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GENERALIZED d-KOSZUL MODULES

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Abstract. Generalized d-Koszul modules are introduced to solve an open problem: the odd Ext-module $\mathrm{E}^{\mathrm{odd}}(M)$ of a d-Koszul module M over a d-Koszul algebra Λ is a Koszul module over the even Yoneda algebra $\mathrm{E}^{\mathrm{ev}}(\Lambda)$.

Introduction

For an integer $d \geq 2$, a d-Koszul algebra was introduced and studied by R. Berger [B1], and developed by E. L. Green et al. [GMMZ] to the nonlocal case. If $d = 2$ it is the usual Koszul algebra. This class of generalized Koszul structures turns out to be important for example in theory of the Artin-Shelter algebras, the Calabi-Yau algebras, and the Yang-Mills algebras (see e.g. [B1], [B2], [CD]).

Let Λ be a d-Koszul algebra and M a d-Koszul Λ-module. It was shown in Theorem 6.1 of [GMMZ] that the even Ext-algebra $\mathrm{E}^{\mathrm{ev}}(\Lambda)$ is a Koszul algebra and the even Ext-module $\mathrm{E}^{\mathrm{ev}}(M)$ is a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module. This generalizes the corresponding result of J. Backelin and R. Fröberg [BF] on the local Koszul algebras. An open problem was raised by E. L. Green et al. [GMMZ], Section 6: Is the odd Ext-module $\mathrm{E}^{\mathrm{odd}}(M)$ also a Koszul module over $\mathrm{E}^{\mathrm{ev}}(\Lambda)$? E. N. Marcos and R. Martínez-Villa [MM] proved that this is the case if the orthogonal algebra $\Lambda^!$ is also a d-Koszul algebra. However, in general $\Lambda^!$ is not a d-Koszul algebra (see [B1]; also Example 2 in [MM]). So the problem remains to be open.

In this paper we introduce the so-called generalized d-Koszul modules. This is a natural class of graded modules. For example, the syzygies of a d-Koszul module are generalized d-Koszul modules up to shifts. Also for each i, J^iM is a generalized d-Koszul module up to shift, where M is a generalized d-Koszul module over a d-Koszul algebra, and J is the graded Jacobson radical.

Our main result is as follows.

Main Theorem. *Let Λ be a d-Koszul algebra, and M a generalized d-Koszul Λ-module. Then $\mathrm{E}^{\mathrm{ev}}(M)$ is a Koszul module over the Koszul algebra $\mathrm{E}^{\mathrm{ev}}(\Lambda)$.*

As a consequence, we have

Corollary. *Let Λ be a d-Koszul algebra, and M a d-Koszul Λ-module. Then $\mathrm{E}^{\mathrm{odd}}(M)$ is a Koszul module over the Koszul algebra $\mathrm{E}^{\mathrm{ev}}(\Lambda)$.*

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This answers in the affirmative the open problem mentioned above.

1. Preliminaries

We fix the notations and recall some facts frequently used later. For the details we refer to [BGS], [GM], and [GMMZ].

1.1. Throughout Λ is a *standardly graded algebra* over a field k (see [GM], p.250), i.e., $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is a positively graded k-algebra satisfying the following three conditions:

(*i*) $\Lambda_0 = k^r$ for some integer $r \geq 1$,

(*ii*) $\dim_k \Lambda_i < \infty, \ \forall\ i \geq 0$,

(*iii*) $\Lambda_i\Lambda_j = \Lambda_{i+j}, \ \forall\ i, j \geq 0$.

A left *graded* Λ*-module* M is a Λ-module together with a decomposition of k-spaces $M = \bigoplus\limits_{i\in\mathbb{Z}} M_i$ such that $\Lambda_i M_j \subseteq M_{i+j}, \ \forall\ i, j \in \mathbb{Z}$. Let M and N be graded Λ-modules. A Λ-homomorphism $f : M \to N$ is *a graded homomorphism* if $f(M_i) \subseteq N_i, \ \forall\ i \in \mathbb{Z}$. For $M \in \mathrm{Gr}(\Lambda)$, let $M[n]$ denote the graded module with $M[n]_i = M_{i-n}$. Let Λ-Mod be the category of the left Λ-modules, $\mathrm{Gr}(\Lambda)$ the category of the left graded Λ-modules and graded homomorphisms, and $\mathrm{gr}(\Lambda)$ the full subcategory of $\mathrm{Gr}(\Lambda)$ consisting of finitely generated Λ-modules. Then Λ-Mod and $\mathrm{Gr}(\Lambda)$ are abelian categories; and $\mathrm{gr}(\Lambda)$ is abelian if Λ is noetherian. Let $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}$ and $\mathrm{Ext}^i_{\mathrm{Gr}(\Lambda)}$ denote the homomorphisms and extensions in $\mathrm{Gr}(\Lambda)$, as opposed to the usual Hom_Λ and Ext^i_Λ in Λ-Mod.

