g$ in the $C^\infty$ topology. Suppose $U$ is a convex (with respect to $g$) open set in $M$ and $(x_0, v_0) \in T^1U$. Then there exists an open neighborhood $\mathcal{V}$ of $(x_0, v_0)$ in $T^1U$ such that for any $(x_i, v_i) \in \mathcal{V}$, $i = 1, 2$, if $\gamma_i : [\hat{r}_i, \hat{s}_i] \to M$ are $g$-geodesics that satisfy (2.9) and $(\gamma_i(t_0), \gamma_i'(t_0)) = (x_i, v_i)$, for $i = 1, 2$, then there exists $\tilde{g} \in \mathcal{N}$ which agrees with $g$ on $M \setminus U$, and a $\tilde{g}$-geodesic $\gamma$ that merges $\gamma_1$ into $\gamma_2$ within $U$.*

*Proof.* Let $\gamma_0 : [\hat{r}_0, \hat{s}_0] \to M$ be a $g$-geodesic such that $(\gamma_0(t_0), \gamma_0'(t_0)) = (x_0, v_0)$ and (2.9) is satisfied for $i = 0$ and some choice of $r_0, s_0$. By replacing $U$ by a smaller convex open neighborhood of $x_0$, if necessary, we may assume there exist $C^\infty$ orthonormal vector fields $E_1, \ldots, E_n$ on $U$ such that $E_n(\gamma_0(t)) = \gamma_0'(t)$ for all $t \in (r_0, s_0)$. We may assume that $t_0 = 0$. Choose $T$ such that $0 < T < |r_0|$ and $\tilde{x}_0 := \gamma_0(-T)$ is not conjugate to $x_0$ along $\gamma_0|[-T, 0]$. For $u \in U$ and $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$, let

$$\Phi(u, \mathbf{z}) := z_1 E_1(u) + \cdots + z_n E_n(u) \in T_u U. \tag{2.10}$$

Define $\varphi : \{\mathbf{p} = (p_1, \ldots p_{n-1}) \in \mathbb{R}^{n-1} : |\mathbf{p}| < 1\} \to \{\mathbf{w} \in \mathbb{R}^n : |\mathbf{w}| = 1\}$ by

$$\varphi(\mathbf{p}) := (p_1, \ldots p_{n-1}, 1 - (p_1^2 + \cdots + p_{n-1}^2)^{1/2}). \tag{2.11}$$

Since $\tilde{x}_0$ and $x_0$ are not conjugate along $\gamma_0|[-T, 0]$ , there exist $\tilde{a}, \tilde{b} > 0$ such that the map

$$f_0(t, \mathbf{p}) := \exp_{\tilde{x}_0}(\Phi(\tilde{x}_0, (t+T)\varphi(\mathbf{p}))),$$

defined for $(t, \mathbf{p}) \in I_{\tilde{a}} \times B_{\tilde{b}}$, is a $C^\infty$ diffeomorphism onto its image, and its image is contained in $U$. Note that $f_0(0, \mathbf{0}) = x_0$. Moreover, there exist $a, b$ with $0 < a < \tilde{a}$, $0 < b < \tilde{b}$, an open neighborhood $\mathcal{A}$ of $Id$ in $SO(n)$, and an open neighborhood $\tilde{U}$ of $\tilde{x}_0$ in $U$ such that for $\tilde{x} \in \tilde{U}$ and $A \in \mathcal{A}$, the map

$$f(t, \mathbf{p}) := \exp_{\tilde{x}}(\Phi(\tilde{x}, (t+T)A(\varphi(\mathbf{p})))),$$

defined for $(t, \mathbf{p}) \in I_a \times B_b$ is a $C^\infty$ diffeomorphism onto its image, and its image is in $U$. Now choose $\mathcal{V}$ to be an open neighborhood of $(x_0, v_0)$ in $T^1U$ such that for each $(x, v) \in \mathcal{V}$, the geodesic $\tilde{\gamma}$ with $(\tilde{\gamma}(0), \tilde{\gamma}'(0)) = (x, v)$ satisfies $\tilde{x} := \tilde{\gamma}(-T) \in \tilde{U}$ and there exists $A \in \mathcal{A}$ with $\Phi(\tilde{x}, A(\varphi(\mathbf{0}))) = \tilde{\gamma}'(-T)$. We also require $\mathcal{V}$ to be small enough so that $\tilde{x}$ is sufficiently close to $\tilde{x}_0$ and $A$ can be chosen sufficiently close to $Id$, so that $f$ is in the neighborhood $\mathcal{F}_0$ of $f_0$ given in Lemma 2.3. (The condition (iii) in Definition 2.2 for $f_0$, as well as $f_1, f_2$ defined below, follows from the Gauss Lemma.)

Let $(x_i, v_i) \in \mathcal{V}$, $i = 1, 2$, and suppose $\gamma_i : [\hat{r}_i, \hat{s}_i] \to M$, $i = 1, 2$, are $g$-geodesics such that (2.9) is satisfied and $(\gamma_i(0), \gamma_i'(0)) = (x_i, v_i)$. Let $r_i, s_i$, $i = 1, 2$, be as in (2.9). For $i = 1, 2$, define

$$f_i(t, \mathbf{p}) := \exp_{\tilde{x}_i}(\Phi(\tilde{x}_i, (t+T)A_i(\varphi(\mathbf{p})))),$$

for $(t, \mathbf{p}) \in I_a \times B_b$, where $\tilde{x}_i := \gamma_i(-T)$, and $A_i \in \mathcal{A}$ is such that $\Phi(\tilde{x}_i, A_i(\varphi(\mathbf{0}))) = \gamma_i'(-T)$. Then $f_i(t, \mathbf{0}) = \gamma_i(t)$ for $t \in I_a$. From Lemma 2.3, we obtain $\tilde{g} \in \mathcal{N}$ which agrees with $g$ on $M \setminus U$ so that conclusion (2) of Lemma 2.3 holds. Finally, we define the required $\tilde{g}$-geodesic $\gamma : [\hat{r}_1, \hat{s}_2] \to M$ as

$$\gamma(t) := \begin{cases} \gamma_1(t), & \text{if } t \in [\hat{r}_1, -a]; \\ \gamma_{\mathbf{0}}(t), & \text{if } t \in (-a, a); \\ \gamma_2(t), & \text{if } t \in [a, \hat{s}_2], \end{cases}$$

where $\gamma_{\mathbf{0}}$ is as in Lemma 2.3(2). □

Lemma 2.7 below allows us to destroy conjugate points along a geodesic by making a small perturbation of the metric. A two-dimensional version of this lemma is contained in [6].

