ON RATES IN MEAN ERGODIC THEOREMS

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ABSTRACT. We create a general framework for the study of rates of decay in mean ergodic theorems. As a result, we unify and generalize results due to Assani, Cohen, Cuny, Derriennic, and Lin dealing with rates in mean ergodic theorems in a number of cases. In particular, we prove that the Cesàro means of a power-bounded operator applied to elements from the domain of its abstract one-sided ergodic Hilbert transform decay logarithmically, and this decay is best possible under natural spectral assumptions.

1. Introduction

In recent years considerable emphasis has been put on the study of convergence rates in limit theorems arising in probability and ergodic theory, see e.g. [13, 12, 10] and the references therein. The basic result of interest here is of course the ergodic theorem in its two forms regarding almost everywhere convergence and convergence in norm. The latter is the famous "Mean Ergodic Theorem" of von Neumann (1931), subsequently generalized by Riesz, Yosida, Kakutani, Lorch and Eberlein towards a general theory of (norm-)convergence of the Cesàro averages

$$(1.1) \qquad A_n(T)x := \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \qquad x \in X,$$

for a bounded linear operator T on a Banach space X [19, Chapter 2]. A natural assumption on the operator T in this context is that T is *power-bounded*, i.e., that $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$. For a power-bounded operator T it is simple operator theory to show that for $x, y \in X$

$$A_n(T)x \to 0 \quad \Longleftrightarrow \quad x \in \overline{\mathrm{ran}}(I - T)$$

and

$$A_n(T)x \to y \quad \Longrightarrow \quad y \in \ker(I - T).$$

It follows that $\ker(I - T) \oplus \overline{\mathrm{ran}}(I - T)$ is precisely the subspace of X on which the Cesàro averages converge strongly, and the operator T is called *mean ergodic* if this is already the whole space X. A *mean ergodic theorem* gives conditions under which this is the case, and Lorch's famous generalization of von Neumann's theorem simply says that every power-bounded operator on a reflexive space is mean ergodic [19, Chapter 2, Theorem 1.2].

In this paper we shall be concerned with the study of convergence rates for $A_n(T)x$, where T is a power-bounded operator. If $A_n(T)x \to y$ then $A_n(T)x - y = A_n(T)(x - y)$, and hence in the study of rates for the convergence of Cesàro averages one can restrict oneself to the convergence to zero on the space $\overline{\mathrm{ran}}(I - T)$. Since one can always consider the restriction of T to this (obviously T-invariant) subspace, often it is no loss of generality in assuming that $X = \overline{\mathrm{ran}}(I - T)$ in the first place. However, we shall always make this assumption explicit when it is needed. Let us collect some known facts.

Proposition 1.1. *Let T be a power-bounded operator on a Banach space X. Then the following statements hold:*

a) *If $\|A_n(T)x\| = \mathrm{o}(1/n)$ then $x = 0$.*

b) *If $x \in \mathrm{ran}(I - T)$ then $\|A_n(T)x\| = \mathrm{O}(1/n)$, and the converse is true if X is reflexive.*

c) *If there exists a sequence $(r_n)_{n \geq 1}$ of positive numbers such that $r_n \searrow 0$ and $\|A_n(T)x\| = \mathrm{O}(r_n)$ for every $x \in X$, then $I - T$ is invertible.*

Part a) is due to Butzer and Westphal [5]. It tells us that we cannot expect better convergence rates than $\mathrm{O}(1/n)$. The first assertion in part b) is trivial, and the second is due to Browder [4] (but appears also in [5]). The proof of c) rests on the principle of uniform boundedness, by virtue of which one first concludes that $A_n(T) \to 0$ in norm. This yields $\ker(I - T) = \{0\}$, and T is a so-called *uniformly ergodic* operator. Dunford [15] has shown that for such operators $\mathrm{ran}(I - T)$ must be closed, cf. [20], whence finally $X = \overline{\mathrm{ran}}(I-T) = \mathrm{ran}(I-T)$, and $I-T$ is invertible. Uniformly ergodic operators are quite special, and Derriennic [13, p.144] remarks that a T induced by an ergodic measure preserving transformation on a non-atomic measure space is never uniformly ergodic. Hence by Proposition 1.1 there are plenty of examples where the Cesàro averages fail to converge with a uniform rate. For intricate results on the failing of rates see also the recent work [3].

The absence of a global uniform convergence rate leads naturally to the problem of identifying elements x of X for which one has a specified rate for $A_n(T)x$ and of describing a possible rate if such elements are given. Results in this direction have importance for almost everywhere convergence theorems, see e.g. [2, 6, 8, 9, 10, 11, 14], and central limit theorems for Markov chains, see e.g. [12] and the references therein.

