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REMARKS ON $\mathbb{A}^1$-HOMOTOPY GROUPS OF SMOOTH TORIC MODELS

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Abstract. We extend previous results on $\mathbb{A}^1$-homotopy groups of smooth proper toric varieties to the case of smooth proper toric models, i.e., smooth proper equivariant compactifications of possibly non-split tori, in characteristic 0.

1. Statement of results

Fix a field k having characteristic 0, and let $\mathcal{S}m_k$ denote the category of schemes that are separated, smooth and have finite type over k. Suppose X is a smooth proper k-scheme. Let $\mathcal{H}(k)$ denote the $\mathbb{A}^1$-homotopy category of k-schemes as constructed in [8, §3.2]. Assume $X(k)$ is non-empty, and fix $x \in X(k)$. One can study the $\mathbb{A}^1$-homotopy sheaves of groups $\pi_i^{\mathbb{A}^1}(X,x)$ (the Nisnevich sheaves of groups on $\mathcal{S}m_k$ denoted $a\underline{\pi}_i^{\mathbb{A}^1}(X,x)$ on [8, p. 110]). Our aim in this short note is to show that the "geometric" decomposition of $\mathbb{A}^1$-homotopy sheaves of groups of smooth proper "split" toric varieties, i.e., equivariant compactifications of ${\mathbb{G}_m}^{\times n}$, studied in [1] and [9] extends to "non-split" toric varieties, i.e., equivariant compactifications of tori T over k. We will refer to equivariant compactifications of tori T over k as toric T-models [6, §5].

Let $\overline{k}$ denote a fixed algebraic closure of k and let G_k denote the Galois group of $\overline{k}$ over k. For a k-scheme Y, let $Y_{\overline{k}}$ denote the variety obtained by extending scalars to $\overline{k}$. Suppose X is a smooth proper toric T-model. One knows that $Pic(X_{\overline{k}})$ is a finitely generated G_k-module, and we denote the associated dual k-torus—the Neron-Severi torus—by $T_{NS(X)}$. With any toric T-model, one can associate a fan Σ in $X^*(T_{\overline{k}})$ that is G_k-invariant. Cox's construction [5] realizing any "split" smooth proper toric variety as a geometric quotient of an open subscheme of affine space by a free action of $T_{NS(X)}$ can be generalized to the non-split case: if X is a smooth proper toric T-model, there are a $T_{NS(X)}$-torsor $f: U \to X$ and an open immersion $U \hookrightarrow \mathbb{A}^n_k$ ($n = \dim T + \dim T_{NS}(X)$) [6, Proposition 5.6]. Let $\mathcal{H}^1_{\text{ét}}(T_{NS(X)})$ denote the Nisnevich sheafification of the presheaf (on $\mathcal{S}m_k$) $U \mapsto H^1_{\text{ét}}(U, T_{NS(X)})$.

Theorem 1.1. *Assume k is a field having characteristic 0 and T is a k-torus. Suppose X is a smooth proper toric T-model, and let x denote the k-rational point of X corresponding to $1 \in T(k)$. The $T_{NS(X)}$-torsor $f: U \to X$ above is an $\mathbb{A}^1$-cover. In particular, if $\tilde{x}$ is any lift of x, there is a short exact sequence (of Nisnevich sheaves of groups)*

$$1 \longrightarrow \pi_1^{\mathbb{A}^1}(U,\tilde{x}) \longrightarrow \pi_1^{\mathbb{A}^1}(X,x) \longrightarrow T_{NS(X)} \longrightarrow 1,$$

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and, for each integer $i > 1$, there are isomorphisms $\pi_i^{\mathbb{A}^1}(U,\tilde{x}) \xrightarrow{\sim} \pi_i^{\mathbb{A}^1}(X,x)$. Finally, f induces a morphism of sheaves $\pi_0^{\mathbb{A}^1}(X) \to \mathcal{H}^1_{\acute{e}t}(T_{NS(X)})$ that is an isomorphism on sections over finitely generated separable extensions L/k.

Remark 1.2. There are examples of k-tori T and smooth proper toric T-models X for which $\pi_0^{\mathbb{A}^1}(X)(k)$ is non-trivial. Thus, over non separably closed fields, we have the interesting phenomenon that a smooth proper $\mathbb{A}^1$-disconnected space can have $\mathbb{A}^1$-connected covering spaces! For a manifestation of this phenomenon for non-proper smooth varieties, one can consider the morphism $\mathbb{A}^m \setminus 0 \to \mathbb{A}^m \setminus 0/\mu_n$ [1, Remark 3.13].