**Lemma 2.7.** *Let $(M, g)$ be a closed $C^\infty$ Riemannian manifold of dimension $n \geq 2$, and let $\mathcal{N}$ be an open neighborhood of $g$ in the $C^\infty$ topology. Let $x, y \in M$ and suppose $\gamma : [0, L] \to M$ is a $g$-geodesic from $x$ to $y$. Let $0 = t_0 < t_1 < \cdots < t_\ell = L$, where $\ell \geq 1$, and define $z_k := \gamma(t_k)$ for $k = 0, \ldots, \ell$. Suppose $s_0 \in (t_j, t_{j+1})$ for some $j \in \{0, \ldots, \ell - 1\}$ and $u_0 := \gamma(s_0)$ is not a self-intersection point of $\gamma$ (i.e., $u_0 \notin \gamma([0, L] \setminus \{s_0\})$). Let $U_0$ be an open neighborhood of $u_0$. Then there exists $\hat{g} \in \mathcal{N}$ that agrees with $g$ on $M \setminus U_0$ such that the following conditions hold:*

(1) *$\gamma$ is also a unit speed geodesic for $\hat{g}$.*

(2) *If* $k_1$ *and* $k_2$ *are integers such that* $0 \le k_1 \le j$ *and* $j+1 \le k_2 \le \ell$*, then* $z_{k_1}$ *is not conjugate to* $z_{k_2}$ *along* $\gamma|[t_{k_1}, t_{k_2}]$ *in the* $\hat{g}$ *metric.*

*Proof.* It suffices to prove the lemma for the case $\ell = 1$ and $0 = t_0 < s_0 < t_1 = L$, because we can then obtain (2) in the general case through a finite sequence of perturbations of the metric (within $\mathcal{N}$) corresponding to each possible pair $(k_1, k_2)$ with $0 \le k_1 \le j$ and $j+1 \le k_2 \le \ell$. Each successive perturbation adds one more pair $(k_1, k_2)$ such that $z_{k_1}$ is not conjugate to $z_{k_2}$ along $\gamma|[t_{k_1}, t_{k_2}]$, and the perturbations can be taken small so that no new conjugacies are introduced between such pairs of points.

We now assume $\ell = 1$ and $0 = t_0 < s_0 < t_1 = L$ . By perturbing $s_0$ slightly, if necessary, we may assume that $x$ is not conjugate to $u_0$ along $\gamma|[0, s_0]$. We may also assume that the open neighborhood $U_0$ of $u_0$ is chosen so that $\{t \in [0, L] : \gamma(t) \in U_0\} = (s_0 - \eta, s_0 + \eta)$ for some $\eta$ with $0 < \eta < \min(s_0, L - s_0)$. Let $U$ be an open neighborhood of $x$ disjoint from $U_0$. Suppose $\tau \in (0, s_0 - \eta)$ is such that $\gamma|[0, \tau]$ is one-to-one, and whenever $0 < t \le \tau$, $x$ is not conjugate to $\gamma(t)$ along $\gamma|[0, t]$, and $\gamma(t)$ is not conjugate to $y$ along $\gamma|[t, L]$. Let $E_1, \ldots, E_n$ be $C^\infty$ vector fields along $\gamma|[0, \tau]$ with $\gamma'(t) = E_n(\gamma(t))$ for $t \in [0, \tau]$. Let $\Phi$ and $\varphi$ be as in (2.10) and (2.11) for $u \in \gamma([0, \tau])$. Since $x$ is not conjugate to $u_0$ along $\gamma|[0, s_0]$, there exist $\tilde{a}, \tilde{b} > 0$ such that the map

$$f_1(t, \mathbf{p}) := \exp_{x,g}(\Phi(x, (t + s_0)\varphi(\mathbf{p})),$$

defined for $(t, \mathbf{p}) \in I_{\tilde{a}} \times B_{\tilde{b}}$, is a $C^\infty$ diffeomorphism onto its image, and its image is in $U_0$. (The '$g$' in the subscript indicates we are referring to the exponential map for the metric $g$. ) There exist $a, b, \tilde{\delta}$ with $0 < a < \tilde{a}$, $0 < b < \tilde{b}$, $0 < \tilde{\delta} < \tau$, such that the map

$$f_2(t, \mathbf{p}) := \exp_{\tilde{x},g}(\Phi(\tilde{x}, (t + s_0 - \delta)\varphi(\mathbf{p})),$$

defined for $(t, \mathbf{p}) \in I_a \times B_b$ is a $C^\infty$ diffeomorphism onto its image, and its image is in $U_0$ for any $\tilde{x} := \gamma(\delta)$ with $0 < \delta < \tilde{\delta}$. Let $f_0$ be the restriction of $f_1$ to $I_a \times B_b$, and let $\mathcal{F}_0$ be as in Lemma 2.3. We choose $\delta$ sufficiently small so that $f_2 \in \mathcal{F}_0$. Since $f_1(I_{a/2} \times B_{b/2}) \cap f_2(I_{a/2} \times B_{b/2})$ is a subset of $U_0$, Lemma 2.3 implies that there is a $\hat{g} \in \mathcal{N}$ which agrees with $g$ on $M \setminus U_0$ and Lemma 2.3(2) holds with $\tilde{g}$ replaced by $\hat{g}$. We also obtain Lemma 2.3(3) with $\tilde{g}$ replaced by $\hat{g}$, because $f_1(t, \mathbf{0}) = f_2(t, \mathbf{0})$ for $t \in I_a$. Therefore $\gamma$ is also a geodesic for $\hat{g}$. For $\mathbf{p} \in B_{b/4}$, let $\gamma_{\mathbf{p}}$ be as in Lemma 2.3(2) and define $\sigma_{\mathbf{p}} : [0, L] \to M$ by

$$\sigma_{\mathbf{p}}(t) := \begin{cases} \exp_{x,g}(\Phi(x, t\varphi(\mathbf{p})), & \text{if } t \in [0, s_0 - a]; \\ \gamma_{\mathbf{p}}(t - s_0), & \text{if } t \in (s_0 - a, s_0 + a); \\ \exp_{\tilde{x},g}(\Phi(\tilde{x}, (t - \delta)\varphi(\mathbf{p})), & \text{if } t \in [s_0 + a, L]. \end{cases} \tag{2.12}$$