Polynomial rates of decay were studied thoroughly in [14] in case of arbitrary Banach space contractions while in the special case of unitary and normal contractions on Hilbert spaces more general rates were investigated in [2] and [9]. For instance, it was proved in [14] that for a power-bounded and mean ergodic operator T and $s \in (0, 1)$ one has

$$(1.2) \qquad \|A_n(T)x\| = \mathrm{o}(n^{-s}) \qquad \text{as } n \to \infty,$$

for every $x \in \mathrm{ran}(I-T)^s$. Moreover, it was shown there that under the same assumptions the condition $x \in \mathrm{ran}(I - T)^s$ is equivalent to the strong and also to the weak convergence of the series

$$\sum_{n=0}^{\infty} \alpha_n^{(-s)} T^n x,$$

where $(1 - z)^{-s} = \sum_{n \geq 0} \alpha_n^{(-s)} z^n$ is the power series representation of $(1 - z)^{-s}$.

Towards logarithmic rates it was proved in [9] that if T is a normal contraction on a Hilbert space, then

$$(1.3) \qquad H_T x := \sum_{k=1}^{\infty} \frac{T^k x}{k} \text{ converges} \quad \implies \quad \|A_n(T)x\| = \mathrm{O}(1/\log n),$$

but for more general operators T the validity of (1.3) remained an open problem. (Note that it was proved in [1] that the converse implication in (1.3) does not, in general, hold even for unitary T.) The operator H_T appearing here (with its natural domain) is called the *one-sided ergodic Hilbert transform*, and was studied thoroughly in [2, 8, 9, 7, 14]. It was also an open question for some years whether in general one has $-H_T = \log(I-T)$ if $\overline{\mathrm{ran}}(I-T) = X$, where the latter operator is the generator of the C_0-semigroup $((I-T)^s)_{s \geq 0}$. It was solved (in the affirmative) recently, independently in [7] and [16].

The main result of the present paper is that the implication (1.3) holds for every power-bounded operator T on a Banach space (Section 4). Moreover, this is actually only a special case within a general approach to establishing rates in the mean ergodic theorem (Section 5).

2. Preliminaries

2.1. Some Notations and Definitions. For a closed linear operator A on a complex Banach space X we denote by $\mathrm{dom}(A)$, $\mathrm{ran}(A)$, $\ker(A)$, and $\sigma(A)$ the *domain*, the *range*, the *kernel*, and the *spectrum* of A, respectively. The norm-closure of the range is written as $\overline{\mathrm{ran}}(A)$. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. The *open unit disc* is denoted by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the *torus* by $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, and $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of *natural numbers*. For positive sequences $(r_n)_{n \geq 0}$ and $(s_n)_{n \geq 0}$ we write $r_n \sim s_n$ if there is $c > 0$ such that $r_n/c \leq s_n \leq cr_n$ for all $n \in \mathbb{N}$. Finally, $\delta_{m,n}$ denotes the Kronecker function on integers $m, n \geq 0$, i.e. $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise.

2.2. Functional Calculus. We denote by $\mathrm{A}_+^1(\mathbb{D})$ the algebra of holomorphic functions on $\mathbb{D}$ that have absolutely summable Taylor coefficients:

$$\mathrm{A}_+^1(\mathbb{D}) := \left\{ g \mid g(z) = \sum_{k \geq 0} \alpha_k z^k, \ \sum_{k \geq 0} |\alpha_k| < \infty \right\}$$

with the norm

$$\|g\|_{\mathrm{A}_+^1} := \|\alpha\|_1 = \sum_{k \geq 0} |\alpha_k| \qquad \text{for} \quad g(z) = \sum_{k \geq 0} \alpha_k z^k \in \mathrm{A}_+^1(\mathbb{D}).$$

It is well-known (and easy to see) that for each power-bounded operator T on X the assignment

$$g = \sum_{k \geq 0} \alpha_k z^k \ \mapsto \ g(T) := \sum_{k \geq 0} \alpha_k T^k$$

is a continuous algebra homomorphism (a *functional calculus*) of $\mathrm{A}_+^1(\mathbb{D})$ into $\mathcal{L}(X)$, satisfying

$$(2.1) \qquad \|g(T)\| \leq \left(\sup_{n \geq 0} \|T^n\| \right) \|g\|_{\mathrm{A}_+^1(\mathbb{D})} \qquad (g \in \mathrm{A}_+^1(\mathbb{D})).$$

For this functional calculus one has a spectral mapping theorem; however, we need only the following weaker statement, which we prove for the convenience of the reader.

Lemma 2.1 (Spectral inclusion theorem). *Let $g \in A^1_+(\mathbb{D})$ and let $T \in \mathcal{L}(X)$ be a power-bounded operator on X. Then*

$$g(\sigma(T)) = \{g(\lambda) \mid \lambda \in \sigma(T)\} \subseteq \sigma(g(T)).$$

Proof. Let $\lambda \in \sigma(T)$. Since T is power-bounded, $|\lambda| \leq 1$. Then

$$g(\lambda) - g(T) = \lim_{n \to \infty} \sum\nolimits_{k=1}^n \alpha_k(\lambda^k - T^k) \qquad \text{in } \mathcal{L}(X).$$

Hence if $g(\lambda) - g(T)$ is invertible then for large $n \in \mathbb{N}$ also

$$\sum\nolimits_{k=1}^n \alpha_k(\lambda^k - T^k) = (\lambda - T)\sum\nolimits_{k=1}^n \alpha_k \sum\nolimits_{j=0}^{k-1} \lambda^{k-j}T^j$$

is invertible. This implies that $\lambda - T$ is invertible, contradicting our assumption. □

Lemma 2.2. *Let $f = \sum_{k=0}^\infty \alpha_k z^k$ be a holomorphic function on $\mathbb{D}$, and let T be a power-bounded operator on X. Suppose that $x \in X$ is such that*

$$y := \sum_{k=0}^\infty \alpha_k T^k x \quad \text{converges weakly in } X.$$

If $1/f \in A^1_+(\mathbb{D})$, then $(1/f)(T)y = x$.