2. Torus torsors as $\mathbb{A}^1$-covering spaces

The word *space*, will mean "object of $\Delta^{\circ}\mathcal{S}hv_{Nis}(\mathcal{S}m_k)$" (the category of simplicial Nisnevich sheaves on $\mathcal{S}m_k$); we use caligraphic letters (e.g., $\mathcal{X}, \mathcal{Y}$) to denote such objects. We set $[\mathcal{X},\mathcal{Y}]_s := \hom_{\mathcal{H}_s((\mathcal{S}m_k)_{Nis})}(\mathcal{X},\mathcal{Y})$, where $\mathcal{H}_s((\mathcal{S}m_k)_{Nis})$ is as on [8, p. 49] and $[\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1} := \hom_{\mathcal{H}(k)}(\mathcal{X},\mathcal{Y})$. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of k-spaces is an $\mathbb{A}^1$-*cover* (*cf.* [7, Section 4.1]) if it has the *unique* right lifting property with respect to morphisms that are simultaneously $\mathbb{A}^1$-weak equivalences and monomorphisms of sheaves, i.e., $\mathbb{A}^1$-*acyclic cofibrations.*

Proposition 2.1. *Let T be a multiplicative group over a field k having characteristic 0. If X is a smooth scheme, and $\pi : U \to X$ is a T-torsor locally trivial in the étale topology, then π is an $\mathbb{A}^1$-cover and, in particular, an $\mathbb{A}^1$-fibration.*

Let BT denote the simplicial classifying space of T viewed as a Nisnevich sheaf of groups, and let $BT_{\text{ét}}$ denote the simplicial classifying space of T viewed as an étale sheaf of groups. Let $\alpha : (\mathcal{S}m_k)_{\text{ét}} \to (\mathcal{S}m_k)_{Nis}$ be the morphism of sites induced by the identity functor. Set $B_{\text{ét}}T := \mathbf{R}\alpha_* BT_{\text{ét}}$; see [8, §4.1] for more details.

Lemma 2.2 (*cf.* [2, Lemma 4.2.4])**.** *The space $B_{\acute{e}t}T$ is $\mathbb{A}^1$-local.*

Proof. By adjunction, one has canonical bijections

$$\hom_{\mathcal{H}_s((\mathcal{S}m_k)_{Nis})}(U, B_{\text{ét}}T) \xrightarrow{\sim} \hom_{\mathcal{H}_s^{\text{ét}}(k)}(U, BT_{\text{ét}}).$$

Choosing a fibrant model for $BT_{\text{ét}}$, and using [8, §2 Proposition 3.19 and §4 Proposition 1.16], to check that $B_{\text{ét}}T$ is $\mathbb{A}^1$-local, it suffices to prove that that the maps

$$H^i_{\text{ét}}(U,T) \longrightarrow H^i_{\text{ét}}(U \times \mathbb{A}^1, T)$$

are bijections for $i = 0, 1$. For $i = 0$, this a consequence of étale descent: if k'/k is a separable extension splitting T, then it suffices to observe that any morphism $U \times \mathbb{A}^1 \to \mathbb{G}_m{}^{\times n}$ factors through a morphism $U \to \mathbb{G}_m{}^{\times n}$. For $i = 1$ one could apply [2, Lemma 4.3.7 and Proposition 4.4.3]. For a direct proof, observe that [4, Lemma 2.4], establishes the result for affine X (Grothendieck showed that étale and flat cohomology coincide *Ibid.* p.159). We reduce the case of general X to the affine case by comparing the exact sequences of low degree terms for the Leray spectral sequences associated with an open affine cover $u : U \to X$ and the corresponding cover $u \times id : U \times \mathbb{A}^1 \to X \times \mathbb{A}^1$. □

Proof of Proposition 2.1. After Lemma 2.2, the proof is essentially [7, Lemma 4.5(2)]; here are the details. Start with an $\mathbb{A}^1$-acyclic cofibration $j : \mathcal{A} \to \mathcal{B}$ fitting into a diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{s_0} & U \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi} \\ \mathcal{B} & \longrightarrow & X. \end{array}$$

Now, since $B_{\text{ét}}T$ is $\mathbb{A}^1$-local, the natural maps $[\mathcal{B}, B_{\text{ét}}T]_s \to [\mathcal{A}, B_{\text{ét}}T]_s$ and $[\mathcal{B}, T]_s \to [\mathcal{A}, T]_s$ are bijections. The pullback of π to $\mathcal{A}$ admits a section and is therefore a trivial torsor. By the first bijection just mentioned, it follows that the pullback of π to $\mathcal{B}$ is also trivial, and thus also admits a section, which we denote by s. The composite morphism $j \circ s$ need not be equal to s_0, but if it is not, then there is an element $t_0 \in [\mathcal{A}, T]_s$ such that $t_0 \cdot s = s_0$. By the second bijection mentioned at the beginning of this paragraph, the element t_0 determines a unique element t of $[\mathcal{B}, T]_s$. The product $t^{-1} \cdot s$ is a new section of π pulled back to $\mathcal{B}$. By construction this new section gives back s_0 upon restriction to $\mathcal{A}$ and thus provides the necessary (unique) lift. □

Proof of Theorem 1.1. We return to the notation of the introduction: X is a smooth proper toric T-model, $T_{NS(X)}$ is the associated Neron-Severi torus and $f : U \to X$ is the $T_{NS(X)}$-torsor constructed in [6, Proposition 5.6].