Then $\sigma_{\mathbf{p}}$ is a $\hat{g}$-geodesic that merges, within $U_0$, a $g$-geodesic originating at $x$ with initial velocity $\Phi(x, \varphi(\mathbf{p}))$ into a $g$-geodesic that is at $\tilde{x}$ with velocity $\Phi(\tilde{x}, \varphi(\mathbf{p}))$ at time $\delta$. Thus, for $\mathbf{p} \in B_{b/4}$,

$$\exp_{x,\hat{g}}(\Phi(x, t\varphi(\mathbf{p}))) = \exp_{\tilde{x},g}(\Phi(x, (t - \delta)\varphi(\mathbf{p})) \tag{2.13}$$

for $s_0 + a \le t \le L$. Since $\tilde{x}$ is not conjugate to $y$ along $\gamma|[\delta, L]$ in the metric $g$, $\exp_{\tilde{x},g}$ is locally a diffeomorphism near $(L - \delta)\gamma'(\delta)$. By (2.13), this implies that $\exp_{x,\hat{g}}$ is locally a diffeomorphism near $L\gamma'(0)$. Therefore $x$ is not conjugate to $y$ along $\gamma$ in the $\hat{g}$ metric. □

A *geodesic lasso* is defined to be a closed curve $\gamma : [0, L] \to M$ which is a geodesic, but $\gamma'(0) \neq \gamma'(L)$. The following Lemma 2.8 allows us to perturb a geodesic so that it avoids a finite set of points on $M$, and it also allows us to change a closed geodesic to a geodesic lasso.

**Lemma 2.8.** *Let $(M, g)$ be a closed $C^\infty$ Riemannian manifold of dimension at least two, and let $\mathcal{N}$ be an open neighborhood of $g$ in the $C^\infty$ topology. Let $x, y \in M$ and suppose $\gamma : [0, L] \to M$ is a $g$-geodesic from $x$ to $y$. Let $Z$ be a finite set of points in $M$ such that $x, y \in Z$. Let $\{t \in [0, L] : \gamma(t) \in Z\} = \{t_k : k = 0, \ldots, \ell\}$, where $0 = t_0 < \cdots < t_\ell = L$, $\ell \geq 1$, and define $z_k := \gamma(t_k)$, for $k = 0, \ldots \ell$. Assume that*

(i) *$x$ is not conjugate to $z_k$ along $\gamma|[0, t_k]$, for $k = 1, \ldots, \ell$.*
(ii) *$z_k$ is not conjugate to $y$ along $\gamma|[t_k, L]$, for $k = 0, \ldots, \ell - 1$.*

*Suppose $s_0 \in (0, L)$, $u_0 := \gamma(s_0)$ is not a self-intersection point of $\gamma$, and $u_0 \notin Z$. Let $U_0$ be an open neighborhood of $u_0$. Then there exist open neighborhoods $W_1$ and $W_2$ of $\gamma'(0)$ and $\gamma'(L)$ in $T^1_x M$ and $T^1_y M$ , respectively, such that for any $w_1 \in W_1 \setminus \{\gamma'(0)\}$ and any $w_2 \in W_2 \setminus \{\gamma'(L)\}$, there exists $\tilde{g} \in \mathcal{N}$ that agrees with $g$ on $M \setminus U_0$ and a $\tilde{g}$-geodesic $\tilde{\gamma} : [0, L] \to M$ from $x$ to $y$ such that $\tilde{\gamma}'(0) = w_1$, $\tilde{\gamma}'(L) = w_2$, $\tilde{\gamma}((0, L)) \cap Z = \emptyset$, and $x$ is not conjugate to $y$ along $\tilde{\gamma}$ for $\tilde{g}$.*

*Proof.* We may assume that $Z \subset \gamma([0, L])$. By replacing $U_0$ by a smaller open neighborhood of $u_0$ if necessary, we may assume that $U_0$ is convex for $g$, $U_0 \cap Z = \emptyset$, and $\{t \in [0, L] : \gamma(t) \in U_0\} = (s_0 - \eta, s_0 + \eta)$, for some $\eta > 0$.

Since $x$ is not conjugate to $z_k$ along $\gamma|[0, t_k]$ for $k = 1, \ldots, \ell$, and $\exp_{x,g}$ is locally a diffeomorphism near $0 \in T_x M$, there exist open neighborhoods $V_k$ of $t_k\gamma'(0)$ in $T_x M$, for $k = 0, \ldots, \ell$, such that the maps $\exp_{x,g} : V_k \to M$ are diffeomorphisms onto their images. Also,

$$Z \cap \exp_{x,g}(\{t\gamma'(0) : t \in [0, L]\} \setminus (V_0 \cup \cdots \cup V_\ell)) = \emptyset, \tag{2.14}$$

because $(\exp^{-1}_{x,g} Z) \cap \{t\gamma'(0) : t \in [0, L]\} = \{t_0\gamma'(0), \ldots, t_\ell\gamma'(0)\}$. By the continuity of $\exp_{x,g}$, we can choose $W_1$ sufficiently small so that (2.14) still holds for $\gamma$ replaced by any $g$-geodesic $\gamma_1 : [0, L] \to M$ with $\gamma_1(0) = x$ and $\gamma_1'(0) \in W_1$. If $\gamma_1'(0) \in W_1 \setminus \{\gamma'(0)\}$, then $\{t\gamma_1'(0) : t \in (0, L]\} \cap (V_0 \cup \cdots \cup V_\ell)$ does not contain any of $t_k\gamma'(0)$, $k = 0, \ldots, \ell$. Thus, (2.14) for $\gamma_1$ implies that $\gamma_1((0, L]) \cap Z = \emptyset$. Similarly, if $W_2$ is sufficiently small, then for any $g$-geodesic $\gamma_2 : [0, L] \to M$ with $\gamma_2(L) = y$ and $\gamma_2'(L) \in W_2 \setminus \gamma'(L)$ , we have $\gamma_2([0, L)) \cap Z = \emptyset$.

Let $v_0 = \gamma'(s_0)$ and let $\mathcal{V}$ be an open neighborhood of $(u_0, v_0)$ in $T^1 U_0$ satisfying the conclusion of Lemma 2.6 (with $U$ replaced by $U_0$ and $x_0$ replaced by $u_0$). In addition to the requirements of the preceding paragraph, we require $W_1$ and $W_2$ to be sufficiently small so that if $\gamma_i : [0, L] \to M$, $i = 1, 2$, are such that $\gamma_1(0) = x$, $\gamma_1'(0) \in W_1$, $\gamma_2(L) = y$, and $\gamma_2'(L) \in W_2$, then there exist $r_i, s_i$ with $0 < r_i < s_0 < s_i < L$, such that $\{t \in [0, L] : \gamma_i(t) \in U_0\} = (r_i, s_i)$ and $(\gamma_i(s_0), \gamma_i'(s_0)) \in \mathcal{V}$.