Proof. Let $C > 0$ be such that $\left\| \sum_{k=n}^m \alpha_k T^k x \right\| \leq C$ for all $0 \leq n \leq m$. Write $1/f(z) = \sum_{j=0}^\infty \beta_j z^j$, i.e.,

$$\sum_{j=0}^n \alpha_{n-j}\beta_j = \delta_{n0}, \qquad n \geq 0.$$

Since $\sum_{j=0}^M \beta_j T^j \to (1/f)(T)$ in operator norm, it follows that

$$\left(\sum_{j=0}^M \beta_j T^j \right)\left(\sum_{k=0}^M \alpha_k T^k x \right) \to (1/f)(T)y \qquad \text{weakly,} \quad \text{as } M \to \infty.$$

On the other hand, fix $\epsilon > 0$ and find $M_0 \in \mathbb{N}$ such that $\sum_{j > M_0} |\beta_j|\, \|T^j\| \leq \epsilon$. Then for $M > M_0$ write

$$\left(\sum_{j=0}^M \beta_j T^j \right)\left(\sum_{k=0}^M \alpha_k T^k x \right) = \sum_{n=0}^M \left(\sum_{j=0}^n \alpha_{n-j}\beta_j \right)T^n x + \sum_{j=0}^M \beta_j T^j \sum_{k=M-j+1}^M \alpha_k T^k x$$

$$= x + \sum_{j=0}^{M_0} \beta_j T^j \sum_{k=M-j+1}^M \alpha_k T^k x + \sum_{j=M_0+1}^M \beta_j T^j \sum_{k=M-j+1}^M \alpha_k T^k x.$$

In the last sum, the second summand tends weakly to zero as $M \to \infty$, since $(\sum_{k=0}^m \alpha_k T^k x)_{m \geq 0}$ is weakly Cauchy; the third has norm less than $C\epsilon$. □

2.3. Admissible functions. A holomorphic function $f(z) := \sum_{k=0}^{\infty} \alpha_k z^k$ on $\mathbb{D}$ (or, equivalently, the sequence $(\alpha_k)_{k \geq 0}$) is called *admissible* if f has no zeroes in $\mathbb{D}$ and

$$\frac{1}{f(z)} = \gamma_0 - \sum_{k=1}^{\infty} \gamma_k z^k \qquad (z \in \mathbb{D})$$

with $\gamma_k \geq 0$ for all $k \geq 0$.

We shall review briefly the basic analytic properties of admissible functions. Using induction and the identities linking the Taylor coefficients of f and $g := 1/f$ one finds that if f is admissible, then $\alpha_0 > 0$ and hence $\gamma_0 = \alpha_0^{-1} > 0$; moreover $\alpha_k \geq 0$ for all $k \geq 1$. In particular, $f(t)$ is positive and increasing for $0 \leq t < 1$, and this yields that $\sum_{k=1}^{\infty} \gamma_k \leq \gamma_0$. In particular, g has absolutely summable Taylor coefficients, i.e., $g \in A_+^1(\mathbb{D})$. So two cases can occur, described by the equivalences

$$(2.2) \qquad \sum_{k \geq 1} \gamma_k < \gamma_0 \iff g(1) > 0 \iff f(1) < \infty \iff f \in A_+^1(\mathbb{D})$$

and

$$(2.3) \qquad \sum_{k \geq 1} \gamma_k = \gamma_0 \iff g(1) = 0 \iff f(1) = \infty \iff f \notin A_+^1(\mathbb{D}),$$

where $f(1) := \lim_{t \uparrow 1} f(t)$.

We remark that every function g that satisfies the natural conditions

$$g(z) = \gamma_0 - \sum_{k=1}^{\infty} \gamma_k z^k, \quad \gamma_k \geq 0 \quad (k \geq 1), \quad \sum_{k=1}^{\infty} \gamma_k \leq \gamma_0, \quad \gamma_0 > 0,$$

can be written as $g = 1/f$ for some admissible function f. (This follows from the fact that 1 is an extremal point of $\mathbb{D}$.) This allows one to construct admissible functions by virtue of their inverses. A much harder question is whether a function f given by its Taylor expansion is admissible. The best known criterion is the following result of Kaluza [18] (see also [17, p. 68, Theorem 22]).

Proposition 2.3 (Kaluza). *Let $\alpha = (\alpha_k)_{k \geq 0}$ be a positive and decreasing sequence. Suppose that α is log-convex, i.e. $\alpha_0 > 0$ and $\alpha_k^2 \leq \alpha_{k-1}\alpha_{k+1}, k \geq 1$, and let $f(z) := \sum_{k=0}^{\infty} \alpha_k z^k$, $z \in \mathbb{D}$. Then f is admissible.*

Examples 2.4.   1)   From Kaluza's theorem it is immediate that the function

$$f(z) = 2 - \log(1 - z) = 2 + \sum_{n=1}^{\infty} z^n / n$$

is admissible. This fact will be used later.