Let I be a subset of $\mathbb{Z}$, and $M \in \mathrm{Gr}(\Lambda)$. M is *generated in degrees in* I, if $M = \Lambda(\bigoplus\limits_{j\in I} M_j)$; M is *generated in degree* i if $M = \Lambda M_i$; M is *supported above degree* n if $M_j = 0$ for $j < n$; and M is *concentrated in degrees in* I if $M_i = 0$ for $i \notin I$.

Let J be the ideal $\bigoplus\limits_{i\geqslant 1} \Lambda_i$ of Λ. The trivial Λ-module Λ_0 is the lift of the Λ_0-module Λ_0 via the k-algebra homomorphism $\Lambda \to \Lambda/J = \Lambda_0$. It is a graded Λ-module concentrated in degree 0. We need the following well-known fact.

Lemma 1.1. *Let* $M \in \mathrm{Gr}(\Lambda)$*, and* I *be a subset of* $\mathbb{Z}$*. If* $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ *for* $j \notin I$*, then* M *is generated in degrees in* I*.*

Proof. For the convenience of the reader we include a justification. Put $L := M/\Lambda(\bigoplus\limits_{i\in I} M_i)$. If $L \neq 0$, then $L/JL \neq 0$. While L/JL is a graded module over the semisimple algebra Λ_0, it follows that $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(L/JL, \Lambda_0[j]) \neq 0$ for some $j \notin I$, and hence $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) \neq 0$ for some $j \notin I$, contrary to the assumption. □

1.2. Denote by $\mathrm{E}(\Lambda)$ the Ext-algebra $\bigoplus\limits_{i\geqslant 0} \mathrm{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$, with the multiplication given by the Yoneda product. We also consider the even Ext-algebra

$$\mathrm{E}^{\mathrm{ev}}(\Lambda) := \bigoplus_{i\geqslant 0} \mathrm{Ext}^{2i}_\Lambda(\Lambda_0, \Lambda_0),$$

which is a positively graded algebra with grading $\mathrm{E}^{\mathrm{ev}}(\Lambda)_n := \mathrm{Ext}_\Lambda^{2n}(\Lambda_0, \Lambda_0)$. For a Λ-module M, let $\mathrm{E}(M)$ be the graded $\mathrm{E}(\Lambda)$-module $\bigoplus_{i \geqslant 0} \mathrm{Ext}_\Lambda^i(M, \Lambda_0)$. We also consider the even Ext-module $\mathrm{E}^{\mathrm{ev}}(M) := \bigoplus_{n \geqslant 0} \mathrm{Ext}_\Lambda^{2n}(M, \Lambda_0)$ over $\mathrm{E}(\Lambda)$, and the odd Ext-module $\mathrm{E}^{\mathrm{odd}}(M) := \bigoplus_{n \geqslant 0} \mathrm{Ext}_\Lambda^{2n+1}(M, \Lambda_0)$ over $\mathrm{E}^{\mathrm{ev}}(\Lambda)$: they are graded modules with gradings

$$\mathrm{E}^{\mathrm{ev}}(M)_n := \mathrm{Ext}_\Lambda^{2n}(M, \Lambda_0), \quad \text{and} \quad \mathrm{E}^{\mathrm{odd}}(M)_n := \mathrm{Ext}_\Lambda^{2n+1}(M, \Lambda_0), \ \forall\, n \geq 0.$$

Every graded Λ-module M has a graded projective resolution

$$\mathbf{Q}^\bullet : \ \cdots \to Q^i \to \cdots \to Q^1 \to Q^0 \to M \to 0. \tag{1}$$

If each Q^i is finitely generated, then we say that $\mathbf{Q}^\bullet$ is *a finitely generated graded projective resolution* of M. If $M \in \mathrm{gr}(\Lambda)$, then M admits a *minimal* graded projective resolution (1) in the sense that $\mathrm{Im}(Q^i \to Q^{i-1}) \subseteq JQ^{i-1}$, $\forall\, i \geq 1$ (see Propositions 2.3 and 2.4 in [GM]).

If $M \in \mathrm{gr}(\Lambda)$, then for each $N \in \mathrm{Gr}(\Lambda)$, $\mathrm{Hom}_\Lambda(M, N)$ is a graded k-space with *the shift-grading*: $\mathrm{Hom}_\Lambda(M, N)_i = \mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(M, N[i])$, i.e., $\mathrm{Hom}_\Lambda(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(M, N[i])$.

If M has a finitely generated graded projective resolution, then for each $N \in \mathrm{Gr}(\Lambda)$ and each $n \geq 1$, $\mathrm{Ext}_\Lambda^n(M, N)$ is a graded k-space with *the shift grading*: $\mathrm{Ext}_\Lambda^n(M, N)_i = \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^n(M, N[i])$, i.e., $\mathrm{Ext}_\Lambda^n(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}_{\mathrm{Gr}(\Lambda)}^n(M, N[i])$.