Suppose $w_1 \in W_1 \setminus \{\gamma'(0)\}$ and $w_2 \in W_2 \setminus \{\gamma'(L)\}$, and let $\gamma_i : [0, L] \to M$, $i = 1, 2$, be $g$-geodesics such that $\gamma_1(0) = x$, $\gamma_1'(0) = w_1$, $\gamma_2(L) = y$, and $\gamma_2'(L) = w_2$. By Lemma 2.6, there exists a metric $\tilde{g} \in \mathcal{N}$ that agrees with $g$ on $M \setminus U_0$ and a

$\tilde{g}$-geodesic $\tilde{\gamma} : [0, L] \to M$ that merges $\gamma_1$ into $\gamma_2$ within $U_0$. Since $U_0 \cap Z = \emptyset$ and $\gamma_1((0, L)) \cap Z = \emptyset = \gamma_2((0, L)) \cap Z$, we have $\tilde{\gamma}((0, L)) \cap Z = \emptyset$. By Lemma 2.7 we can make a small additional perturbation of the metric $\tilde{g}$ within $U_0$, if necessary, to arrange for $x$ and $y$ to not be conjugate along $\tilde{\gamma}$. $\square$

The following lemma will be used to obtain strong insecurity in Theorem 1.1 in the case of a manifold of dimension at least three.

**Lemma 2.9.** *Let $(M, g)$ be a closed $C^\infty$ Riemannian manifold of dimension at least three, and let $\mathcal{N}$ be an open neighborhood of $g$ in the $C^\infty$ topology. Let $x, y \in M$ and suppose $\gamma : [0, L] \to M$ is a $g$-geodesic from $x$ to $y$. Let $Z$ be the union of the images of finitely many $C^1$ curves from compact intervals to $M$ such that $x, y \in Z$. Suppose that $\{t \in [0, L] : \gamma(t) \in Z\}$ is finite. Define $\ell$ , $t_k$, and $z_k$ as in Lemma 2.8, and assume (i) and (ii) from Lemma 2.8. Assume that there exist points in the image of $\gamma$ that are not self-intersection points of $\gamma$. Then there exist open neighborhoods $W_1$ and $W_2$ of $\gamma'(0)$ and $\gamma'(L)$ in $T^1_x M$ and $T^1_y M$, respectively, and dense open subsets $\hat{W}_i$ of $W_i$, $i = 1, 2$, such that for any $w_i \in \hat{W}_i$, there exist $\hat{g} \in \mathcal{N}$ that agrees with $g$ on an open set containing $Z$, and a $\hat{g}-$geodesic $\hat{\gamma} : [0, L] \to M$ from $x$ to $y$ such that $\hat{\gamma}'(0) = w_1$, $\hat{\gamma}'(L) = w_2$, $\hat{\gamma}((0, L)) \cap Z = \emptyset$, and $x$ is not conjugate to $y$ along $\hat{\gamma}$ for $\hat{g}$. In addition, $\hat{g}$ and $\hat{\gamma}$ may be chosen such that the following two conditions hold.*

(1) If $x \neq y$, then $\hat{\gamma}$ has no self-intersection points.
(2) If $x = y$, then $x \notin \hat{\gamma}((0, L))$, and $\hat{\gamma}$ has no self-intersection points except $x$.

*Proof.* The proof is similar to the proof of Lemma 2.8. We indicate the modifications that are needed. As in the proof of Lemma 2.8, $x$ not being conjugate to $y$ can be arranged at the end. Thus it is enough to obtain $\hat{\gamma}$ and $\hat{g}$ satisfying the other conditions in the conclusion of the present lemma.

We no longer assume that $Z \subset \gamma([0, L])$. We choose $s_0$, $u_0$, and $U_0$ and $V_k$, $k = 0, \dots \ell$, as in Lemma 2.8 and its proof. We also require the closure of $U_0$ to be disjoint from $Z$. We again have (2.14) and we can choose an open neighborhood $W_1$ of $\gamma'(0)$ in $T^1_x M$ sufficiently small so that (2.14) still holds for $\gamma$ replaced by any $g$-geodesic $\gamma_1 : [0, L] \to M$ with $\gamma_1(0) = x$ and $\gamma_1'(0) \in W_1$. Now let $\hat{Z} = \{w_1 \in W_1 : w_1 = v/||v||,$ for $v \in (\exp^{-1}_{x,g} Z) \cap (V_0 \cup \cdots \cup V_\ell)\}$ and let $\hat{W} = W_1 \setminus \hat{Z}$. (Here $|| \cdot ||$ denotes the norm with respect to the metric $g$.) Then $\hat{Z}$ is relatively closed in $W_1$, and $\hat{Z}$ is the union of at most countably many $C^1$ curves. Since $T^1_x M$ is at least two dimensional, $\hat{W}_1$ is dense in $W_1$. If $\gamma_1 : [0, L] \to M$ is a $g$-geodesic with $\gamma_1'(0) \in \hat{W}_1$, then $\gamma_1((0, L]) \cap Z = \emptyset$. Similarly, there is an open neighbhorhood $W_2$ of $\gamma'(L)$ in $T^1_y M$ and a dense open subset $\hat{W}_2$ of $W_2$ such that for any $g$-geodesic $\gamma_2 : [0, L] \to M$ with $\gamma_2(L) = y$ and $\gamma_2'(L) \in \hat{W}_2$, we have $\gamma_2([0, L)) \cap Z = \emptyset$.

We now apply Lemma 2.6 as in the third and fourth paragraphs in the proof of Lemma 2.8 , except we take $w_i \in \hat{W}_i$, $i = 1, 2$, instead of $w_1 \in W_1 \setminus \{\gamma'(0)\}$ and $w_2 \in W_2 \setminus \gamma'(L)$. As in the proof of Lemma 2.8, we obtain a metric $\tilde{g} \in \mathcal{N}$ that agrees with $g$ on $M \setminus U_0$ and a $\tilde{g}-$geodesic $\tilde{\gamma} : [0, L] \to M$ from $x$ to $y$ such that $\tilde{\gamma}((0, L)) \cap Z = \emptyset$.