2)   Consider the functions $f_s(z) := (1 - z)^{-s}, s \in (0, 1)$. They are admissible, since the binomial series yields

$$g_s(z) = 1/f_s(z) = (1 - z)^s = \sum_{n=0}^{\infty} \alpha_n^{(s)} z^n$$

with $\alpha_0^{(s)} = 1$ and $\alpha_n^{(s)} = \binom{s}{n}(-1)^n \leq 0$ for $n \in \mathbb{N}$. (One can also use Kaluza's theorem here.)

3. Estimating Rates in Terms of Taylor Coefficients

The main idea behind the proof of (1.3) is to note that by Lemma 2.2 one has $\operatorname{dom}(H_T) \subseteq \operatorname{ran}(g(T))$ for $g = 1/f$ and $f(z) = 2 - \log(1-z)$ is admissible. In view of the possible generalizations in Section 5 we shall therefore investigate the asymptotic behaviour of the Cesàro means $A_n(T)$ on spaces of the form $\operatorname{ran}(g(T))$, where $g = 1/f$ for some admissible function f.

First we note that the case $f(1) < \infty$ is uninteresting here, since then $f \in \mathrm{A}^1_+(\mathbb{D})$ and $\operatorname{ran}(g(T)) = X$ is the whole space; and according to the discussion in the introduction one cannot expect a general convergence rate for $(A_n(T)x)_{n \geq 1}$. So it is reasonable to restrict the considerations to the case that $f(1) = \infty$, i.e., $g(1) = 0$. This means that one has

$$(3.1) \qquad g(z) = \gamma_0 - \sum_{k=1}^{\infty} \gamma_k z^k, \quad \gamma_k \geq 0 \quad (k \geq 1), \quad \sum_{k=1}^{\infty} \gamma_k = \gamma_0 > 0,$$

which will be our standard assumptions for g. We note first the following fact.

Lemma 3.1. *Let f be admissible with $f(1) = \infty$, so that $g := 1/f$ satisfies (3.1). Then $\operatorname{ran}(g(T)) \subseteq \overline{\operatorname{ran}}(I - T)$ for every power-bounded operator T on X.*

Proof. We have

$$g(T) = \lim_{n \to \infty} \sum_{k=1}^{n} \gamma_k (I - T^k) = \lim_{n \to \infty} (I - T) \sum_{k=1}^{n} \gamma_k \sum_{j=0}^{k-1} T^j.$$

$\square$

By Lemma 3.1 and the mean ergodic theorem, one has $A_n(T)x \to 0$ as $n \to \infty$ whenever $x \in \operatorname{ran}(g(T))$. We shall see below that this convergence even happens with a certain rate, which is identified in the following lemma.

Lemma 3.2. *Let f be admissible such that $f(1) = \infty$, and let $g = 1/f$, i.e., g satisfies (3.1). Then*

$$\|A_n \cdot g\|_{\mathrm{A}^1_+(\mathbb{D})} = 2r_n \qquad (n \in \mathbb{N}),$$

where $\quad A_n(z) := \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} z^k, \qquad (n \in \mathbb{N},\ z \in \mathbb{D}), \qquad$ *and*

$$(3.2) \qquad r_n = r_n[g] := \frac{1}{n} \sum_{k=1}^{n} k\gamma_k + \sum_{k>n} \gamma_k = \sum_{k=1}^{\infty} \min(k/n, 1)\gamma_k.$$

Proof. If $n = 1$ then clearly $\|A_1 \cdot g\|_{\mathrm{A}^1_+(\mathbb{D})} = 2\sum_{k \geq 1} \gamma_k = 2r_1$. For $z \in \mathbb{D}$ and $n \geq 2$ we have

$$h_n(z) := nA_n(z)g(z) = \left(1 + z + \cdots + z^{n-1}\right)g(z) = \left(\sum_{j=0}^{n-1} z^j\right) \cdot \left(\gamma_0 - \sum_{k=1}^{\infty} \gamma_k z^k\right)$$

$$= \gamma_0 + \sum_{k=1}^{n-1}\left(\gamma_0 - \sum_{j=0}^{k-1}\gamma_{k-j}\right)z^k - \sum_{k=n}^{\infty}\left(\sum_{j=0}^{n-1}\gamma_{k-j}\right)z^k$$

$$= \sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{\infty}\gamma_j\right)z^k - \sum_{k=n}^{\infty}\left(\sum_{j=k-n+1}^{k}\gamma_j\right)z^k.$$

Since $h_n(1) = ng(1) = 0$, we must have

$$\sum_{k=n}^{\infty} \sum_{j=k-n+1}^{k} \gamma_j = \sum_{k=0}^{n-1} \sum_{j=k+1}^{\infty} \gamma_j$$

and hence

$$\|h_n\|_{A_+^1(\mathbb{D})} = \sum_{k=0}^{n-1} \sum_{j=k+1}^{\infty} \gamma_j + \sum_{k=n}^{\infty} \sum_{j=k-n+1}^{k} \gamma_j = 2 \sum_{k=0}^{n-1} \sum_{j=k+1}^{\infty} \gamma_j$$

$$= 2 \sum_{j=1}^{\infty} \left(\sum_{k=0}^{\min(j,n)-1} 1 \right) \gamma_j = 2 \sum_{j=1}^{\infty} \min(j,n) \gamma_j = 2nr_n.$$

$\square$

Observe that the sequence of functions $u_n := \min(\cdot/n, 1)$ decreases to zero pointwise on $\mathbb{R}_+$, hence $r_n \searrow 0$ by the second equality in (3.2) and the monotone convergence theorem.