Fix a minimal graded projective resolution of the trivial Λ-module Λ_0:

$$\mathbf{P}^\bullet : \cdots \to P^n \to \cdots \to P^1 \to P^0 \to \Lambda_0 \to 0. \tag{2}$$

We need the following fact.

Lemma 1.2. ([GMMZ], Lemma 3.2) *Let M be a graded module supported above degree* 0 *with a minimal graded projective resolution* (1). *For any integer $n \geq 1$, if P^n in* (2) *is supported above degree s, then so is Q^n.*

1.3. For the theory of the Koszul algebras and the Koszul modules we refer to A. Beilinson, V. Ginzburg and W. Soergel [BGS], and E. L. Green and R. Martínez-Villa [GM].

Definition 1.3 ([GMMZ], [MM])**.** *Let $d \geq 2$ be an integer. A graded Λ-module M is a d-Koszul module if M admits a finitely generated graded projective resolution* (1) *such that each Q^i is generated in degree $\delta(i)$, where*

$$\delta(i) := \begin{cases} nd, & \text{if } i = 2n, \\ nd + 1, & \text{if } i = 2n+1. \end{cases}$$

If the trivial Λ-module Λ_0 is a d-Koszul module, then we call Λ a d-Koszul algebra.

Theorem 1.4. ([GMMZ], Theorem 6.1) *Let Λ be a d-Koszul algebra and M a d-Koszul Λ-module. Then $\mathrm{E}^{\mathrm{ev}}(\Lambda)$ is a Koszul algebra, and $\mathrm{E}^{\mathrm{ev}}(M)$ is a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module.*

2. Generalized d-Koszul modules

2.1. Let $d \geq 2$ be an integer. For each integer $i \geq 0$ we assign a subset $\Delta(i)$ of $\mathbb{N}_0$ as

$$\Delta(i) := \begin{cases} \{nd\}, & \text{if } i = 2n; \\ \{nd+1, \cdots, nd+d-1\}, & \text{if } i = 2n+1. \end{cases}$$

Definition 2.1. *A graded Λ-module M is called a generalized d-Koszul module if M admits a finitely generated graded projective resolution $\mathbf{Q}^\bullet$ such that each Q^i is generated in degrees in $\Delta(i)$, i.e., $Q^i = \Lambda(\bigoplus\limits_{j\in\Delta(i)} Q^i_j)$, $i \geq 0$.*

Remark 2.2. (i) *As remarked by Beilinson-Ginzburg-Soergel [BGS] (p.476) in the Koszul situation, $\mathbf{Q}^\bullet$ in Definition 2.1 is unique up to isomorphism.*

More precisely, if $\mathbf{L}^\bullet$ is another graded projective resolution of M such that each L^i is also generated in degrees in $\Delta(i)$ (it is not assumed to be finitely generated), then $\mathbf{L}^\bullet \cong \mathbf{Q}^\bullet$ as complexes. In fact, $\mathbf{L}^\bullet$ is homotopy equivalent to $\mathbf{Q}^\bullet$; while any chain maps $\mathbf{Q}^\bullet \to \mathbf{Q}^\bullet$ and $\mathbf{L}^\bullet \to \mathbf{L}^\bullet$, which respect the grading on Q^i and L^i and are homotopic to zero must themselves be zero (since any element in $\Delta(i)$ is strictly smaller than any element in $\Delta(i+1)$, and Q^i and L^i are both generated in degrees in $\Delta(i)$). It follows that $\mathbf{L}^\bullet \cong \mathbf{Q}^\bullet$ as complexes.

(ii) *We emphasize that, as in the d-Koszul situation, here Q^i is also required to be* **finitely generated***: it is for the application of the shift grading on $\mathrm{Ext}^n_\Lambda(M, -)$.*

(iii) *If M is a generalized d-Koszul module, then such a graded projective resolution $\mathbf{Q}^\bullet$ in the definition is minimal, and each syzygy $\Omega^i(M)$ is a graded Λ-module finitely generated in degrees in $\Delta(i)$. In particular, M is finitely generated in degree 0.*

(iv) *A d-Koszul module is always generalized d-Koszul; and a generalized 2-Koszul module is a finitely generated Koszul module (if Λ is noetherian, then a generalized 2-Koszul Λ-module is exactly a finitely generated Koszul Λ-module).*

Example 2.3. *Let A be the algebra given by the quiver*

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 3$$

with relations α^3, $\gamma\beta\alpha$. Then the simple (left) module $S(1)$ has a minimal graded projective resolution

$$\cdots \to P(1)[4] \oplus P(2)[5] \to P(1)[3] \oplus P(3)[3] \to P(1)[1] \oplus P(2)[1] \to P(1) \to S(1) \to 0$$

where $\Omega^4 S(1) = (\Omega^2 S(1))[3]$. Thus $S(1)$ is a generalized 3-Koszul A-module. Since $Q^3 = P(1)[4] \oplus P(2)[5]$ is generated in degrees 4 and 5, but not generated in degree 4, it follows that $S(1)$ is **not** *a 3-Koszul A-module (by an argument in Remark 2.2(i)).*

2.2. We have the following characterization for a d-Koszul module and for a generalized d-Koszul module, which is the corresponding version of Proposition 2.14.2 in Beilinson - Ginzburg - Soergel [BGS] for the Koszul modules.