Since $x, y \in Z$, $\tilde{\gamma}((0, L)) \cap \{x, y\} = \emptyset$. Thus the set $\mathcal{S}$ of self-intersection points of $\tilde{\gamma}$ is finite. We will modify $\tilde{\gamma}$ to eliminate self-intersection points. If $\{s \in (0, L) : \tilde{\gamma}(s) \in$

$\mathcal{S}\} = \emptyset$ , then either $\mathcal{S} = \{x\}$ (in the case $x = y$) or $\mathcal{S} = \emptyset$ (in the case $x \neq y$), and in both of these cases, $\tilde{\gamma}$ already satisfies the conditions required of $\hat{\gamma}$. Hence we may assume $\{s \in (0, L) : \tilde{\gamma}(s) \in \mathcal{S}\} = \{s_k : k = 1, \dots, m\}$, where $0 < s_1 < \cdots < s_m < L$ and $m \geq 1$. Choose $\tilde{s}_i$, $i = 1, 2$, such that $0 < \tilde{s}_1 < s_1 < \tilde{s}_2 < s_2$ and $\tilde{\gamma}(\tilde{s}_1)$ is not $\tilde{g}$-conjugate to $\tilde{\gamma}(\tilde{s}_2)$ along $\tilde{\gamma}|[\tilde{s}_1, \tilde{s}_2]$. We will show that there exist a metric $\tilde{g}_1 \in \mathcal{N}$ that agrees with $g$ on an open set containing $Z$ and a $\tilde{g}_1$-geodesic $\tilde{\gamma}_1 : [0, L] \to M$ from $x$ to $y$ such that $\tilde{\gamma}_1$ agrees with $\tilde{\gamma}$ on $[0, \epsilon] \cup [L - \epsilon, L]$, for some positive $\epsilon$, $\tilde{\gamma}_1((0, L)) \cap Z = \emptyset$, and we have

(2.15) $\{s \in (0, L) : \tilde{\gamma}_1(s)$ is a self-intersection point of $\tilde{\gamma}_1$ on $[0, L]\} \subset \{s_2, \dots, s_m\}$.

By applying this procedure at most $m$ times, we obtain a metric $\hat{g} \in \mathcal{N}$ that agrees with $g$ on an open set containing $Z$ and a $\hat{g}$-geodesic $\hat{\gamma}$ from $x$ to $y$ satisfying all of the conditions in the lemma.

For $i = 1, 2$, let $U_i$ be an open neighborhood of $\tilde{\gamma}(\tilde{s}_i)$ whose closure does not intersect $Z$. Since $\tilde{\gamma}(\tilde{s}_1)$ and $\tilde{\gamma}(\tilde{s}_2)$ are not self-intersection points of $\tilde{\gamma}$, we may assume that $U_1$ and $U_2$ are chosen so that $\{s \in [0, L] : \tilde{\gamma}(s) \in U_i\} = (\tilde{s}_i - \tau_i, \tilde{s}_i + \eta_i)$, for some $\tau_i, \eta_i > 0$, where $\{0, s_1, s_2, \dots, s_m, L\} \cap (\tilde{s}_i - \tau_i, \tilde{s}_i + \eta_i) = \emptyset$ for $i = 1, 2$. We also choose a point $\tilde{u}_0 \in \tilde{\gamma}((\tilde{s}_1, s_1))$ and a convex open neighborhood $\tilde{U}_0$ of $\tilde{u}_0$ whose closure does not intersect $Z \cup \tilde{\gamma}([0, \tilde{s}_1] \cup [\tilde{s}_2, L])$. We may assume that $\tilde{U}_0$, $U_1$, and $U_2$ are pairwise disjoint.

By applying the same procedure as in the above construction of $\tilde{\gamma}$, we can find a metric $\tilde{g}_0 \in \mathcal{N}$ that agrees with $\tilde{g}$ outside $\tilde{U}_0$, and a $\tilde{g}_0$-geodesic $\sigma : [\tilde{s}_1, \tilde{s}_2] \to M$ from $\tilde{\gamma}(\tilde{s}_1)$ to $\tilde{\gamma}(\tilde{s}_2)$ such that $\sigma((\tilde{s}_1, \tilde{s}_2)) \cap \tilde{\gamma}([0, \tilde{s}_1] \cup [\tilde{s}_2, L]) = \emptyset$. Moreover, $\tilde{g}_0$ and $\sigma$ can be chosen so that $\sigma$ is as close as we like to $\tilde{\gamma}|[\tilde{s}_1, \tilde{s}_2]$ in the $C^0$ topology. In particular, we may assume that $\sigma([\tilde{s}_1, \tilde{s}_2]) \cap Z = \emptyset$. We then use Lemma 2.6 to smooth out the broken geodesic that is equal to $\tilde{\gamma}$ on $[0, \tilde{s}_1] \cup [\tilde{s}_2, L]$ and is equal $\sigma$ on $[\tilde{s}_1, \tilde{s}_2]$. That is, we merge $\tilde{\gamma}$ into $\sigma$ within $U_1$, and we merge $\sigma$ back into $\tilde{\gamma}$ within $U_2$, letting $t_0$ in Definition 2.4 be $\tilde{s}_1$ and $\tilde{s}_2$, respectively. The merging procedure results in a new metric $\tilde{g}_1 \in \mathcal{N}$ that agrees with $\tilde{g}_0$ on $M \setminus (U_1 \cup U_2)$ (and therefore agrees with $g$ on $M \setminus (U_0 \cup \tilde{U}_0 \cup U_1 \cup U_2)$), and a $\tilde{g}_1$-geodesic $\tilde{\gamma}_1 : [0, L] \to M$ such that $\tilde{\gamma}_1$ agrees with $\tilde{\gamma}$ on $[0, \epsilon] \cup [L - \epsilon, L]$, for some positive $\epsilon$, and $\tilde{\gamma}_1((0, L)) \cap Z = \emptyset$. It follows from the proof of Lemma 2.6 that $\tilde{\gamma}_1$ can be constructed so that it has no self-intersections in $U_1 \cup U_2$. Thus (2.15) holds. □

## 3. Proof of Theorem 1.1

We now use the results of Section 2 to prove Theorem 1.1. The notation $\operatorname{tr}(\gamma)$ will mean the trace of a curve $\gamma : I \to M$, i.e., $\operatorname{tr}(\gamma) = \{\gamma(t) : t \in I\}$.

*Proof.* Let $(x, y, g) \in M \times M \times \mathbb{G}$, and let $n \in \mathbb{N}$. We consider the statement $S(x, y, n, g)$ : there exist $g$-geodesics $\gamma_i : [0, L_i] \to M$ from $x$ to $y$, $i = 1, \dots, n$, which satisfy the following four properties:

(i) If $x \neq y$, then the tangent vectors

$$\gamma_1'(0), \gamma_2'(0), \dots, \gamma_n'(0)$$

at $x$ are pairwise linearly independent, and the tangent vectors

$$\gamma_1'(L_1), \gamma_2'(L_2), \dots, \gamma_n'(L_n)$$

at $y$ are pairwise linearly independent. If $x = y$, then the tangent vectors

$$\gamma_1'(0), \gamma_1'(L_1), \gamma_2'(0), \gamma_2'(L_2), \ldots, \gamma_n'(0), \gamma_n'(L_n)$$

are pairwise linearly independent. Thus we cannot join $\gamma_i$ to $\gamma_j$ smoothly at $x$ or at $y$, for any $i, j \in \{1, \ldots, n\}$.