Theorem 3.3. *Let $g(z) = \gamma_0 - \sum_{k=1}^{\infty} \gamma_k z^k$ be as in (3.1), let $r = r[g]$ be as in (3.2), and let T be a power-bounded operator on X. If $x \in \mathrm{ran}(g(T))$, then*

$$\|A_n(T)x\| = \mathrm{O}(r_n) \qquad as\ n \to \infty;$$

and if in addition $nr_n \to \infty$ and T is mean ergodic, then

$$\|A_n(T)x\| = \mathrm{o}(r_n) \qquad as\ n \to \infty.$$

Proof. The first assertion follows from Lemma 3.2 and the functional calculus estimate

$$\|A_n(T)g(T)\| \le M \|A_n g\|_{A_+^1(\mathbb{D})} \le 2Mr_n \qquad (n \in \mathbb{N}),$$

where $M := \sup_{n \ge 0} \|T^n\|$. If $(I - T)y = 0$, then $g(T)y = g(1)y = 0$. And if $y = (I - T)z \in \mathrm{ran}(I - T)$, then

$$r_n^{-1} A_n(T)g(T)y = r_n^{-1} A_n(T)g(T)(I - T)z = (nr_n)^{-1} g(T)(I - T^n)z \to 0$$

since $nr_n \to \infty$. As $\sup_n r_n^{-1} \|A_n(T)g(T)\| < \infty$, it follows that $r_n^{-1} A_n(T)g(T)y \to 0$ for all $y \in \overline{\mathrm{ran}}(I - T) \oplus \ker(I - T)$. The latter direct sum equals X by mean ergodicity of T. $\square$

4. The One-Sided Ergodic Hilbert Transform

Let T be a power-bounded operator on X. Recall from the Introduction that the (unbounded) operator H_T, defined by

$$\mathrm{dom}(H_T) := \left\{ x \in X \mid \sum_{k=1}^{\infty} \frac{T^k x}{k} \text{ converges} \right\}, \quad H_T x := \sum_{k=1}^{\infty} \frac{T^k x}{k}$$

is called the *one-sided ergodic Hilbert transform* associated with T. This operator has recently obtained greater attention [8, 9, 7, 11, 16]. In particular, it was conjectured that for $x \in \mathrm{dom}\, H_T$ one would have a logarithmic decay of the Cesàro means $(A_n(T)x)_{n \ge 1}$. The following theorem shows that this is indeed the case; it sharpens and generalizes (1.3) proved in [9, Proposition 3.4] for normal contractions on Hilbert spaces.

Theorem 4.1. *Let T be a power-bounded operator on X with $M := \sup_{n \ge 0} \|T^n\|$.*

a) *If $x \in X$ is such that $z := \sum_{k=1}^{\infty}(1/k)T^k x$ converges weakly in X, then*

$$\|A_n(T)x\| \leq \frac{2eM}{2 + \log n} \, \|z + 2x\| \qquad (n \geq 1),$$

where $M := \sup_{n \geq 0} \|T^n\|$. Moreover,

$$\|A_n(T)x\| = \mathrm{o}(1/\log n) \qquad as \ n \to \infty.$$

b) *Suppose that T is mean ergodic and $z = 1$ is an accumulation point of*

$$\sigma(T) \cap \{z \in \mathbb{D} \mid \arg(1 - z) \leq \theta\}$$

for some $\theta \in [0, \pi/2)$. Then for any sequence $(\epsilon_n)_{n \geq 1}$ with $\epsilon_n \searrow 0$ there is $x \in \mathrm{dom}(H_T)$ such that

$$\|A_n(T)x\| \neq \mathrm{O}(\epsilon_n/\log n) \qquad as \ n \to \infty.$$

We note that for fixed $x \in X$ the weak convergence of the series $\sum_{k=1}^{\infty}(1/k)T^k x$ is equivalent to its strong convergence, cf. [7, Theorem 3.2] or [16, Theorem 6.2]; and the strong convergence clearly implies $\lim_{n \to \infty} A_n(T)x = 0$ by Kronecker's Lemma, cf. [17, p. 73, Theorem 26]. The novelty of Theorem 4.1 is that a specific rate for this latter convergence is identified and that this rate is recognized as optimal under mild spectral conditions.