Lemma 2.4. *Let M be a graded Λ-module with a finitely generated graded projective resolution. Then*

(i) M is d-Koszul if and only if $\mathrm{Ext}^i_\Lambda(M, \Lambda_0)_j = \mathrm{Ext}^i_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0, \ \forall\ j \neq \delta(i)$.

(ii) M is generalized d-Koszul if and only if $\mathrm{Ext}^i_\Lambda(M, \Lambda_0)$ *is concentrated in degrees in $\Delta(i)$, with the shift grading, i.e.,* $\mathrm{Ext}^i_\Lambda(M, \Lambda_0)_j = \mathrm{Ext}^i_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0, \ \forall\ j \notin \Delta(i)$.

Proof. They can be similarly proved as Proposition 2.14.2 in [BGS]. For the convenience of the reader we include a justification of (ii).

Assume that M is generalized d-Koszul. Then M has a graded projective resolution $\mathbf{Q}^\bullet$ such that each Q^i is generated in degrees in $\Delta(i)$, and $\mathrm{Ext}^i_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j])$ is the i-th cohomology group of the complex $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(\mathbf{Q}^\bullet, \Lambda_0[j])$. Since Q^i is generated in degrees in $\Delta(i)$, and $\Lambda_0[j]$ is concentrated in degree j, it follows that $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(Q^i, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$, and hence $\mathrm{Ext}^i_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$.

Conversely, assume that $\mathrm{Ext}^i_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$. We construct inductively a graded projective resolution $\mathbf{L}^\bullet$ of M such that each L^i is generated in degrees in $\Delta(i)$. Since $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \neq 0$, by Lemma 1.1, M is generated in degree 0, and hence we have a surjective graded Λ-homomorphism $L^0 \to M$ such that L^0 is generated in degree 0. Denote by K^1 its kernel. Then $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(K^1, \Lambda_0[j]) = \mathrm{Ext}^1_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(1)$, and hence by Lemma 1.1, K^1 is generated in degrees in $\Delta(1)$. Thus we have a surjective graded Λ-homomorphism $L^1 \to K^1$ such that L^1 is generated in degrees in $\Delta(1)$. Repeating this process we are done.

By assumption we have already a finitely generated graded projective resolution $\mathbf{Q}^\bullet$. By the argument in Remark 2.2(i), there are chain maps $f : \mathbf{L}^\bullet \to \mathbf{Q}^\bullet$ and $g : \mathbf{Q}^\bullet \to \mathbf{L}^\bullet$ such that $gf = \mathrm{Id}_{\mathbf{L}^\bullet}$, which means that $\mathbf{L}^\bullet$ is a direct summand of $\mathbf{Q}^\bullet$. Thus $\mathbf{L}^\bullet$ is also a finitely generated resolution. By definition M is generalized d-Koszul. □

2.3. For a d-Koszul module M, in general $\Omega^i M$ and $J^i M$ are **not** d-Koszul modules, up to shifts (see Proposition 5.2 in [GMMZ] for some special cases); however, they turn out to be generalized d-Koszul, after proper shifts. In the rest of this section we precisely state and prove these results, which will be important in the proof of Main Theorem and Corollary.

Lemma 2.5. *Let M be a d-Koszul Λ-module. Then*

(i) $\mathrm{E}^{\mathrm{odd}}(M) \cong \mathrm{E}^{\mathrm{ev}}(\Omega M)$ *as graded* $\mathrm{E}(\Lambda)$*-modules.*

(ii) $(\Omega^i M)[-\delta(i)]$ *is a generalized d-Koszul module for each* $i \geq 0$.

Proof. (i) By definition we have an isomorphism of graded $\mathrm{E}(\Lambda)$-modules

$$\mathrm{E}^{\mathrm{odd}}(M) = \bigoplus_{n\geqslant 0} \mathrm{Ext}_\Lambda^{2n+1}(M, \Lambda_0) \cong \bigoplus_{n\geqslant 0} \mathrm{Ext}_\Lambda^{2n}(\Omega M, \Lambda_0) = \mathrm{E}^{\mathrm{ev}}(\Omega M).$$

(ii) Taking a graded projective resolution $\mathbf{Q}^\bullet$ of M such that each Q^i is finitely generated in degree $\delta(i)$, we see that $(\Omega^i M)[-\delta(i)]$ has a graded projective resolution

$$\mathbf{L}^\bullet: \ \cdots \to L^j \to \cdots \to L^1 \to L^0 \to (\Omega^i M)[-\delta(i)] \to 0$$

where $L^j = Q^{i+j}[-\delta(i)]$ is finitely generated in degree $\delta(i+j) - \delta(i)$ for $j \geq 0$.