(ii) For each $i = 1, \ldots, n$, we have $\gamma_i((0, L_i)) \cap \{x, y\} = \emptyset$. That is, $\gamma_i$ meets $x$ and $y$ only at its endpoints.

(iii) Any three of $\gamma_1, \ldots, \gamma_n$ are concurrent only at $x$ and at $y$.

(iv) The point $x$ is not conjugate to $y$ in the metric $g$ along $\gamma_i|[0, L_i]$, for $i = 1, \ldots, n$.

We define $\mathcal{H}_n(x, y) := \{g \in \mathbb{G} : S(x, y, n, g) \text{ is satisfied}\}$. We make the following claim:

*Claim* 3.1. $(a)$ $\mathcal{H}_n(x, y)$ is $C^\infty$-dense in $\mathbb{G}$ and $(b)$ there is a $C^1$-open neighborhood $\mathcal{G}_n(x, y)$ of $\mathcal{H}_n(x, y)$ in $\mathbb{G}$ such that parts (i), (ii), and (iii) of $S(x, y, n, g)$ are satisfied for all $g \in \mathcal{G}_n(x, y)$.

Claim 3.1 implies that the set $\bigcap \mathcal{G}_n(x, y)$ is the intersection of a countable collection of sets that are $C^1$ open and $C^\infty$ dense in $\mathbb{G}$. Suppose $P \subseteq M \setminus \{x, y\}$ is a set with $m$ points, and $g \in \bigcap \mathcal{G}_n(x, y)$. Since $g \in \mathcal{G}_{2m+1}$, we can find $2m + 1$ $g$-geodesics that satisfy (iii). If $P$ were a blocking set for $(x, y)$, then by the pigeonhole principle, at least three of these geodesics would pass through the same point in $P$, which leads to a contradiction. Hence there is no finite blocking set for $(x, y)$, and Theorem 1.1(1) follows from Claim 3.1.

Similarly, if we define $\widetilde{\mathcal{H}}_n := \{(x, y, g) \in M \times M \times \mathbb{G} : S(x, y, n, g) \text{ is satisfied}\}$ and $\widehat{\mathcal{H}}_n := \{(x, g) \in M \times \mathbb{G} : S(x, x, n, g)\} \text{ is satisfied}\}$, and we prove the following claims, then Theorem 1.1(2),(3) will follow by considering $\bigcap \widetilde{\mathcal{G}}_n$ and $\bigcap \widehat{\mathcal{G}}_n$, respectively.

*Claim* 3.2. $(a)$ $\widetilde{\mathcal{H}}_n$ is $C^\infty$-dense in $M \times M \times \mathbb{G}$ and $(b)$ there is a $C^1$-open neighborhood $\widetilde{\mathcal{G}}_n$ of $\widetilde{\mathcal{H}}_n$ in $M \times M \times \mathbb{G}$ such that (i), (ii), and (iii) of $S(x, y, n, g)$ are satisfied for all $(x, y, g) \in \widetilde{\mathcal{H}}_n$.

*Claim* 3.3. $(a)$ $\widehat{\mathcal{H}}_n$ is $C^\infty$-dense in $M \times \mathbb{G}$ and $(b)$ there is a $C^1$-open neighborhood $\widehat{\mathcal{G}}_n$ of $\widehat{\mathcal{H}}_n$ in $M \times \mathbb{G}$ such that (i), (ii), and (iii) of $S(x, x, n, g)$ are satisfied for all $(x, g) \in \widehat{\mathcal{H}}_n$.

We now prove Claim 3.1$(a)$ by mathematical induction. For $n = 1$, let $\mathcal{N}$ be any non-empty $C^\infty$-open set in $\mathbb{G}$, and let $g \in \mathcal{N}$. Let $\gamma : [0, L] \to M$ be a $g$-geodesic from $x$ to $y$. By restricting the domain of $\gamma$, if necessary, we may assume that $\gamma((0, L)) \cap \{x, y\} = \emptyset$. Then we let $\ell = 1$ and $0 = t_0 < t_1 = L$ in Lemma 2.7. By Lemma 2.7, there exists $\hat{g} \in \mathcal{N}$ such that $\gamma$ is also a unit speed geodesic for $\hat{g}$ and $x$ is not conjugate to $y$ along $\gamma$. If $(x, \gamma'(0)) \neq (y, \gamma'(L))$, then we let $g_1 = \hat{g}$, and $\gamma_1 = \gamma$. If $(x, \gamma'(0)) = (y, \gamma'(L))$, that is, $\gamma$ is a closed geodesic, then we apply Lemma 2.8 to obtain $g_1 \in \mathcal{N}$ and a $g_1$-geodesic lasso $\gamma_1 : [0, L] \to M$ with $\gamma_1(0) = \gamma_1(L) = x$ but $\gamma_1'(0) \neq \gamma_1'(L)$, and $x \notin \gamma_1((0, L))$. Then (i) , (ii), and (iv) are satisfied, and (iii) is vacuous. Since $\mathcal{N}$ is arbitrary, $\mathcal{H}_1(x, y)$ is $C^\infty$-dense.

Next we suppose $\mathcal{H}_{n-1}(x, y)$ is $C^\infty$-dense for some $n \geq 2$, and we will prove that $\mathcal{H}_n(x, y)$ is $C^\infty$-dense. Let $\mathcal{N}$ be any non-empty $C^\infty$-open set in $\mathbb{G}$. There exist $g_{n-1} \in \mathcal{H}_{n-1}(x,y) \cap \mathcal{N}$ and $g_{n-1}$-geodesics $\gamma_i : [0, L_i] \to M$ from $x$ to $y$, $i = 1, \dots, n-1$, so that properties (i) - (iv) are satisfied with $n$ replaced by $n-1$. By Theorem 2.1, there exists a $g_{n-1}$-geodesic $\gamma : [0, L] \to M$ from $x$ to $y$, distinct from $\gamma_1, \dots, \gamma_{n-1}$. If $x = y$, we also require $\gamma$ to be distinct from $-\gamma_1, \dots, -\gamma_{n-1}$, where $-\gamma_i$ is $\gamma_i$ traversed in the opposite direction.