Proof. The proof of part a) proceeds in several steps. In the *first step* we replace the operator H_T by $2I + H_T$ and consider the admissible function $f(z) := 2 - \log(1 - z)$. Let us write

$$g(z) = 1/f(z) = \frac{1}{2 - \log(1 - z)} = \frac{1}{2} - \sum_{n=1}^{\infty} \gamma_n z^n.$$

By admissiblility of f one has $\gamma_n \geq 0$ for all $n \geq 1$. And since $f(1) = \infty$ one also has $\sum_{n=1}^{\infty} \gamma_n = 1/2$.

Now suppose that $z = \sum_{n=1}^{\infty}(1/n)T^n x$ weakly and let $y := 2x + z$. In the *second step* we observe that then $x = g(T)y$ by Lemma 2.2. It follows that we can estimate

$$\|A_n(T)x\| = \|A_n(T)g(T)y\| = \|(A_n g)(T)y\| \leq M \, \|y\| \, \|A_n g\|_{\mathrm{A}_+^1(\mathbb{D})}$$

by the functional calculus.

In the *third step* we apply Lemma 3.2 to obtain

$$\|A_n g\|_{\mathrm{A}_+^1(\mathbb{D})} = 2r_n \quad \text{with} \quad r_n = \sum_{k=1}^{\infty} \min(k/n, 1)\gamma_k \qquad (n \in \mathbb{N}),$$

This yields the estimate $\|A_n(T)x\| \leq 2M \, \|y\| \, r_n$, but the rate $(r_n)_{n \in \mathbb{N}}$ is not very explicit. Thus in the *fourth step* we prove that

(4.1)
$$\frac{r_n}{e} \leq \frac{1}{2 + \log n} \leq 2r_n \qquad (n \in \mathbb{N}).$$

To this aim, write $z_n := 1 - 1/n$ and note that

$$g(z_n) = \frac{1}{2 - \log(1 - z_n)} = \frac{1}{2 + \log n} \qquad (n \in \mathbb{N}).$$

To establish (4.1) write

$$g(z) = \frac{1}{2} - \sum_{k=1}^{\infty} \gamma_k z^k = \sum_{k=1}^{\infty} \gamma_k - \sum_{k=1}^{\infty} \gamma_k z^k$$

$$= \sum_{k=1}^{\infty} \gamma_k(1 - z^k) = \sum_{k=1}^{n} \gamma_k(1 - z) \sum_{j=0}^{k-1} z^j + \sum_{k=n+1}^{\infty} \gamma_k(1 - z^k),$$

so that $|g(z)| \leq |1 - z| \sum_{k=1}^{n} \gamma_k k + 2 \sum_{k=n+1}^{\infty} \gamma_k$ for $|z| \leq 1$. Inserting $z_n = 1 - 1/n$ yields $|g(z_n)| \leq 2r_n$. For the converse estimate we use that $z_n^j \geq z_n^k$ for $j \leq k$ and hence

$$g(z_n) = \sum_{k=1}^{\infty} \gamma_k(1 - z_n^k) \geq \sum_{k=1}^{n} \gamma_k(1 - z_n^k) + (1 - z_n^n) \sum_{k>n} \gamma_k.$$

Now, for $1 \leq k \leq n$ one has $\quad 1 - z_n^k = (1 - z_n) \sum_{j=0}^{k-1} z_n^j \geq (k/n) z_n^{n-1} \quad$ and one can see by elementary calculus that $z_n^{n-1} \geq 1/e$ for $n \geq 2$.

At this stage, the first part of Theorem 4.1.a is proved. For the second part note that

$$nr_n \sim \frac{n}{2 + \log n} \to \infty \qquad \text{as } n \to \infty.$$

Hence $r_n^{-1} A_n(T)g(T)(I-T) = (nr_n)^{-1}g(T)(I-T^n) \to 0$ in norm. A density argument yields $r_n^{-1} A_n(T)g(T)w \to 0$ for all $w \in \overline{\text{ran}}(I - T)$. But since $A_n(T)x \to 0$, one has $x \in \overline{\text{ran}}(I - T)$, and since this space is T-invariant, it follows that also $y \in \overline{\text{ran}}(I - T)$. This completes the proof of Theorem 4.1.a.

Towards the proof of part b) we apply the spectral inclusion theorem (Lemma 2.1) to the function $A_n g$, and obtain the estimate

$$\|A_n(T)g(T)\| = \|(A_n g)(T)\| \geq \sup_{\lambda \in \sigma((A_n g)(T))} |\lambda| \geq \sup_{\lambda \in \sigma(T)} |(A_n g)(\lambda)|.$$

By hypothesis there is an infinite set $J \subseteq \mathbb{N}$ such that for each $n \in J$ one has $\cos \theta \geq 1/n$ and one finds $z_n \in \sigma(T)$ such that

$$\arg(1 - z_n) \leq \theta \quad \text{and} \quad \frac{1}{2n} \leq |1 - z_n| \leq \frac{1}{n}.$$