If i is even, then L^j is generated in degree $\delta(j)$. That is, $(\Omega^i M)[-\delta(i)]$ is a d-Koszul module, and hence a generalized d-Koszul module.

Assume that i is odd. Then L^j is generated in degree nd if $j = 2n$, and L^j is generated in degree $nd+d-1$ if $j = 2n+1$. By definition $(\Omega^i M)[-\delta(i)]$ is generalized d-Koszul. □

Theorem 2.6. *Let Λ be a d-Koszul algebra and M a generalized d-Koszul Λ-module. Then*

(i) *$(J^i M)[-i]$ is generalized d-Koszul for each $i \geq 1$.*

(ii) *For each $n \geq 1$ we have k-isomorphisms*

$$\mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[nd]) \cong \mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(J^2 M, \Lambda_0[nd]) \cong \cdots \cong \mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(J^{d-1} M, \Lambda_0[nd]).$$

Proof. (i) It suffices to prove that $(JM)[-1]$ is generalized d-Koszul. Since $M = \bigoplus_{i\geqslant 0} M_i$ is finitely generated in degree 0, $JM = \bigoplus_{i\geqslant 1} M_i$ is finitely generated in degree 1.

We first prove the following claim: $JM[-1]$ admits a graded projective resolution $\mathbf{Q}_1^\bullet$ such that Q_1^i is generated in degrees in $\Delta(i)$. By the proof of Lemma 2.4(ii), it suffices for each $n \geq 0$ to prove that $\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JM[-1], \Lambda_0[j]) = \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0, \quad \forall\ j \neq nd$, and that $\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0, \quad \forall\ j \notin \Delta(2n+1) = \{nd+1, \cdots, nd+d-1\}$.

Applying $\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(-, \Lambda_0[j+1])$ to the graded exact sequence $0 \to JM \to M \to M/JM \to 0$ we get the following exact sequence of k-spaces

$$\begin{aligned} &\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) \to \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) \\ \to &\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) \to \mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \mathrm{Ext}^{2(n+1)}_{\mathrm{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]). \end{aligned}$$

Since Λ is a d-Koszul algebra, P^{2n} in (2) is supported above degrees nd, and hence by Lemma 1.2, Q^{2n} is supported above degrees nd, where $\mathbf{Q}^\bullet$ is a minimal graded projective resolution of $JM[-1]$. Thus $\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$ for $j < nd$. Similarly, $\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$ for $j < nd+1$.

Since M is generalized d-Koszul, by Lemma 2.4(ii), $\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$ if $j \neq nd-1$, and $\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$ if $j \notin \{nd, \cdots, nd+d-2\}$.

Note that M/JM is a Λ/J-module and $\Lambda/J = \Lambda_0$ is a semisimple algebra. Thus M/JM is a direct summand of a finite direct sum of copies of the trivial Λ-module

Λ_0. In particular, M/JM is a d-Koszul module. By Lemma 2.4(i),

$$\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0 \ \text{ if } j \neq nd,$$

and $\mathrm{Ext}^{2(n+1)}_{\mathrm{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0$ if $j \neq (n+1)d - 1$.

Now if $j \neq nd$, then by the exact sequence above we have the following exact sequence

$$\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) \to \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0,$$

where if $j \neq nd-1$ then $\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$, and hence $\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$; and if $j = nd-1 < nd$, then we already know $\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$.

Let $j \notin \Delta(2n+1) = \{nd+1, \cdots, nd+d-1\}$. Then by the exact sequence above we have the following exact sequence

$$\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) \to \mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) \to \mathrm{Ext}^{2(n+1)}_{\mathrm{Gr}(\Lambda)}(M/JM, \Lambda_0[j+1]) = 0,$$

where if $j \notin \{nd, \cdots, nd+d-2\}$ then $\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(M, \Lambda_0[j+1]) = 0$, and hence $\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$; and if $j \in \{nd, \cdots, nd+d-2\}$, then $j = nd < nd+1$, and in this case we already know $\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[j+1]) = 0$. This proves the claim.