Since X is proper, it follows from, e.g., [5, Lemma 1.4] that U has complement of codimension ≥ 2 in the affine space in which it sits since the same thing is true upon passing to a separable closure. Since k has characteristic 0 and is thus infinite, it follows that U is even connected by lines. (In fact, [1, Proposition 5.12] gives conditions guaranteeing that this complement has codimension $\geq d$, depending only on the fan of $X_{\overline{k}}$.) In any case, we can choose a point $\tilde{x}$ lifting x.

By Proposition 2.1, π is an $\mathbb{A}^1$-cover and thus an $\mathbb{A}^1$-fibration. Consider the long exact sequence in $\mathbb{A}^1$-homotopy groups of π, which exists by a formal argument in the theory of model categories (*cf.* [1, Remark 3.2]). The higher ($i > 1$) homotopy (sheaves of) groups of $B_{\text{ét}}T_{NS(X)}$ are trivial, and $\pi_1^{\mathbb{A}^1}(B_{\text{ét}}T_{NS(X)}) = T_{NS(X)}$ (again, see [8, §4 Proposition 1.16]). We then have a long exact sequence of groups (and pointed sets)

$$\begin{aligned} 1 \longrightarrow &\pi_1^{\mathbb{A}^1}(U, \tilde{x}) \longrightarrow \pi_1^{\mathbb{A}^1}(X, x) \longrightarrow T_{NS(X)} \\ \longrightarrow &\pi_0^{\mathbb{A}^1}(U) \longrightarrow \pi_0^{\mathbb{A}^1}(X) \longrightarrow \pi_0^{\mathbb{A}^1}(B_{\text{ét}}T_{NS(X)}), \end{aligned}$$

and for each $i > 1$, we have isomorphisms $\pi_i^{\mathbb{A}^1}(U, \tilde{x}) \xrightarrow{\sim} \pi_i^{\mathbb{A}^1}(X, x)$.

For the case $i = 0$, observe that the morphism $X \to B_{\text{ét}}T_{NS(X)}$ classifying f induces the morphism $\pi_0^{\mathbb{A}^1}(X) \to \pi_0^{\mathbb{A}^1}(B_{\text{ét}}T_{NS(X)})$. Using the $\mathbb{A}^1$-weak equivalence $X \to Sing_*^{\mathbb{A}^1}(X)$, there is an induced epimorphism $\pi_0^s(Sing_*^{\mathbb{A}^1}(X)) \to \pi_0^{\mathbb{A}^1}(X)$ by [8, §2 Corollary 3.22]. Again using the fact that X is proper, we conclude $\pi_0^s(Sing_*^{\mathbb{A}^1}(X))(L)$ is $X(L)/R$.

Since $B_{\text{ét}}T_{NS(X)}$ is $\mathbb{A}^1$-local, $\pi_0^{\mathbb{A}^1}(B_{\text{ét}}T) = \mathcal{H}^1_{\text{ét}}(T_{NS(X)})$. Taking sections over finitely generated separable extensions L/k determines a morphism of functors (on field extensions)

$$X(L)/R \longrightarrow H^1_{\text{ét}}(L, T_{NS}(X))$$

that coincides with the "obvious" such morphism gotten by restricting the $T_{NS}(X)$-torsor $f : U \to X$ to L-points of X. The torus $T_{NS(X)}$ is flasque (see, e.g., [3, Proposition 6]) so [3, §5 Corollaire 1] implies that the restriction map $X(L)/R \to H^1_{\text{ét}}(L, T_{NS}(X))$ is a bijection. It follows that $\pi_0^{\mathbb{A}^1}(X) \to \mathcal{H}^1_{\text{ét}}(T_{NS(X)})$ is an isomorphism on sections over separable finitely generated L/k. □

Remark 2.3. The statement in Theorem 1.1 involving $\pi_0^{\mathbb{A}^1}$ provides an alternate proof of [2, Theorem 2.4.3] in the special case of smooth proper toric models. Furthermore, this statement can be strengthened slightly. Indeed, the multiplication morphism $T \times T \to T$ gives rise to a rational map $X \times X \to X$. Resolving indeterminacy, we get a morphism $X' \to X \times X$ (that is a composite of blow-ups). One can check that this induces a composition on $\pi_0^{\mathbb{A}^1}(X)(L)$ for any L/k (coinciding with the composition on R-equivalence classes). The map of the proposition is in fact a homomorphism of abelian groups. One would like to show that $\pi_0^{\mathbb{A}^1}(X)$ can be equipped with the structure of a Nisnevich sheaf of abelian groups and that the map $\pi_0^{\mathbb{A}^1}(X) \to \mathcal{H}^1_{\text{ét}}(T_{NS(X})$ is an isomorphism of sheaves.

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