Using the elementary inequality

$$1 - |z| \geq 1 - |z|^2 \geq |1 - z| (2 \cos \theta - |1 - z|), \qquad (z \in \mathbb{D}, \arg(1 - z) \leq \theta)$$

we conclude that

$$1 - |z_n| \geq \frac{1}{2n}(2 \cos \theta - \frac{1}{n}) \geq \frac{\cos \theta}{2n}$$

for each $n \in J$. Hence

$$\sup_{\lambda \in \sigma(T)} |(A_n g)(\lambda)| \geq |A_n(z_n)g(z_n)| = \frac{|1 - z_n^n|}{n|1 - z_n|} |g(z_n)|$$

$$\geq \frac{1 - |z_n|^n}{n|1 - z_n|} |g(z_n)| \geq \frac{\cos \theta}{2e} \cdot \frac{n}{n} \cdot \frac{\cos \theta}{2e} r_n = \left(\frac{\cos \theta}{2e}\right)^2 r_n$$

for each $n \in J$. Consequently, if $(\epsilon_n)_{n \geq 1}$ is any positive sequence such that $\epsilon_n \to 0$, then the family of operators $\frac{1}{\epsilon_n r_n} A_n(T)g(T), n \in \mathbb{N}$, is not uniformly norm bounded. By the principle of uniform boundedness there must exist a vector $y \in X$ such that

$$\sup_{n \in \mathbb{N}} \frac{1}{\epsilon_n r_n} \|A_n(T)g(T)y\| = \infty.$$

Hence, with $x = g(T)y \in \operatorname{ran} g(T)$ we have

$$\|A_n(T)x\| \neq \mathrm{O}(\epsilon_n r_n) \qquad \text{as } n \to \infty,$$

as was to be shown. To obtain Theorem 4.1.b in its actual formulation, one has to use that $\operatorname{dom}(H_T) = \operatorname{ran}(g(T))$ for a mean-ergodic operator T [16, Theorem 5.6]. $\square$

5. Generalization via Admissible Functions

The arguments of the previous section can be adapted to generalize Theorem 4.1 towards other convergence rates. If f is an admissible function and $g = 1/f$, then Theorem 3.3 yields the rate of decay $(r_n)_{n \geq 1}$ for $\|A_n(T)g(T)\|$ in terms of the Taylor coefficients of g. However, in situations of interest these are often unknown, so it seems desirable to be able to read off the r_n (or at least their asymptotic behaviour) from the *values* of f or g at certain points as in the previous section. The following lemma helps to achieve this.

Lemma 5.1. *Let* $0 < \alpha \leq 1$ *and* $n \in \mathbb{N}$. *If* $z \in \mathbb{D}$ *is such that*

$$|1 - z| \leq \frac{1}{n} \quad and \quad 1 - |z| \geq \frac{\alpha}{n}$$

then $\quad 1 - |z|^k \geq (\alpha/e) \min(k/n, 1) \quad$ *for all* $k \geq 1$.

To prove the lemma it suffices to observe that if $k \geq n$ then $1 - |z|^k \geq 1 - |z|^n$ and for $1 \leq k \leq n$ one has

$$1 - |z|^k = (1 - |z|) \sum_{j=0}^{k-1} |z|^j \geq (\alpha k/n) \left(1 - |1 - z|\right)^{n-1} \geq \frac{\alpha k}{en}.$$

Let $g(z) = \gamma_0 - \sum_{k=1}^{\infty} \gamma_k z^k$ satisfy our standard assumptions (3.1), and let $r = r[g]$ as in (3.2). Then we can write

$$g(z) = \sum_{k=1}^{\infty} \gamma_k(1 - z^k) = \sum_{k=1}^{n} \gamma_k(1 - z) \sum_{j=0}^{k-1} z^j + \sum_{k=n+1}^{\infty} \gamma_k(1 - z^k)$$

for all $z \in \overline{\mathbb{D}}$ and all $n \geq 1$. Hence if we know that $|1 - w| \leq \beta/n$ for some $w \in \overline{\mathbb{D}}$, $n \in \mathbb{N}$ and $\beta > 0$, then we can estimate

$$|g(w)| \leq \sum_{k=1}^{n} \gamma_k |1 - w| \sum_{j=0}^{k-1} |w|^j + \sum_{k=n+1}^{\infty} \gamma_k |1 - w^k|$$

$$\leq \beta \sum_{k=1}^{n} (k/n)\gamma_k + 2 \sum_{k=n+1}^{\infty} \gamma_k \leq \max(\beta, 2) \, r_n.$$

And if we know that $|1 - w| \leq 1/n$ and $1 - |w| \geq \alpha/n$, then

$$|g(w)| \geq \operatorname{Re} g(w) = \sum_{k=1}^{\infty} \gamma_k(1 - \operatorname{Re} z^k) \geq \sum_{k=1}^{\infty} \gamma_k(1 - |z|^k) \geq (\alpha/e)r_n$$

by Lemma 5.1. So by choosing $w = w_n$ in a suitable way, we are able to read off the rate $r = r[g]$ from the values $|g(w_n)|$, as it is stated in the next theorem.

Theorem 5.2. *Let f be admissible such that $f(1) = \infty$, let $g = 1/f$ and let $r = r[g]$ be the associated rate sequence as in (3.2). If $0 < \alpha \leq 1$ and $(w_n)_{n \in \mathbb{N}} \subseteq \mathbb{D}$ are such that $|1 - w_n| \leq 1/n$ and $1 - |w_n| \geq \alpha/n$ for all n, then*

$$(1/2)\,|g(w_n)| \leq r_n \leq (e/\alpha)\,|g(w_n)| \qquad (n \in \mathbb{N}).$$

Combining Lemma 2.2 with Theorems 3.3 and 5.2 we then obtain the following corollary.