By (i) and (ii), we have $\mathrm{tr}(\gamma) \nsubseteq \mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})$. However, it may happen that $\mathrm{tr}(\gamma)$ contains one (or more) of the sets $\mathrm{tr}(\gamma_1), \dots, \mathrm{tr}(\gamma_{n-1})$. If $x = y$, then we can restrict the domain of $\gamma$, if necessary, so that $\mathrm{tr}(\gamma)$ does not contain any of the sets $\mathrm{tr}(\gamma_1), \dots, \mathrm{tr}(\gamma_{n-1})$. If $x \neq y$, then we can restrict the domain of $\gamma$, if necessary, to obtain a $g_{n-1}$-geodesic from $x$ to $y$ such that one of the following happens: (a) $\mathrm{tr}(\gamma)$ does not contain any of the sets $\mathrm{tr}(\gamma_1), \dots, \mathrm{tr}(\gamma_{n-1})$; (b) $\gamma$ consists of one of $\gamma_1, \dots, \gamma_{n-1}$ preceded by a $g_{n-1}$-geodesic from $x$ to $x$; (c) $\gamma$ consists of one of $\gamma_1, \dots, \gamma_{n-1}$ followed by a $g_{n-1}$-geodesic from $y$ to $y$. If (a) holds, then we assume that $\mathrm{tr}(\gamma)$ does not contain any of the sets $\mathrm{tr}(\gamma_1), \dots, \mathrm{tr}(\gamma_{n-1})$, and the rest of this paragraph can be skipped. So assume that one of cases (b) or (c) hold, and assume that the domain of $\gamma$ has been restricted so that cases (b) and (c) do not hold for any further restriction to a proper closed subinterval of the domain. Let $u_0 \in \mathrm{tr}(\gamma) \setminus [\mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})]$ be such that $u_0$ is not a self-intersection point of $\gamma$, and let $U_0$ be an open neighborhood of $u_0$ such that $U_0 \cap [\mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})] = \emptyset$. By Lemma 2.7, we can make a perturbation of the $g_{n-1}$ metric within $U_0$ such that $\gamma$ remains a geodesic, the new metric is in $\mathcal{N}$, and neither of $x$ or $y$ is conjugate to either of $x$ or $y$ along an arc of $\gamma$. Then Lemma 2.8 applies with $Z = \{x, y\}$. Thus we may again perturb the metric within $U_0$ to produce a new metric $\hat{g} \in \mathcal{N}$ and a $\hat{g}$-geodesic $\hat{\gamma}$ close to $\gamma$ and different from $\gamma_1, \dots, \gamma_{n-1}$, such that $\hat{\gamma}$ meets $x$ and $y$ only at its endpoints. In particular, $\mathrm{tr}(\gamma)$ does not contain any of the sets $\mathrm{tr}(\gamma_1), \dots, \mathrm{tr}(\gamma_{n-1})$. Since $U_0 \cap [\mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})] = \emptyset$, $\gamma_1, \dots, \gamma_{n-1}$ remain geodesics for $\hat{g}$.

From the preceding paragraph, we have a metric $\hat{g} \in \mathcal{N}$ and a $\hat{g}$-geodesic $\hat{\gamma} : [0, L] \to M$ from $x$ to $y$ such that $\gamma_1, \dots, \gamma_{n-1}$ are $\hat{g}$-geodesics and $\mathrm{tr}(\hat{\gamma})$ does not contain any of the sets $\mathrm{tr}(\gamma_1), \dots, \mathrm{tr}(\gamma_{n-1})$. Then $\mathrm{tr}(\hat{\gamma}) \cap [\mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})]$ is a finite set. If $n = 2$, let $Z = \{x, y\}$; if $n > 2$, let $Z$ be the collection of all intersection points between the trace of any two of $\gamma_1, \dots, \gamma_{n-1}$. From (i) and (ii), we know that $Z$ is a finite set. We also have $x, y \in Z$. We want to perturb $\hat{\gamma}$ so that it does not meet $Z$ except at its endpoints. Let $\hat{\gamma}^{-1}(Z) \cap [0, L] = \{t_0, \dots, t_l\}$, where $0 = t_0 < \cdots < t_\ell = L$, and denote $z_k := \hat{\gamma}(t_k)$, for $k = 0, \dots, \ell$. Let $s_1 \in (t_0, t_1)$ , $s_2 \in (t_{\ell-1}, t_\ell)$, $s_1 < s_2$, $u_1 := \hat{\gamma}(s_1)$, $u_2 := \hat{\gamma}(s_2)$ be such that $u_1, u_2 \notin \mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})$ and $u_1, u_2$ are not self-intersection points of $\hat{\gamma}$. We can apply Lemma 2.7 twice with $s_0 = s_i$ and $U_0 = U_i$ for $i = 1, 2$, where $(U_1 \cup U_2) \cap [\mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})] = \emptyset$. Thus we obtain a metric $\bar{g} \in \mathcal{N}$ such that $\gamma_1, \dots, \gamma_{n-1}$ are $\bar{g}$-geodesics, and conditions (i) and (ii) in Lemma 2.8 hold for $g$ replaced by $\bar{g}$ and $\gamma$ replaced by $\hat{\gamma}$. Hence, by Lemma 2.8, there is a metric $\tilde{g} \in \mathcal{N}$ such that $\gamma_1, \dots, \gamma_{n-1}$ are $\tilde{g}$-geodesics, and there is a $\tilde{g}$-geodesic $\tilde{\gamma}$ from $x$ to $y$ that is different from $\gamma_1, \dots, \gamma_{n-1}$, and does not meet any point of $Z$ except at its endpoints. Moreover, by Lemma 2.8, we may choose $\tilde{g}$ and $\tilde{\gamma}$ so that $x$ and $y$ are not conjugate along $\tilde{\gamma}$ in the $\tilde{g}$-metric. All of the perturbations of the metric can be done outside a neighborhood of $\mathrm{tr}(\gamma_1) \cup \cdots \cup \mathrm{tr}(\gamma_{n-1})$. We let $g_n = \tilde{g}$. Then $\gamma_1, \dots, \gamma_{n-1}$ are $g_n$-geodesics, and (iv) remains true for $\gamma_1, \dots, \gamma_{n-1}$ with the metric $g_n$. Thus properties (i)-(iv) hold for $\gamma_1, \dots, \gamma_n$, where $\gamma_n = \tilde{\gamma}$, and $g$ is replaced by

$g_n$. Since $\mathcal{N}$ is arbitrary, we conclude that $\mathcal{H}_n(x,y)$ is $C^\infty$-dense. This completes the proof of Claim 3.1(a).

Claim 3.2($a$) and Claim 3.3($a$) follow from Claim 3.1(a), because $\tilde{\mathcal{H}}_n$ is $C^\infty$- dense in each fiber $\{(x,y)\} \times \mathbb{G}$, and $\hat{\mathcal{H}}_n$ is $C^\infty$-dense in each fiber $\{x\} \times \mathbb{G}$.