Since M/JM is a d-Koszul module, M/JM has a finitely generated graded projective resolution, say $\mathbf{Q}_3^\bullet$, such that $\mathbf{Q}_3^i$ is generated in degrees in $\delta(i)$. By the graded version of the Horseshoe Lemma, we get a graded projective resolution $\mathbf{Q}_2^\bullet$ of M, such that $\mathbf{Q}_2^i = \mathbf{Q}_1^i \oplus \mathbf{Q}_3^i$ for each i. Thus $\mathbf{Q}_2^i$ is also generated in degrees in $\Delta(i)$. Since M is a generalized d-Koszul module, by Remark 2.2(i), we know that $\mathbf{Q}_2^\bullet$ is finitely generated, and hence $\mathbf{Q}_1^\bullet$ is finitely generated. By definition $JM[-1]$ is generalized d-Koszul.

(ii) Let $d \geq 3$. Applying $\mathrm{Hom}_\Lambda(-, \Lambda_0)$ to the graded exact sequence $0 \to J^2M \to JM \to JM/J^2M \to 0$, we get the following exact sequence

$$\mathrm{Ext}^{2n-1}_\Lambda(JM/J^2M, \Lambda_0) \to \mathrm{Ext}^{2n-1}_\Lambda(JM, \Lambda_0) \to \mathrm{Ext}^{2n-1}_\Lambda(J^2M, \Lambda_0) \to \mathrm{Ext}^{2n}_\Lambda(JM/J^2M, \Lambda_0).$$

Since $(JM/J^2M)[-1]$ is d-Koszul, by Lemma 2.4(i), $\mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JM/J^2M, \Lambda_0[j]) = 0$ if $j \neq nd - d + 2$, and $\mathrm{Ext}^{2n}_\Lambda(JM/J^2M, \Lambda_0[j]) = 0$ if $j \neq nd+1$. Taking the nd-th homogeneous components of the exact sequence above, we obtain that $\mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JM, \Lambda_0[nd]) \cong \mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(J^2M, \Lambda_0[nd])$. Repeating the process one gets (ii). □

3. Proofs of Main Theorem and Corollary

3.1. We begin with a lemma, which seems to be of independent interest.

Lemma 3.1. *Let A be an arbitrary Koszul algebra and $\mathcal{C}$ a full subcategory of* $\mathrm{Gr}(A)$. *Suppose that for any $X \in \mathcal{C}$, there exist exact sequences in* $\mathrm{Gr}(A)$

$$0 \to \Omega \to P^0 \to X \to 0, \tag{3}$$

$$0 \to X'' \to X' \to \Omega[-1] \to 0, \tag{4}$$

such that P^0 is a graded projective A-module generated in degree 0 *and $X', X'' \in \mathcal{C}$. Then all modules in $\mathcal{C}$ are Koszul A-modules.*

Proof. By Proposition 2.14.2 in Beilinson - Ginzburg - Soergel [BGS], it suffices to prove that for each $X \in \mathcal{C}$, $\mathrm{Ext}^i_{\mathrm{Gr}(A)}(X, A_0[j]) = 0$ unless $j = i$. We use induction on i.

The sequence (3) implies that X is generated in degree 0, and hence $\mathrm{Hom}_{\mathrm{Gr}(A)}(X, A_0[j]) = 0$ unless $j = 0$. The sequence (4) implies that Ω is a graded A-module and is generated in degree 1, since $X' \in \mathcal{C}$ is generated in degree 0. By $\mathrm{Ext}^1_{\mathrm{Gr}(A)}(X, A_0[j]) \cong \mathrm{Hom}_{\mathrm{Gr}(A)}(\Omega, A_0[j])$, we see that $\mathrm{Ext}^1_{\mathrm{Gr}(A)}(X, A_0[j]) = 0$ unless $j = 1$.

Let $n \geq 1$. Assume that for each $X \in \mathcal{C}$ and for each positive integer i with $i \leq n$, $\mathrm{Ext}^i_{\mathrm{Gr}(A)}(X, A_0[j]) = 0$ unless $j = i$. The exact sequence (4) implies the following exact sequence for every integer j

$$\mathrm{Ext}^{n-1}_{\mathrm{Gr}(A)}(X'', A_0[j]) \to \mathrm{Ext}^n_{\mathrm{Gr}(A)}(\Omega[-1], A_0[j]) \to \mathrm{Ext}^n_{\mathrm{Gr}(A)}(X', A_0[j]).$$

By the inductive hypothesis, we have $\mathrm{Ext}^{n-1}_{\mathrm{Gr}(A)}(X'', A_0[j]) = 0$ unless $j = n-1$, and $\mathrm{Ext}^n_{\mathrm{Gr}(A)}(X', A_0[j]) = 0$ unless $j = n$. Let $\mathbf{Q}^\bullet$ be a minimal graded projective resolution of $\Omega[-1]$ (it exists since $\Omega \subseteq P^0$ is supported above 0). By Lemma 1.2, Q^n is supported above degree n, which implies $\mathrm{Ext}^n_{\mathrm{Gr}(A)}(\Omega[-1], A_0[j]) = 0$ for $j < n$. It follows from the exact sequence above that $\mathrm{Ext}^n_{\mathrm{Gr}(A)}(\Omega[-1], A_0[j]) = 0$ unless $j = n$. Thus $\mathrm{Ext}^{n+1}_{\mathrm{Gr}(A)}(X, A_0[j]) = \mathrm{Ext}^n_{\mathrm{Gr}(A)}(\Omega, A_0[j]) = \mathrm{Ext}^n_{\mathrm{Gr}(A)}(\Omega[-1], A_0[j-1]) = 0$ unless $j = n+1$. This completes the proof. □