Corollary 5.3. *Let $f(z) = \sum_{k \geq 0} \alpha_k z^k$ be an admissible function with $f(1) = \infty$. Let T be a power-bounded operator on X, let $M := \sup_{n \geq 0} \|T^n\|$, and let $x \in X$ be such that*

$$y := \sum_{k=0}^{\infty} \alpha_k T^k x \quad \text{converges weakly.}$$

Then $x = (1/f)(T)y$ *and* $\|A_n(T)x\| \leq \dfrac{2eM}{f(1 - 1/n)}\,\|y\|$ $(n \in \mathbb{N})$. *If in addition $n/f(1 - 1/n) \to \infty$ as $n \to \infty$, then*

$$\|A_n(T)x\| = \mathrm{o}\left(\frac{1}{f(1 - 1/n)}\right) \qquad \text{as } n \to \infty.$$

Proof. We apply Lemma 2.2 to conclude that $x = g(T)y$. Then the estimate

$$\|A_n(T)x\| \leq \frac{2eM}{f(1 - 1/n)}\,\|y\|$$

follows readily from the proof of Theorem 3.3 and from Theorem 5.2. Now suppose that $n/f(1 - 1/n) \to \infty$. Then, by Theorem 5.2, $nr_n \to \infty$. Furthermore, $x \in Y := \overline{\operatorname{ran}}(I - T)$ (Lemma 3.1) and Y is T-invariant, so $y \in Y$ as well. This means that we can suppose without loss of generality that T is mean ergodic. Hence the second part of Theorem 3.3 yields that

$$\|A_n(T)x\| = \mathrm{o}(r_n) = \mathrm{o}(1/f(1 - 1/n)) \quad \text{as } n \to \infty,$$

again by Theorem 5.2. □

Thus Theorem 3.3 gives an asymptotic estimate for the Cesàro averages $A_n(T)x$, when $x \in \operatorname{ran}(g(T))$, and Theorem 5.2 allows to identify $r_n \sim g(1 - 1/n)$, for instance. The obvious question now is, under which conditions on T the rate $r[g]$ is actually *optimal* on $\operatorname{ran}(g(T))$. That an extra condition is needed is readily seen by letting $T = 0$. It seems therefore reasonable to require that T has spectrum on a set where we can reconstruct the rate $r[g]$ from values of g. The following is the best we can do at this moment. Its proof follows the lines of the proof of Theorem 4.1.b above.

Theorem 5.4. *Let f be an admissible function on $\mathbb{D}$ such that $f(1) = \infty$, let $g = 1/f$ and let $r[g] = (r_n)_{n \geq 1}$ the associated rate sequence. Let T be a power-bounded operator on X. Suppose that $z = 1$ is an accumulation point of $\sigma(T) \cap \{z \in \mathbb{D} \mid \arg(1-z) \leq \theta\}$ for some $\theta \in [0, \pi/2)$. Then for any sequence $(\epsilon_n)_{n \geq 1}$ with $\epsilon_n \searrow 0$ there is some $x \in \operatorname{ran}(g(T))$ such that*

$$\|A_n(T)x\| \neq \mathrm{O}(\epsilon_n r_n) \qquad \text{as } n \to \infty.$$

Theorem 4.1 on logarithmic rates can formally be obtained from Theorems 3.3, 5.2 and 5.4 by considering $f(z) = 2 - \log(1-z)$ as above. Moreover, one can also deduce results on polynomial rates obtained originally by a different method in [14, Theorem 2.11 and Corollary 2.15]. Indeed, fix $s \in (0,1)$ and consider the function

$$f_s(z) = (1-z)^{-s} = \sum_{n=0}^{\infty} \alpha_n^{(-s)} z^n.$$

Then f_s is admissible (see Example 2.4.2). By Theorem 5.2 one has $r_n \sim 1/f_s(1 - 1/n) = n^{-s}$; from Theorem 3.3 it follows that

$$\|A_n(T)x\| = O(n^{-s}) \qquad \text{as } n \to \infty$$

for every $x \in \operatorname{ran}(I - T)^s$ and every power-bounded operator T on a Banach space. And since $s \in (0,1)$, one has $nr_n \sim n^{1-s} \to \infty$, whence if T is moreover mean ergodic, then

$$\|A_n(T)x\| = o(n^{-s}) \qquad \text{as } n \to \infty$$

for every $x \in \operatorname{ran}(I - T)^s$. Lemma 2.2 finally shows that if $y := \sum_{n \geq 0} \alpha_n^{(-s)} T^n x$ is weakly convergent, then $x = (I - T)^s y$.

Apart from these known facts, Theorem 5.4 adds a (previously unknown) condition on T under which the rate n^{-s} is optimal for $A_n(T)$ on $\operatorname{ran}(I - T)^s$. This is true, e.g., if T is a non-invertible isometry, like the one-sided shift on ℓ^p, $1 \leq p \leq \infty$.

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