Next we want to prove Claim 3.1($b$). Let $g \in \mathcal{H}_n(x,y)$, and suppose $\gamma_1, \dots, \gamma_n$ are $g$-geodesics that satisfy properties (i)-(iv).

For the purpose of defining $C^1$ distances on $M$ between the given geodesics $\gamma_i$ and nearby curves $\tilde{\gamma}_i$, we extend the domain of $\gamma_i$ to $[0, L_i + 1]$. The distance between tangent vectors of $\gamma_i$ and tangent vectors of $\tilde{\gamma}_i$ will be measured with respect to the natural metric on $TM$ induced by $g$. If we consider geodesics as curves in $TM$, then they are solutions to a system of first order ordinary differential equations whose coefficients are continuous functions of the metric and its first derivatives. For any $\epsilon > 0$ there exists a $C^1$ neighborhood $\mathcal{N}_1$ of $g$ in $\mathbb{G}$ and a $\delta = \delta(\epsilon) > 0$ such that: if $\hat{g} \in \mathcal{N}_1$ and $\tilde{\gamma}_i$ is a $\hat{g}$-geodesic with $\tilde{\gamma}_i(0) = \gamma_i(0)$ and $|\tilde{\gamma}_i'(0) - \gamma_i'(0))| < \delta$, then the $C^1$ distance on $M$ between $\tilde{\gamma}_i|[0, L_i+1]$ and $\gamma_i|[0, L_i+1]$ is less than $\epsilon$. We choose $\epsilon > 0$ such that if the $C^1$ distance between $\tilde{\gamma}_i|[0, L_i+1]$ and $\gamma_i|[0, L_i+1]$ is less than $\epsilon$, $|L_i - \tilde{L}_i| < \epsilon$, and $\tilde{\gamma}_i(\tilde{L}_i) = y$, then conditions (i)-(iii) hold with $\gamma_i$ replaced by $\tilde{\gamma}_i$, and $L_i$ replaced by $\tilde{L}_i$.

By (iv), $y$ is not $g$-conjugate to $x$ along any of $\gamma_1, \dots, \gamma_n$. We choose open neighborhoods $U_1, \dots, U_n$ of $L_1\gamma_1'(0), \dots, L_n\gamma_n'(0)$ in $T_xM$, respectively, and an open neighborhood $U$ of $y$ in $M$, so that

$$\exp_{x,g} : U_i \to U$$

is a diffeomorphism, for $i = 1, \dots, n$. By replacing $U$ and $U_i$ by smaller open neighhorhoods, if necessary, we may assume that if $\tilde{\gamma}_i'(0)\tilde{L}_i \in U_i$, then $|L_i - \tilde{L}_i| < \epsilon$ and $|\tilde{\gamma}_i'(0) - \gamma_i'(0)| < \delta$.

If $B_i \subset U_i$ is an open ball centered at $L_i\gamma_i'(0)$ with $\overline{B_i} \subset U_i$, then $y \notin \exp_{x,g}(\partial B_i)$ and the topological degree of $\exp_{x,g}|\partial B_i$ is nonzero at $y$. Any continuous map $f_i : \overline{B_i} \to U$ that is sufficiently close to $\exp_{x,g}|\overline{B_i}$ in the $C^0$ topology also satisfies $y \notin f_i(\partial B_i)$ , and the topological degree of $f_i|\partial B_i$ is nonzero at $y$. This implies $y \in f_i(B_i)$. (See, for instance, Theorem 1.1 of [4].) Now we choose a $C^1$-open neighborhood $\mathcal{N}_2 \subset \mathcal{N}_1$ of $g$ such that if $\tilde{g} \in \mathcal{N}_2$, then $\exp_{x,\tilde{g}}$ is sufficiently $C^0$-close to $\exp_{x,g}$ on $\overline{B_i}$, $i = 1, \dots, n$, so that there exist $y_i \in B_i$ with $\exp_{x,\tilde{g}} y_i = y$. For $\tilde{g} \in \mathcal{N}_2$, let $\tilde{\gamma}_i, i = 1, \dots, n$, be $\tilde{g}$-geodesics defined on $[0, \tilde{L}_i]$ such that $\tilde{\gamma}_i'(0)\tilde{L}_i = y_i$. Then conditions (i)-(iii) hold for $\gamma_i, L_i, g$ replaced by $\tilde{\gamma}_i, \tilde{L}_i, \tilde{g}$, respectively. Thus there exists a $C^1$-open neighborhood $\mathcal{G}_n$ of $\mathcal{H}_n$ such that conditions (i)-(iii) hold for all $\tilde{g} \in \mathcal{G}_n$.

This finishes the proof of Claim 3.1(b), and thus the proof of Theorem 1.1(1). The proofs of Claims 3.2(b) and 3.3(b) are similar to the proof of Claim 3.1(b), except we do not assume that $\tilde{\gamma}_i(0) = \gamma_i(0)$. This completes the proof of statements (1), (2), and (3) of Theorem 1.1.

Now suppose that $M$ has dimension at least three. We indicate the changes that are needed in the above proof to sharpen "fails to have the finite blocking property" to "is strongly insecure" in (1), (2), and (3). We let $T(x,y,n,g)$ be the statement obtained from statement $S(x,y,n,g)$ by replacing condition (iii) by

(iii′) The sets $\gamma_1((0,L_1)), \dots \gamma_n((0,L_n))$ are pairwise disjoint.

and adding the condition

(v) If $x \neq y$, then $\gamma_1, \ldots, \gamma_n$ are non-self-intersecting; and if $x = y$, then $\gamma_1, \ldots, \gamma_n$ have no self-intersection points except $x$.

The argument proceeds as before, with $T(x, y, n, g)$ replacing $S(x, y, n, g)$ throughout. Once we obtain a metric $\hat{g} \in \mathcal{N}$ and a $\hat{g}$-geodesic $\hat{\gamma} : [0, L] \to M$ from $x$ to $y$ such that $\gamma_1, \ldots, \gamma_{n-1}$ are $\hat{g}$-geodesics and $tr(\hat{\gamma}) \cap [tr(\gamma_1) \cup \cdots \cup tr(\gamma_{n-1})]$ is a finite set, then we change the previous definition of $Z$ to $Z = tr(\gamma_1) \cup \cdots \cup tr(\gamma_{n-1})$, and apply Lemma 2.9 instead of Lemma 2.8. $\square$

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Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
*E-mail address*: gerber@indiana.edu

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
*E-mail address*: wku@indiana.edu