3.2. Proof of Main Theorem. By Theorem 1.4, $\mathrm{E}^{\mathrm{ev}}(\Lambda)$ is a Koszul algebra. Put

$$\mathcal{C} := \{\mathrm{E}^{\mathrm{ev}}(N) \in \mathrm{Gr}(\mathrm{E}^{\mathrm{ev}}(\Lambda)) \mid N \text{ is a generalized } d\text{-Koszul } \Lambda\text{-module}\}.$$

It suffices to prove that all the conditions in Lemma 3.1 are satisfied.

The graded exact sequence $0 \to JN \to N \to N/JN \to 0$ induces the following exact sequence of graded k-spaces for each $n \geq 0$

(5)

$$\mathrm{Ext}^{2n-1}_\Lambda(N, \Lambda_0) \to \mathrm{Ext}^{2n-1}_\Lambda(JN, \Lambda_0) \to \mathrm{Ext}^{2n}_\Lambda(N/JN, \Lambda_0) \to \mathrm{Ext}^{2n}_\Lambda(N, \Lambda_0) \to \mathrm{Ext}^{2n}_\Lambda(JN, \Lambda_0).$$

Since N and $JN[-1]$ are generalized d-Koszul, by Lemma 2.4(ii), we have $\mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(N, \Lambda_0[nd]) = 0 = \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(JN, \Lambda_0[nd])$. Taking the nd-th homogeneous components of (5) we get the following exact sequence for each $n \geq 0$

(6)

$$0 \to \mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JN, \Lambda_0[nd]) \to \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(N/JN, \Lambda_0[nd]) \to \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(N, \Lambda_0[nd]) \to 0.$$

Since N is generalized d-Koszul and N/JN is d-Koszul, by Lemma 2.4, we have

$$\mathrm{E}^{\mathrm{ev}}(N/JN) = \bigoplus_{n\geq 0} \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(N/JN, \Lambda_0[nd]), \quad \mathrm{E}^{\mathrm{ev}}(N) = \bigoplus_{n\geq 0} \mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(N, \Lambda_0[nd]).$$

By taking direct sum of (6), we get the following short exact sequence in $\mathrm{Gr}(\mathrm{E}^{\mathrm{ev}}(\Lambda))$:

$$0 \to \Omega \to \mathrm{E}^{\mathrm{ev}}(N/JN) \to \mathrm{E}^{\mathrm{ev}}(N) \to 0, \tag{7}$$

where $\Omega := \bigoplus_{n\geq 0} \mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JN, \Lambda_0[nd])$. In particular, Ω is a graded $\mathrm{E}^{ev}(\Lambda)$-module with grading $\Omega_n := \mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JN, \Lambda_0[nd])$. (One can also prove this directly as follows:

since Λ is d-Koszul algebra, it follows from Lemma 2.4(i) that

$$\mathrm{Ext}^{2m}_{\Lambda}(\Lambda_0,\Lambda_0)\,\mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JN,\Lambda_0[nd]) = \mathrm{Ext}^{2m}_{\mathrm{Gr}(\Lambda)}(\Lambda_0,\Lambda_0[md])\,\mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JN,\Lambda_0[nd]) \subseteq \mathrm{Ext}^{2(n+m)-1}_{\mathrm{Gr}(\Lambda)}(JN,\Lambda_0[(n+m)d]).)$$

By Theorem 1.4, $\mathrm{E}^{\mathrm{ev}}(N/JN)$ is a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module, in particular it is generated in degree 0. Since N/JN is a direct summand of finite direct sum of copies of the trivial Λ-module Λ_0, $\mathrm{E}^{\mathrm{ev}}(N/JN)$ is a projective $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module.

Similarly, the graded exact sequence $0 \to J^dN \to J^{d-1}N \to J^{d-1}N/J^dN \to 0$ induces the following exact sequence of graded k-spaces for each $n \geq 0$

$$\mathrm{Ext}^{2n}_{\Lambda}(J^{d-1}N,\Lambda_0) \to \mathrm{Ext}^{2n}_{\Lambda}(J^{d}N,\Lambda_0) \to \mathrm{Ext}^{2n+1}_{\Lambda}(J^{d-1}N/J^dN,\Lambda_0) \to \mathrm{Ext}^{2n+1}_{\Lambda}(J^{d-1}N,\Lambda_0) \to \mathrm{Ext}^{2n+1}_{\Lambda}(J^{d}N,\Lambda_0).$$

Note that by Theorem 2.6(i), $J^{d-1}N[-(d-1)]$ and $J^dN[-d]$ are generalized d-Koszul Λ-modules, and that $(J^{d-1}N/J^dN)[-(d-1)]$ is a d-Koszul module. Taking the $(n+1)d$-th homogeneous components, and by the same arguments we get another exact sequence in $\mathrm{Gr}(\mathrm{E}^{\mathrm{ev}}(\Lambda))$:

$$0 \to \bigoplus_{n\geq 0}\mathrm{Ext}^{2n}_{\mathrm{Gr}(\Lambda)}(J^dN,\Lambda_0[(n+1)d]) \to \bigoplus_{n\geq 0}\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(J^{d-1}N/J^dN,\Lambda_0[(n+1)d]) \to \bigoplus_{n\geq 0}\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(J^{d-1}N,\Lambda_0[(n+1)d]) \to 0,$$

or equivalently,

$$0 \to \mathrm{E}^{\mathrm{ev}}(J^dN) \to \mathrm{E}^{\mathrm{odd}}(J^{d-1}N/J^dN) \to \Omega[-1] \to 0, \tag{8}$$

where

$$\Omega[-1] = (\bigoplus_{n\geq 0}\mathrm{Ext}^{2n-1}_{\mathrm{Gr}(\Lambda)}(JN,\Lambda_0[nd]))[-1] = \bigoplus_{n\geq 0}\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(JN,\Lambda_0[(n+1)d]) \cong \bigoplus_{n\geq 0}\mathrm{Ext}^{2n+1}_{\mathrm{Gr}(\Lambda)}(J^{d-1}N,\Lambda_0[(n+1)d]),$$

where the last isomorphism follows from Theorem 2.6(ii).

Since $(J^{d-1}N/J^dN)[-(d-1)]$ is d-Koszul, by Lemma 2.5(ii), $\Omega(J^{d-1}N/J^dN)[-d]$ is generalized d-Koszul, and by Lemma 2.5(i), we have

$$\mathrm{E}^{\mathrm{odd}}(J^{d-1}N/J^dN) = \mathrm{E}^{\mathrm{odd}}((J^{d-1}N/J^dN)[-(d-1)]) \cong \mathrm{E}^{\mathrm{ev}}(\Omega(J^{d-1}N/J^dN)[-d]),$$

from which we see $\mathrm{E}^{\mathrm{odd}}(J^{d-1}N/J^dN) \in \mathcal{C}$.

Since N is generalized d-Koszul, by Theorem 2.6(i), $(J^dN)[-d]$ is generalized d-Koszul. Thus $\mathrm{E}^{\mathrm{ev}}(J^dN) = \mathrm{E}^{\mathrm{ev}}((J^dN)[-d]) \in \mathcal{C}$.

Now (7) and (8) shows that all the conditions in Lemma 3.1 are satisfied. This completes the proof. □

3.3. Proof of Corollary. By Lemma 2.5(ii), $(\Omega M)[-1]$ is a generalized d-Koszul module. It follows from Main Theorem that $\mathrm{E}^{\mathrm{ev}}((\Omega M)[-1])$ is a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module. Therefore by Lemma 2.5(i), $\mathrm{E}^{\mathrm{odd}}(M) \cong \mathrm{E}^{\mathrm{ev}}(\Omega M) = \mathrm{E}^{\mathrm{ev}}((\Omega M)[-1])$ is a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module. □

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References

[BF] J. Backelin, R. Fröberg, Koszul algebras, Veronese subrings and rings with linear resolution, Rev. Roumaine Math. Pure Appl. 30(1985), 85-97.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9(1996), 473-527.

[B1] R. Berger, Koszulity for nonquadratic algebras, J. Algebra, 239(2001), 705-734.

[B2] R. Berger, Gerasimov's theorem and N-Koszul algebras, J. Lond. Math. Soc. (2) 79 (2009), 631-648.

[CD] A. Connes, M. Dubois-Violette, Yang-Mills algebra, Lett. Math. Phys. 61(2002), 149-158.

[GM] E. L. Green, R. Martínez-Villa, Koszul and Yoneda algebras, In: Representation theory of algebras (Cocoyoc, 1994), 247-297, CMS Conf. Proc. 18, Amer. Math. Soc., Providence, RI, 1996.

[GMMZ] E. L. Green, E.N. Marcos, R. Martínez-Villa, P. Zhang, D-Koszul algebras, J. Pure Appl. Algebra 193(2004), 141-162.

[MM] E. N. Marcos, R. Martínez-Villa, The odd part of an n-Koszul algebra, Commun. Algebra 33(2005), 101-108.

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